Vol. 89, No. 1 DUKE MATHEMATICAL JOURNAL © 1997

EXPLICIT SIEGEL THEORY: AN ALGEBRAIC APPROACH

LYNNE H. WALLING

To the memory of Martin Eichler

Let Q be a positive definite quadratic form on a $\mathbb{Z}$-lattice L of even rank $m \geqslant 6$; for convenience, assume $Q(L) \subseteq 2\mathbb{Z}$. To gain understanding of the representation numbers

$$r(L, 2n) = \#\{x \in L: Q(x) = 2n\},$$

we study the average representation numbers

$$r(\text{gen } L, 2n) = \frac{1}{\text{mass } L} \sum_{L' \in \text{gen } L} \frac{1}{o(L')} r(L', 2n),$$

since $r(L, 2n)$ is asymptotic to $r(\text{gen } L, 2n)$ as $n \to \infty$. Here L' runs over the distinct isometry classes within gen L, the genus of L; $o(L')$ denotes the order of the orthogonal group of L'; and mass $L = \sum_{L' \in \text{gen } L}(1/o(L'))$.

In the 1930s Siegel used analytic methods to show that $r(\text{gen } L, 2n)$ is a product of "p-adic densities' (see [5]; cf. [2]):

$$r(\text{gen } L, 2n) = c \prod_q \frac{A_q(L, 2n)}{q^{m-1}},$$

where c is an easily computed constant, the product is over all $q = p^a$ with p prime and a sufficiently large, and $A_q(L, 2n)$ is the number of solutions to $Q(x) \equiv 2n \pmod q$, $x \in L/qL$. (Siegel actually shows that the average number of times a definite or indefinite quadratic form of arbitrary level and rank at least 4 represents another quadratic form is the product of p-adic densities.) One could use Hensel's lemma to compute the p-adic densities $((Aq(L, 2n))/(q^{m-1})$, but this gets extremely tedious when L is of arbitrary level.

We use algebraic considerations to obtain a new derivation of Siegel's formula, obtaining a more explicit formula for average representation numbers. We first consider lattices K whose associated theta series $\theta(K; \tau)$ have square-free, odd-level N, and quadratic character χ. Using local considerations, we design

Received 23 February 1996.
Author's work partially supported by National Science Foundation grant number DMS 9103303.

operators on modular forms for which $\theta(\text{gen } K; \tau)$ is an eigenform. We then consider the action of these operators on Eisenstein series, constructing the 1-dimensional simultaneous eigenspace for these operators. Since $\theta(\text{gen } K; \tau)$ is known to lie in the space of Eisenstein series (see [5]; cf. [7]), this allows us to write $\theta(\text{gen } K; \tau)$ as an explicit linear combination of Eisenstein series, giving us our initial formula for average representation numbers (Corollaries 2.6 and 2.7):

$$r(\text{gen } K, 2n) = \frac{\chi(2)}{a_0} \sum_{\substack{D|N \\ d|n}} \frac{\chi_{N/D}(d)\chi_D(n/d)}{c(D)} d^{m/2-1} = \rho_{K,\infty} \prod_{p \text{ prime}} \rho_{K,p}(n),$$

where χ_D, $\chi_{N/D}$ are the unique quadratic characters modulo D, N/D (respectively) so that $\chi_D \chi_{N/D} = \chi$, a_0 and $\rho_{K,\infty}$ are explicit constants, the $c(D)$ are given by simple formulas in terms of the genus invariants of K, and with N' the conductor of χ,

$$\rho_{K,p}(n) = \begin{cases} \dfrac{1 - (\chi(p)p^{m/2-1})^{e+1}}{1 - \chi(p)p^{m/2-1}} (1 - \chi(p)p^{-m/2}) & \text{if } p \nmid N, \\ (c_K(p) + \chi_p(n/p^e)\chi_{N/p}(p^e)p^{e(m/2-1)}) \dfrac{p^{m/2-2} - 1}{p^{m/2-1} - 1} & \text{if } p | N/N', \\ (c_K(p) + \chi_p(n/p^e)\chi_{N/p}(p^e)p^{e(m/2-1)}) p^{-1/2} & \text{if } p | N'. \end{cases}$$

We also show that the average theta series attached to the genera within fam K are linearly independent (Corollary 2.8).

Next, given a lattice L whose theta series has arbitrary (odd) level N', we use lattice constructions and combinatorial arguments to obtain a description of $\theta(\text{gen } L; \tau)$ in terms of partial sums of $\theta(\text{gen } K; \tau)$ where K has square-free level. Using the formulas for $r(\text{gen } K, 2n)$, we prove that $r(\text{gen } L, 2n) = \rho_{L,\infty} \prod_q \rho_{L,q}(n)$ where, for each prime q with $e = \text{ord}_q(n)$ and $\varepsilon = ((n/q^e)/q)$,

$$\rho_{L,q}(n) = v_e(\varepsilon; L, q) + \sum_{0 \leqslant \ell < e} q^{(m/2-1)(e-\ell)} v_\ell(0; L, q);$$

here the quantities v_ℓ are given by simple formulas in terms of the genus invariants of L at q (see Theorem 3.7).

Since it can be shown that the average theta series of a genus is also that of a spinor genus, these formulas describe the average representation numbers of the spinor genus of L.

The lattice techniques used herein are local; thus we can extend these results to lattices of arbitrary rank over totally real number fields (work in progress) and possibly to Siegel modular forms.

The author thanks Bill Duke, Andy Earnest, Kathy Merrill, Tom Shemanske, and Jeff Stopple for helpful and encouraging conversations.

§1. Preliminaries. We review some standard notation and terminology and state some basic results. The reader is referred to [4], [1], and [3].

