UNIVERSITY OF BRISTOL

Examination for the Degree of B.Sc. and M.Sci. (Level C/4)

ANALYSIS 1A

MATH 10003

(Paper Code MATH-10003J)

January 2017 1 hour 30 minutes

This paper contains two sections, Section A and Section B. Please use a separate answer booklet for each section.
Section A contains ***five*** *short questions.* ***ALL*** *answers will be used for assessment. This section is worth 40% of the marks for the paper.*
Section B contains ***two*** *longer questions.* ***ALL*** *answers will be used for assessment. This section is worth 60% of the marks for the paper.*

Calculators are **not** *permitted in this examination.*

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Please note on this exam $\mathbb{N} = \{1, 2, 3, \ldots\}$

Do not turn over until instructed.

Section A: Short Questions

A1. **(4+4 marks)** Let $A = \{1 - n : n \in \mathbb{N}\}$ and $B = \{x \in \mathbb{R} : |x - 1| \geq |x - 2|\}$.

(a) Find $\sup A$ and $\inf A$. Justify your answers.

(b) Find $\sup B$ and $\inf B$. Justify your answers.

A2. **(2+3+3 marks)**

(a) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and $\alpha \in \mathbb{R}$. Define what it means for $\lim_{n\to\infty} a_n = \alpha$.

(b) For each of the following sequences, find the limit (justifying your answer).

i. $(a_n)_{n\in\mathbb{N}}$ where $a_n = \frac{n+7}{n^2+9}$ for all $n \in \mathbb{N}$.

ii. $(b_n)_{n\in\mathbb{N}}$ where $b_n = \frac{2^n+5}{2^n}$ for all $n \in \mathbb{N}$.

A3. **(3+5 marks)**

(a) Define what it means for a sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ to be a Cauchy sequence.

(b) Prove that if $(b_n)_{n\in\mathbb{N}}$ is a convergent sequence of real numbers then it is a Cauchy sequence.

A4. **(2+3+3 marks)**
Determine which of the following series are convergent. Justify your answers, stating clearly any convergence test you are using.

(a) $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+1}$

(b) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

(c) $\sum_{n=1}^{\infty} \frac{2}{2n^2-1}$

A5. **(3+5 marks)**

(a) State the Intermediate Value Theorem.

(b) Show that there exists $x \in (0, 1)$ such that

$$x^3 + x^2 + 4 = 7x.$$

Section B: Longer Questions

B1. (a) **(3+5+6 marks)**

i. State the monotone convergence theorem.

ii. Let $f : (0, 2] \to \mathbb{R}$ be a function which is monotone increasing (i.e. for all $x, y \in (0, 2]$ if $x \leq y$ then $f(x) \leq f(y)$).

A. Show that the sequence $(a_n)_{n\in\mathbb{N}}$ where $a_n = f\left(1 - \frac{1}{2n}\right)$ for all $n \in \mathbb{N}$ is convergent.

B. Show that if f is unbounded then $\lim_{x\to 0} f(x) = -\infty$.

(b) **(3 marks)**
State the Extremal Value Theorem.

(c) **(5 marks)**
Give an example of a function $f : (1, 2) \to \mathbb{R}$ where f is continuous on $(1, 2)$ but is unbounded. Justify your answer.

(d) **(8 marks)** Let $f, g : [0, 1] \to \mathbb{R}$ be functions which are continuous on $[0, 1]$, where $g(x) > 0$ for all $x \in [0, 1]$ and where $f(x) < g(x)$ for all $x \in [0, 1]$. Show that there exists $\alpha < 1$ such that $f(x) \leq \alpha g(x)$ for all $x \in [0, 1]$. (**Hint:** consider the function $h : [0, 1] \to \mathbb{R}$ defined by $h(x) = \frac{f(x)}{g(x)}$ for all $x \in [0, 1]$).

B2. (a) **(3 marks)**
State the Bolzano-Weierstrass Theorem.

(b) **(5 marks)**
Give an example of an unbounded sequence which contains a convergent subsequence.

(c) **(5 marks)**
Show that if $(a_n)_{n\in\mathbb{N}}$ is a sequence such that $\lim_{n\to\infty} |a_n| = \infty$ then all subsequences $(a_{n_k})_{k\in\mathbb{N}}$ satisfy that $\lim_{k\to\infty} |a_{n_k}| = \infty$.

(d) **(6 marks)** Show that a sequence $(a_n)_{n\in\mathbb{N}}$ contains no convergent subsequences if and only if $\lim_{n\to\infty} |a_n| = \infty$.

(e) **(4 marks)** Give an example of a sequence $(a_n)_{n\in\mathbb{N}}$ where $\lim_{n\to\infty} a_n = 0$ but the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(f) **(7 marks)** Show that if a sequence of positive real numbers $(a_n)_{n\in\mathbb{N}}$ satisfies that $\lim_{n\to\infty} a_n = 0$ then it contains a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ where the series $\sum_{k=1}^{\infty} a_{n_k}$ is convergent.

End of examination.