Analysis Workshop
A simple dynamical system and chaos

Story. "Dynamical systems" is an area of mathematics which looks at the evolution of a system over time. In this workshop you will be looking at a discrete time dynamical system which is usually represented mathematically via considering the iterates of a function.

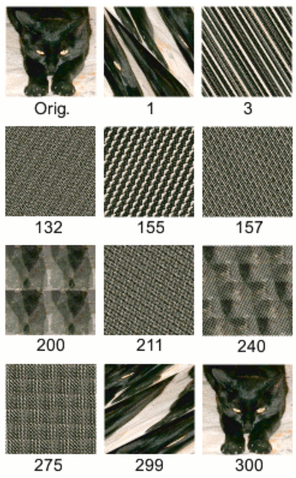
Dynamical systems is a field of mathematics which to a large extent was developed in the twentieth century, much of which took place in the old Soviet Union. Two mathematicians who played significant roles in this are Dmitry Anosov

http://www-history.mcs.st-andrews.ac.uk/Biographies/Anosov.html

and Vladimir Arnold

http://www-history.mcs.st-andrews.ac.uk/Biographies/Arnold.html.

You may well have heard of the term "chaos" which is an informal name given to a system where a small change in an initial condition makes a very big change to the long term behaviour of the system. Examples of this include certain maps which can be categorised in terms of having an expanding direction or of having a contracting direction, and these maps are called Anosov diffeomphisms. One particular example of this is the Arnold cat map, which is a map on the "torus" (being the surface of a doughnut). The name "cat map" is sometimes thought to stand for "Continuous Automorphism of the Torus", but in fact the name comes from Arnold illustrating the behaviour of the map by considering its action on his cat.



(Attribution: pictures by Claudio Rocchini - Own Work (It's not proper Arnold's cat but my black cat, due copyright restrictions), CC BY 2.5, https://commons.wikimedia.org/w/index.php?curid=1350710)

Typical examples of the maps referred to above are given by certain matrices acting on the torus (the hollow doughnut). Here we instead look at similar maps on $\mathbb{R}$; the downside of this is that it means the maps are not chaotic, but the upside is that they are much easier to analyse. Still, the ideas of contraction and expansion are significant in this workshop.

For further exploration: One way of making these dynamical systems show more interesting behaviour (when $|a| > 1$) is to restrict them to a bounded interval. To give an example of how this can be done; for $x \in \mathbb{R}$ let $\lfloor x \rfloor = \sup\{k \in \mathbb{Z} : k \leq x\}$ denote the integer part of x and $\{x\} = x - \lfloor x \rfloor x$. We then define $T : [0, 1) \to [0, 1)$ by $T(x) = \{10x\}$ (often written $10x$ mod 1). Any $x \in [0, 1)$ can be written as a decimal expansion

$$x = \sum_{i=1}^{\infty} a_i 10^{-i}$$

where each $a_i \in \{0, 1, \ldots, 9\}$. Consider the following:

1. What is the decimal expansion of $T(x)$ in terms of the decimal expansion of x?

2. If for some $k \neq l \in \mathbb{N}$ we have $T^k(x) = T^l(x)$ what does this say about the decimal expansion of x? What does this mean in terms of whether x is rational or irrational?