Let V be an m-dimensional vector space over $\mathbb{Q}$; assume m is even. Let Q be a positive definite quadratic form on V with associated symmetric bilinear form B (so $Q(x) = B(x,x)$). Take L to be a lattice on V (i.e., a rank-m $\mathbb{Z}$-submodule of V); for convenience, assume L is even integral, that is, $Q(L) \subseteq 2\mathbb{Z}$. Define the discriminant of L to be $dL = \det(B(x_i, x_j))$, where $\{x_1, \ldots, x_m\}$ is a $\mathbb{Z}$-basis for L. Let $O(V)$ denote the orthogonal group of V (i.e., the collection of all (global) isometries of V) and $O(L)$ that of L. Since Q is positive definite, $O(L)$ is finite. We say a lattice K is isometric to L, written $K \simeq L$, if there is an isometry $\sigma \in O(V)$ so that $\sigma K = L$. For any prime q, let $\mathbb{Z}_{(q)}$ denote the q-adic integers and $L_{(q)} = L \otimes \mathbb{Z}_{(q)}$. Q extends naturally to a quadratic form on $L_{(q)}$.

We say a lattice K is in the genus of L, gen L, if $K_{(q)} \simeq L_{(q)}$ at each prime q (i.e., there is a local isometry at each prime q taking $K_{(q)}$ onto $L_{(q)}$). There are a finite number of (global) isometry classes within gen L. A lattice K is in the family of L, fam L, if K is a lattice on V^{α} for some odd $\alpha \in \mathbb{Z}_+$, and for every prime q there is a q-adic unit u so that $K_{(q)} \simeq L^u_{(q)}$ (see [7]). Here, V^{α} denotes the vector space V scaled by α, that is, V equipped with the quadratic form αQ, and $L^u_{(q)}$ denotes $L_{(q)}$ scaled by u. As shown in Lemma 3.1 of [7], there are 2^r genera in fam L for some $r \in \mathbb{Z}_+$; in Lemma 1.3 below, we give a more precise count.

Say q is an odd prime. Then $L_{(q)}$ can be diagonalized. That is, there is a $\mathbb{Z}_{(q)}$-basis $\{x_1, \ldots, x_m\}$ for $L_{(q)}$ so that $(B(x_i, x_j)) = \operatorname{diag}\{Q(x_1), \ldots, Q(x_m)\}$; we write $L_{(q)} \simeq \langle \alpha_1, \ldots, \alpha_m \rangle$ where $\alpha_i = Q(x_i)$. In fact, $L_{(q)} = J_0 \perp \cdots \perp J_s$ where each J_i is q^i-modular; that is, $J_i \simeq q^i\langle 1, \ldots, 1, \eta_i \rangle$, $\eta_i \in \mathbb{Z}^{\times}_{(q)}$. The J_i are called Jordan components of $L_{(q)}$. These are not uniquely determined by L, but their $\mathbb{Z}_{(q)}$-isometry classes are. Note that the $\mathbb{Z}_{(q)}$-isometry class of a q^i-modular lattice is determined by its rank and its discriminant (up to squares of q-adic units). Thus we have

$$L_{(q)} \simeq \langle 1, \ldots, 1, \eta_0 \rangle \perp q\langle 1, \ldots, 1, \eta_1 \rangle \perp \cdots \perp q^s\langle 1, \ldots, 1, \eta_s \rangle, \text{ and so}$$

$$L \simeq \langle 1, \ldots, 1, \eta_0 \rangle \perp q\langle 1, \ldots, 1, \eta_1 \rangle \perp \cdots \perp q^s\langle 1, \ldots, 1, \eta_s \rangle \pmod{q^t}$$

for any $t \in \mathbb{Z}_+$; that is, relative to some $\mathbb{Z}$-basis $\{x_1, \ldots, x_m\}$ for L, $(B(x_i, x_j)) \equiv \langle 1, \ldots, 1, \eta_0 \rangle \perp \cdots \perp q^s\langle 1, \ldots, 1, \eta_s \rangle \pmod{q^t}$. From this we obtain the following technical lemma.

LEMMA 1.1. *Let L, L' be even integral $\mathbb{Z}$-lattices and q an odd prime. Write $L_{(q)} \simeq \langle 1, \ldots, 1, \eta_0 \rangle \perp \cdots \perp q^s\langle 1, \ldots, 1, \eta_s \rangle$, $L'_{(q)} \simeq \langle 1, \ldots, 1, \eta'_0 \rangle \perp \cdots \perp q^{s'}\langle 1, \ldots, 1, \eta_{s'} \rangle$.*

(1) *Suppose $L \simeq L' \pmod{q^t}$ where $t > s, s'$. Then $L_{(q)} \simeq L'_{(q)}$.*

(2) *Suppose $L_{(q)} \simeq L'_{(q)}$. Then $L \simeq L' \pmod{q^t}$ for any $t \in \mathbb{Z}_+$.*

Assume we have scaled L so that $Q(L) \subseteq 2\mathbb{Z}$, $Q(L) \not\subseteq 2n\mathbb{Z}$ for any $n > 1$. Then with notation as above, L/qL is a $\mathbb{Z}/q\mathbb{Z}$-vector space. Here we use Q and B to denote the quadratic and bilinear forms naturally induced on L/qL; the induced forms take values in $\mathbb{Z}/q\mathbb{Z}$. We call a nonzero vector $\bar{x} \in L/qL$ isotropic if $Q(\bar{x}) = 0$; we call $\bar{x}$ anisotropic if $Q(\bar{x}) \neq 0$. (*Note:* When it will not cause confusion, we use $\bar{x}$ freely to denote the image of x in various reduced lattices L'/qL'.) A subspace of L/qL is called totally isotropic if all its nonzero vectors are isotropic, and it is called anisotropic if it contains no (nonzero) isotropic vectors. We define the radical to be

$$\operatorname{rad} L/qL = \{\bar{x} \in L/qL\colon B(\bar{x}, \bar{y}) = 0 \text{ for all } \bar{y} \in L/qL\}.$$

If $\operatorname{rad} L/qL = \{0\}$, then we say L/qL is regular, and we have $L/qL = H_1 \perp \cdots \perp H_k \perp A$, where A is anisotropic of dimension 0, 1, or 2, and each H_i is a hyperbolic plane; that is, $H_i \simeq \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \simeq \langle 1, -1 \rangle$. Here k is called the Witt index of L/qL. When L/qL is not regular, $L/qL = \bar{U} \perp \operatorname{rad} L/qL$ for some regular subspace $\bar{U}$ whose isometry class is uniquely determined by L/qL. We say the Witt index of L/qL is that of $\bar{U}$. More generally, we say a space of type $(r; d, \mu)$ is a $\mathbb{Z}/q\mathbb{Z}$-quadratic space $\overline{W} = \bar{U} \perp \operatorname{rad} \overline{W}$ such that $\dim \overline{W} = r$, $\dim \bar{U} = d$, and $((-1)^{\ell} d\bar{U}/q) = \mu$, where $\ell = [d/2]$, $d\bar{U} =$ the discriminant of $\bar{U}$, and $(*/*)$ denotes the Legendre symbol. When $r = d$, we simply say the space is type (d, μ). (For instance, a hyperbolic plane is type (2, 1).)

One easily verifies the next result.

PROPOSITION 1.2. *Let L be an even integral lattice and q an odd prime with*

$$L_{(q)} = J_0 \perp \cdots \perp J_s \qquad \text{and} \qquad J_i \simeq q^i \langle 1, \ldots, 1, \eta_i \rangle$$

for some $\eta_i \in \mathbb{Z}_{(q)}^{\times}$.

(1) *Say $\bar{C}$ is a d-dimensional totally isotropic subspace of L/qL such that $\bar{C} \cap \operatorname{rad} L/qL = \{0\}$. Thus $L/qL = (\bar{C} \oplus \bar{D}) \perp \bar{U} \perp \operatorname{rad} L/qL$, where $\bar{C} \oplus \bar{D}$ is hyperbolic (i.e., an orthogonal sum of hyperbolic planes) and $\bar{U} \simeq \langle 1, \ldots, 1, \eta' \rangle$ is an $(r - d, \mu)$ space. Let $M =$ preimage in L of $\bar{C} \oplus \operatorname{rad} L/qL$, and $M' =$ preimage in L of $\bar{C}^{\perp}$, where $\bar{C}^{\perp} = \{\bar{x}\colon B(\bar{x}, \bar{C}) = 0\}$. Then $M_{(q')} = M'_{(q')} = L_{(q')}$ for every prime $q' \neq q$, and*

$$M_{(q)} \simeq q\langle 1, -1, \ldots, 1, -1 \rangle \perp J_1 \perp q^2 \langle 1, \ldots, 1, \eta' \rangle \perp J_2 \perp \cdots \perp J_s,$$

$$M'_{(q)} \simeq \langle 1, \ldots, 1, \eta' \rangle \perp q\langle 1, -1, \ldots, 1, -1 \rangle \perp J_1 \perp J_2 \perp \cdots \perp J_s,$$

where $\langle 1, -1, \ldots, 1, -1 \rangle$ has rank $2d$, $\langle 1, \ldots, 1, \eta' \rangle$ has rank $-2d + \operatorname{rank} J_0$, $\eta' \in \mathbb{Z}_{(q)}^{\times}$, and $(\eta'/q) = ((-1)^d \eta_0/q)$.

(2) *Say $J_0 = C_0 \perp D_0$, $J_1 = C_1 \perp D_1$ (so C_i, D_i are necessarily q^i-modular). Let $M =$ preimage in L of $\bar{C}_0 \perp \bar{C}_1 \subseteq L/qL$. Then $M_{(q')} = M'_{(q')} = L_{(q')}$ for every prime $q' \neq q$, and*

$$M_{(q)} = C_0 \perp C_1 \perp qD_0 \perp J_2 \perp qD_1 \perp J_3 \perp \cdots \perp J_s.$$

(3) *When* $R =$ *preimage in* L *of* rad L/qL, *we have* $R_{(q)} = J_1 \perp qJ_0 \perp J_2 \perp \cdots \perp J_s$.

We define the theta series attached to L to be $\theta(L;\tau) = \sum_{x\in L} e^{\pi i Q(x)\tau}$ where $\tau \in \mathscr{H} = \{\tau' \in \mathbb{C}\colon \Im\tau' > 0\}$. Since Q is positive definite, $\theta(L;\tau)$ is a modular form of weight $m/2$, some level N, and character χ_L, where $\chi_L(d) = (\operatorname{sgn} d)^{m/2}((-1)^{m/2}\, dL/|d|)$, where $(*/*)$ is the Kronecker symbol. We refer to the level of $\theta(L;\tau)$ as the level of L. For odd primes q, $\operatorname{ord}_q N = s$, where s is as above in the Jordan decomposition of $L_{(q)}$. χ_L is a quadratic character modulo N, and for odd primes p not dividing N, $\chi_L(p) = 1$ if and only if L/pL is hyperbolic. We will be assuming $N =$ level of L is odd, so (cf. [6]) we necessarily have

$$L_{(2)} \simeq \begin{pmatrix} 2a_1 & 1 \\ 1 & 2c_1 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 2a_{m/2} & 1 \\ 1 & 2c_{m/2} \end{pmatrix}$$

and $(-1)^{m/2}\, dL \equiv 1 \pmod 4$.

We say a lattice K has minimal level and discriminant at an odd prime q if, for some η, $\eta' \in \mathbb{Z} - q\mathbb{Z}$,

$$K_{(q)} \simeq \begin{cases} \langle 1,\ldots,1,\eta\rangle & \text{or} \\ \langle 1,\ldots,1,\eta\rangle \perp q\langle \eta'\rangle & \text{or} \\ \langle 1,\ldots,1,\eta\rangle \perp q\langle 1,\eta'\rangle & \text{where } \left(\dfrac{(-1)^{m/2-1}\eta}{q}\right) = -1 = \left(\dfrac{-\eta'}{q}\right). \end{cases}$$

In the last case, the condition on Legendre symbols means that neither Jordan component of K is hyperbolic modulo q (where we consider the second Jordan component scaled by $1/q$). When K has minimal (odd) level and discriminant at all odd primes, we simply say K has minimal level and discriminant.

Lemma 1.3. *Suppose* K *has minimal, odd level* N *and minimal discriminant* dK; *let* $q_1,\ldots,q_h$ *be the primes exactly dividing* N. *If* $h = 0$, *then* fam $K =$ gen K; *if* $h > 0$, *then there are* 2^{h-1} *genera in* fam K.

Proof. As the case $h = 0$ is trivial, we suppose $h > 0$; take $K' \in \operatorname{fam} K$, and define $\chi(d) = (\operatorname{sgn} d)^{m/2}((-1)^{m/2}\, dK/|d|)_K$, where $(*)_K$ denotes the Kronecker symbol. (So $(*/d)_K$ is the Jacobi symbol $(*/d)$ when d is positive and odd.) Note that our assumptions on K imply $dK = q_1\cdots q_h \ell^2$ for some $\ell \in \mathbb{Z}$, and $(-1)^{m/2} q_1 \cdots q_h \equiv 1 \pmod 4$ (cf. [6]). Then as described in the proof of Lemma 3.1 of [7], $K' = J^{1/\alpha}$, where J is "connected to K by a prime-sublattice chain." That is, $\alpha = p_1\cdots p_r p_{r+1}^2 \cdots p_{r+s}^2$ where the p_j are odd primes (not necessarily distinct) with $\chi(p_j) = 1$ if $j \leqslant r$, and there exist lattices $J_0 = K, J_1, \ldots, J_{r+s-1}$, $J_{r+s} = J$ such that J_j is a p_j-sublattice of J_{j-1} if $j \leqslant r$, and J_j is a p_j^2-sublattice of J_{j-1} if $j > r$. (A p-sublattice J' of J is the preimage in J of a maximal totally isotropic subspace of the quadratic space J/pJ. A p^2-sublattice J' of J is a

p-sublattice of a p-sublattice of J with $\dim J'/(J' \cap pJ)$ maximal; cf. [6].) Hence $\chi(\alpha)$ must equal 1. Also (cf. [6] and [7]), $K'_{(q)} \simeq K_{(q)}$ for all primes $q \nmid q_1 \cdots q_h$ (for $q = 2$, refer to §82E and 93:16 of [4]) and $K'_{(q_i)} \simeq K_{(q_i)}$ if and only if $(\alpha/q_i) = 1$. Thus we identify gen K' with the vector $((\alpha/q_1), \ldots, (\alpha/q_h))$. As $1 = \chi(\alpha) = (\alpha/N) = (\alpha/q_1 \cdots q_h)$, the value of (α/q_h) is determined by the values of (α/q_j) for $j < h$. Hence there can be at most 2^{h-1} genera within fam K.

On the other hand, choose $\varepsilon_j = \pm 1$ for $1 \leqslant j < h$ and set $\varepsilon_h = \varepsilon_1 \cdots \varepsilon_{h-1}$. Using the Chinese Remainder theorem, we can find an odd prime p such that $(p/q_j) = \varepsilon_j$ for $1 \leqslant j \leqslant h$; notice that quadratic reciprocity implies that $\chi(p) = 1$. Let J be a p-sublattice of K and set $K' = J^{1/p}$; then $K' \in$ fam K and gen K' corresponds to $(\varepsilon_1, \ldots, \varepsilon_h)$. Hence fam K contains 2^{h-1} genera. □

Remark. Consider the group $S = \{(\varepsilon_1, \ldots, \varepsilon_h) \colon \varepsilon_i = \pm 1, \varepsilon_1 \cdots \varepsilon_h = 1\}$; as in [8], let $\{v_1, \ldots, v_{h-1}\}$ be a set of generators of this group. For each j, we can find an odd prime p_j such that $v_j = ((p_j/q_1), \ldots, (p_j/q_h))$. Notice that we necessarily have $1 = (p_j/q_1 \cdots q_h) = \chi(p_j)$. Let $A = p_1 \cdots p_{h-1}$. Then each divisor α of A corresponds to $((\alpha/q_1), \ldots, (\alpha/q_h)) \in \mathscr{S}$. Thus we may index the genera in fam K by the divisors of A.

PROPOSITION 1.4. *Let K be a lattice of level N, and let q be an odd prime such that K has minimal level and discriminant at q. Set R = preimage in K of* rad (K/qK). *Then $O(R) = O(K)$ (where $O(K)$ denotes the orthogonal group of K).*

Proof. Take $\sigma \in O(K)$. Then for $x \in R$, we have

$$B(K, \sigma x) = B(\sigma K, \sigma x) = B(K, x) \equiv 0 \pmod{q}.$$

Hence $\overline{\sigma x} \in \operatorname{rad} K/qK$, so $\sigma x \in R$. Thus $O(K) \subseteq O(R)$. Since qK = preimage in R of rad R/qR (where R is scaled by $1/q$), we also have $O(R) \subseteq O(qK) = O(K)$. □

PROPOSITION 1.5. *Let K, R be as in the preceding proposition. As K' varies over the isometry classes in* gen K*, the corresponding R' varies over the classes in* gen R.

Proof. Suppose $R' \in$ gen R with qK' = preimage in R' of rad R'/qR' (where R' is scaled by $1/q$). Thus R' = preimage in K' of rad K'/qK'. One easily verifies that whenever a (local or global) isometry σ carries R to R', then σ also carries qK to qK' and hence K to K'. □

Given any lattice L, the Fourier coefficients of $\theta(L; \tau)$ are the representation numbers of L:

$$\theta(L; \tau) = \sum_{n \geqslant 0} r(L, 2n) e^{2\pi i n \tau}, \quad \text{where} \quad r(L, 2n) = \#\{x \in L \colon Q(x) = 2n\}.$$

We define the average theta series to be $\theta(\text{gen } L;\tau) = 1/\text{mass } L \sum_{L' \in \text{gen } L} \times (1/o(L'))\theta(L';\tau)$, where L' runs over the isometry classes in gen L, $o(L') = \#O(L') =$ the order of the orthogonal group of L', and $\text{mass } L = \sum_{L' \in \text{gen } L}(1/o(L'))$. Thus

$$\theta(\text{gen } L;\tau) = \sum_{n \geqslant 0} r(\text{gen } L, n) e^{2\pi i n \tau},$$

where

$$r(\text{gen } L, n) = \frac{1}{\text{mass } L} \sum_{L' \in \text{gen } L} \frac{1}{o(L')} r(L', 2n).$$

In [7] we showed $\theta(\text{gen } L;\tau)$ lies in the space of Eisenstein series by examining the action of Hecke operators T_p, $p \nmid 2N$, on $\theta(L;\tau)$. Let $q_1, \ldots, q_h$, A be as in our discussion of fam L following Lemma 1.3. Given $\alpha | A$ with corresponding genus gen K_α and prime p such that $\chi(p) = 1$, we have

$$\theta(\text{gen } K_\alpha;\tau)|T_p = (p^{m/2-1} + 1)\theta(\text{gen } K_\beta;\tau),$$

where $\beta | A$, $\beta \equiv p\alpha \pmod{q_1 \cdots q_h}$ (see Lemma 3.3 of [7]).

In this paper we make use of some other standard operators on weight $m/2$ modular forms. For q an odd prime dividing N, we let

$$B_q = q^{-m/4}\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}, \quad U_q = q^{m/4-1} \sum_{b=1}^{q} \begin{pmatrix} 1 & b \\ 0 & q \end{pmatrix}, \quad R_q = \frac{1}{g_q} \sum_{a=1}^{q} \left(\frac{a}{q}\right) \begin{pmatrix} 1 & a/q \\ 0 & 1 \end{pmatrix}$$

where $g_q = \sum_{b \bmod q}(b/q)e^{2\pi i b/q}$. So for a modular form $f(\tau) = \sum_{n \geqslant 0} a(n)e^{2\pi i n \tau}$, we have

$$f(\tau)|B_q = \sum_{n \geqslant 0} a(n)e^{2\pi i q n \tau}, \quad f(\tau)|U_q = \sum_{n \geqslant 0} a(qn)e^{2\pi i n \tau},$$

and

$$f(\tau)|R_q = \sum_{n \geqslant 0} \left(\frac{n}{q}\right) a(n) e^{2\pi i n \tau}.$$

Notice that for any lattice L, $\theta(L;\tau)|B_q^2 = \theta(qL;\tau)$.

Let $G(\tau; c, d; N)$ denote the Eisenstein series of weight $m/2$, odd, square-free level N, and quadratic character χ, as defined in Chapter IV of [3]. As usual, we assume $(-1)^{m/2} = \chi(-1)$. For $D | N$, set

$$E_D(\tau) = \frac{D^{m/2-1}\Gamma(m/2)}{2(-2\pi i)^{m/2}} \sum_{\substack{a \bmod N \\ b \bmod N/D \\ c \bmod D}} \chi_{N/D}(b)\chi_D(c)e^{-2\pi i a c/D} G(\tau; bD, a; N).$$

Then by Theorem 15 in Chapter IV of [3], we find that each E_D is a simultaneous eigenform for the Hecke operators T_p, $p \nmid 2N$, and $\{E_D\colon D|N\}$ is a basis for the space of Eisenstein series of weight $m/2$, level N, and character χ. From Proposition 17 in Chapter IV of [3], we see that $E_D(\tau) = \sum_{n \geqslant 0} a_D(n) e^{2\pi i n \tau}$, where

$$a_D(0) = \begin{cases} 0 & \text{if } D \neq N, \\ \dfrac{N^{m/2-1}\Gamma(m/2)}{2(2\pi i)^{m/2}} \displaystyle\sum_{\substack{n\geqslant 1 \\ b \bmod N}} \chi(b) e^{2\pi i n b/N} n^{-m/2} & \text{if } D = N, \end{cases}$$

and for $n \geqslant 1$,

$$a_D(n) = \sum_{\substack{d|n \\ d>0}} \chi_{N/D}(n/d)\chi_D(d)\, d^{m/2-1}.$$

Standard techniques for evaluating Gauss sums show that

$$a_N(0) = \frac{N^{m/2-1}\Gamma(m/2)}{(2\pi i)^{m/2}} G(\chi_{N'}, 1)\mu(N/N')L(\chi, m/2) \prod_{\substack{q|N/N' \\ q \text{ prime}}} \frac{1-\chi_{N'}(q)q^{1-m/2}}{1-\chi_{N'}(q)q^{-m/2}},$$

where $N' = \operatorname{cond}\chi$, $G(\chi_{N'}, 1)$ is the standard Gauss sum (modulo N') and $L(\chi, s) = \sum_{n\geqslant 1}\chi(n)n^{-s}$, the standard Dirichlet series for χ (modulo N).

§2. Lattices of minimal level and discriminant. Throughout this section, let K be a lattice of odd level N and discriminant dK. We derive formulas for the average representation numbers of gen K when K has minimal level and discriminant.

Convention. When a lattice J has the property that the first Jordan component of $J_{(q)}$ is q^k-modular, we use the quadratic form $q^{-k}Q$ on the $\mathbb{Z}/q\mathbb{Z}$-space J/qJ.

The proofs of the following two propositions illustrate the main techniques used throughout the paper.

PROPOSITION 2.1. *Suppose $N \neq 1$, where N denotes the level of K; let q be an odd prime dividing N. Suppose K has minimal level and discriminant at q. Thus for any $t \in \mathbb{Z}_+$,*

$$K \simeq \begin{cases} \langle 1, \ldots, 1, \eta\rangle \perp q\langle \eta'\rangle \pmod{q^t} & \text{if } q\|dK, \\ \langle 1, \ldots, 1, \eta\rangle \perp q\langle 1, \eta'\rangle \pmod{q^t} & \text{if } q^2|dK; \end{cases}$$

recall that in the latter case our hypotheses on K imply $((-1)^{m/2-1}\eta/q) = -1 = (-\eta'/q)$. Let

$$R = \textit{preimage in } K \textit{ of } \operatorname{rad} K/qK,$$

and let dK denote the discriminant of K. Let p be an odd prime not dividing the level of K such that

$$\chi_q(p) = \chi_q((-1)^{m/2-1}\eta\eta'), \qquad \left(\frac{p}{q'}\right) = \left(\frac{q}{q'}\right)$$

for all primes $q'|N, q' \neq q$. We refer to p as a prime associated to q. Then $\chi_K(p) = 1$ and

$$\begin{aligned} \theta(\text{gen } R; \tau) &= \theta(\text{gen } K; \tau)|T_{K/R}(q) \\ &= \theta(\text{gen } K; \tau)\left|\left[\frac{q^{m/2-1}+1}{q^{m/2-1}(p^{m/2-1}+1)} B_q T_p - \frac{1}{q^{m/2-1}} U_q B_q\right]\right. . \end{aligned}$$

Remark. When $q \nmid dK$, $R = qK$, and so $\theta(\text{gen } R; \tau) = \theta(\text{gen } K; \tau)|B_q^2$. Also, this proposition extends easily to the case dK by imposing the extra condition $p \equiv q \pmod 8$.

Proof. Let $\bar{C}$ be a maximal totally isotropic subspace of K/qK (so $\operatorname{rad} K/qK \subseteq \bar{C}$), and let

$$K' = \text{preimage in } K \text{ of } \bar{C}.$$

By Proposition 1.2,

$$K' \simeq \begin{cases} q\langle 1, \ldots, 1, (-1)^{m/2-1}\eta'\rangle \perp q^2\langle(-1)^{m/2-1}\eta\rangle \pmod{q^t} & \text{if } q\|dK, \\ q\langle 1, \ldots, 1, \eta\rangle \perp q^2\langle 1, \eta'\rangle \pmod{q^t} & \text{if } q^2|dK \end{cases}$$

for any $t \in \mathbb{Z}_+$. Also, for any prime $q' \neq q$, $K'_{(q')} = K_{(q')}$.

Clearly these sublattices K' are in one-to-one correspondence with these subspaces $\bar{C}$. Using the formulas from [1, p. 146] (cf. Proposition 7.2 of [6]), we find there are

$$(q^{m/2-1}+1)\beta = \begin{cases} (q^{m/2-1}+1)(q^{m/2-2}+1)\cdots(q+1) & \text{if } q\|dK, \\ (q^{m/2-1}+1)(q^{m/2-2}+1)\cdots(q^2+1) & \text{if } q^2|dK \end{cases}$$

ways to choose $\bar{C}$, and exactly β of these contain a given vector $x \in K - R$ provided $q|Q(x)$. When $q \nmid Q(x)$, $x \notin K'$ for every K', and when $x \in R$, we have

$x \in K'$ for each choice of K'. Thus, we find that

$$\theta(K;\tau)|U_qB_q + q^{m/2-1}\theta(R;\tau) = \frac{1}{\beta}\sum_{K'}\theta(K';\tau),$$

where K' varies over all the sublattices constructed as above.

For J, J' lattices on V, let $f(J,J') = \#\{\sigma \in O(V) \colon qJ \subset \sigma J' \subset J\}$. So

$$\theta(K;\tau)|U_qB_q + q^{m/2-1}\theta(R;\tau) = \frac{1}{\beta}\sum_{M' \in \operatorname{gen} K'} \frac{f(K,M')}{o(M')}\,\theta(M';\tau),$$

where K' is any sublattice constructed above, and M' runs over the isometry classes in $\operatorname{gen} K'$. Averaging over the isometry classes in $\operatorname{gen} K$ (and thus the corresponding isometry classes in $\operatorname{gen} R$; see Propositions 1.4 and 1.5), we get

$$\begin{aligned}\theta(\operatorname{gen} K;\tau)&|U_qB_q + q^{m/2-1}\theta(\operatorname{gen} R;\tau)\\ &= \frac{1}{\beta\cdot \operatorname{mass} K}\sum_{M' \in \operatorname{gen} K'}\left(\sum_{M \in \operatorname{gen} K}\frac{f(M,M')}{o(M)}\right)\frac{1}{o(M')}\theta(M';\tau)\\ &= \frac{1}{\beta\cdot \operatorname{mass} K}\sum_{M' \in \operatorname{gen} K'}\left(\sum_{M \in \operatorname{gen} K}\frac{f(M',qM)}{o(qM)}\right)\frac{1}{o(M')}\theta(M';\tau).\end{aligned}$$

Now, $\sum_{M\in \operatorname{gen} K} f(M',qM)/o(qM) = (q^{m/2-1}+1)\beta$, the number of maximal totally isotropic subspaces of M'/qM', so

$$\theta(\operatorname{gen} K;\tau)|U_qB_q + q^{m/2-1}\theta(\operatorname{gen} R;\tau) = (q^{m/2-1}+1)\frac{\operatorname{mass} K'}{\operatorname{mass} K}\theta(\operatorname{gen} K';\tau).$$

Comparing zeroth Fourier coefficients, we find that

$$\theta(\operatorname{gen} K;\tau)|U_qB_q + q^{m/2-1}\theta(\operatorname{gen} R;\tau) = (q^{m/2-1}+1)\theta(\operatorname{gen} K';\tau).$$

If $q' \nmid \operatorname{cond}\chi$, then $K^q_{(q')} \simeq K_{(q')} \simeq K'_{(q')}$, but if $q' | \operatorname{cond}\chi$, then $K^q_{(q')} \simeq K'_{(q')}$ only if $(q/q') = 1$. Thus, we do not necessarily have $\theta(\operatorname{gen} K';\tau) = \theta(\operatorname{gen} K;\tau)|B_q$. However, we claim that

$$\theta(\operatorname{gen} K';\tau) = \frac{1}{p^{m/2-1}+1}\,\theta(\operatorname{gen} K;\tau)|B_qT_p = \frac{1}{p^{m/2-1}+1}\,\theta(\operatorname{gen} K;\tau)|T_pB_q.$$

To verify this claim, first note that

$$\chi(p) = \left(\frac{(-1)^{m/2}\,dK}{p}\right) = \left(\frac{(-1)^{m/2}q_1\cdots q_hN_0^2}{p}\right),$$

where $q_1, \dots, q_h$ are distinct primes and $N_0 \in \mathbb{Z}_+$. Since by assumption, χ is a character of odd level N, we must have $(-1)^{m/2} q_1 \cdots q_h \equiv 1 \pmod 4$, and $\operatorname{cond} \chi = q_1 \cdots q_h$. Hence our constraints on p and quadratic reciprocity imply that $\chi(p) = 1$. Thus by Lemmas 5.2 of [6] and 3.3 of [7], we have

$$\theta(\operatorname{gen} K; \tau) | T_p = (p^{m/2-1} + 1) \frac{\operatorname{mass} M}{\operatorname{mass} K} \theta(\operatorname{gen} M; \tau) = (p^{m/2-1} + 1)\theta(\operatorname{gen} M; \tau),$$

where M is a lattice on $V^{1/p}$, $M_{(p)} \simeq K_{(p)}$, for all primes $q' \neq p$, $M_{(q')} \simeq K^p_{(q')}$, and the last equality follows from comparing zeroth Fourier coefficients. So for $q' \neq p$, our constraints on p imply that

$$M^q_{(q')} \simeq K^{pq}_{(q')} \simeq K_{(q')} \simeq K'_{(q')}.$$

Also, since $p \nmid dK$, $M^q_{(p)} \simeq K^q_{(p)} \simeq K_{(p)} \simeq K'_{(p)}$. Hence, $M^q \in \operatorname{gen} K'$, and

$$\theta(\operatorname{gen} K; \tau) | T_p B_q = (p^{m/2-1} + 1)\theta(\operatorname{gen} K'; \tau).$$

The proposition now follows by solving our earlier equation for $\theta(\operatorname{gen} R; \tau)$. □

Assume still that q is an odd prime dividing N; let p be a prime associated to q as in Proposition 2.1. Define

$$T_K(q) = \begin{cases} U_q^2 - (q^{m/2} + q^{m/2-1}) U_q B_q + \dfrac{q^{m-2} + q^{m/2-1}}{p^{m/2-1} + 1} B_q T_p & \text{if } q^2 | dK, \\ U_q^2 - q^{m/2-1} U_q B_q + \dfrac{q^{m-2} + q^{m/2-1}}{p^{m/2-1} + 1} B_q T_p + \left(\frac{(-1)^{m/2-1} 2\eta}{q}\right) q^{m/2-1} R_q & \text{if } q \| dK, \end{cases}$$

and set $\lambda_K(q) = \begin{cases} q^{m-2} - q^{m/2} + 1 & \text{if } q^2 | dK, \\ q^{m-2} + 1 & \text{if } q \| dK. \end{cases}$

PROPOSITION 2.2. *Suppose K has minimal level and discriminant at the odd prime q dividing N. With notation as above, $\theta(\operatorname{gen} K; \tau) | T_K(q) = \lambda_K(q) \theta(\operatorname{gen} K; \tau)$.*

Proof.

Case 1. Suppose $q^2 | dK$. We perform lattice constructions quite similar to those of the preceding proposition. This time, let $\langle \overline{w} \rangle \oplus \operatorname{rad} K/qK$ be a 3-dimensional totally isotropic subspace of K/qK, and let

$$K' = \text{preimage in } K \text{ of } \langle \overline{w} \rangle \oplus \operatorname{rad} K/qK$$

$$\simeq q\langle 1, 1, 1, -\eta' \rangle \perp q^2 \langle 1, \dots, 1, -\eta \rangle \pmod{q^t}$$

for arbitrary $t \in \mathbb{Z}_+$ (see Lemma 1.1). Let $\langle \overline{y} \rangle \oplus \operatorname{rad} K'/qK'$ be an $(m-3)$-

dimensional totally isotropic subspace of K'/qK', and set

$$K'' = \text{preimage in } K' \text{ of } \langle \bar{y}\rangle \oplus \operatorname{rad} K'/qK'$$
$$\simeq q^2\langle 1,\ldots,1,\eta\rangle \perp q^3\langle 1,\eta'\rangle \ (\mathrm{mod}\ q^t)$$

for arbitrary $t \in \mathbb{Z}_+$. Clearly, K' and K'' are in one-to-one correspondence with the subgroups $\langle \bar{w}\rangle \oplus \operatorname{rad} K/qK$ and $\langle \bar{y}\rangle \oplus \operatorname{rad} K'/qK'$ (respectively). Using the formulas of [1], we see there are $((q^{m/2-1}+1)(q^{m/2-2}-1))/(q-1)$ choices for $\langle \bar{w}\rangle \oplus \operatorname{rad} K/qK$, and q^2+1 choices for $\langle \bar{y}\rangle \oplus \operatorname{rad} K'/qK'$ in each K'. Note that $\bar{w} \notin \operatorname{rad} K'/qK'$.

Say $x \in K - R$ with $q^2|Q(x)$; then $x \in K''$ if and only if $\langle \bar{w}\rangle \oplus \operatorname{rad} K/qK = \langle \bar{x}\rangle \oplus \operatorname{rad} K/qK$ and $\langle \bar{y}\rangle \oplus \operatorname{rad} K'/qK' = \langle \bar{x}\rangle \oplus \operatorname{rad} K'/qK'$.

If $x \in R - qK$, then $q^2 \nmid Q(x)$ so x is never in K''. However, qR is in K'' for all pairs (K', K'').

Say $x \in K - R$; then $\overline{qx} \in \operatorname{rad} K'/qK'$ if and only if $\bar{w} \in \langle \bar{x}\rangle^{\perp}$ in K/qK. If $\overline{qx} \in \operatorname{rad} K'/qK'$, then $qx \in K''$ for each K'' constructed from K'; if $\overline{qx} \notin \operatorname{rad} K'/qK'$, then $qx \in K''$ only when $\langle \bar{y}\rangle \oplus \operatorname{rad} K'/qK' = \langle \overline{qx}\rangle \oplus \operatorname{rad} K'/qK'$. When $q|Q(x)$, $\langle \bar{x}\rangle^{\perp}$ has dimension $m-1$, radical $\langle \bar{x}\rangle \oplus \operatorname{rad} K/qK$, and Witt index $m/2-3$. When $q \nmid Q(x)$, $K/qK \simeq \langle \bar{x}\rangle \oplus \langle \bar{x}\rangle^{\perp}$, and so by Witt cancellation, $\langle \bar{x}\rangle^{\perp}$ has dimension $m-1$ and radical $\operatorname{rad} K/qK$. Thus, using the formulas from [1], the number of 3-dimensional totally isotropic subspaces $\langle \bar{w}\rangle \oplus \operatorname{rad} K/qK$ with $\bar{w} \in \langle \bar{x}\rangle^{\perp}$ is

$$\begin{cases} \dfrac{q^{m-4}-1}{q-1} & \text{if } q \nmid Q(x), \\[2ex] \dfrac{q^{m-4}-q^{m/2-1}+q^{m/2-2}-1}{q-1} & \text{if } q|Q(x). \end{cases}$$

Note that if $\bar{v} \in \operatorname{rad} K'/qK'$, then $v \in K''$ for all K'' constructed from K', and otherwise $v \in K''$ only when $K'' = \text{preimage } \langle \bar{v}\rangle \oplus \operatorname{rad} K'/qK'$. Hence, for $x \in K - R$, the number of pairs (K', K'') with $qx \in K''$ is

$$\begin{cases} \dfrac{(q^2+1)(q^{m-4}-q^{m/2-1}+q^{m/2-2}-1)}{q-1} + q^{m-4} & \text{if } q|Q(x), \\[2ex] \dfrac{(q^2+1)(q^{m-4}-1)}{q-1} + q^{m-4} - q^{m/2-2} & \text{if } q \nmid Q(x). \end{cases}$$

Thus we have

$$\theta(K;\tau)|\left[U_q^2B_q^2 - q^{m/2}U_qB_q^3 + \frac{q^{m-2}+q^{m-3}-q^{m/2-1}+q^{m/2-2}-q^2-q}{q-1}B_q^2\right]$$
$$+ \theta(R;\tau)|q^{m-2}B_q^2 = \sum_{(K',K'')} \theta(K'';\tau),$$

where (K', K'') varies over all the pairs constructed as above. Averaging over gen K and using Proposition 2.1, we get

$$\theta(\text{gen } K;\tau)|T_K(q)B_q^2 = \lambda_K(q)\theta(\text{gen } qK;\tau) = \lambda_K(q)\theta(\text{gen } K;\tau)|B_q^2.$$

Case 2. Now suppose $q^2 \nmid dK$. Similar to case (1), let $\langle \bar{w}\rangle \oplus \text{rad } K/qK$ be a 2-dimensional totally isotropic subspace of K/qK, and let

$$K' = \text{preimage in } K \text{ of } \langle \bar{w}\rangle \oplus \text{rad } K/qK$$

$$\simeq q\langle 1,1,-\eta'\rangle \perp q^2\langle 1,\dots,1,-\eta\rangle \pmod{q^t}$$

for arbitrary $t \in \mathbb{Z}_+$. Let $\langle \bar{y}\rangle \oplus \text{rad } K'/qK'$ be an $(m-2)$-dimensional totally isotropic subspace of K'/qK' (scaled by $1/q$), and set

$$K'' = \text{preimage in } K' \text{ of } \langle \bar{y}\rangle \oplus \text{rad } K'/qK'$$

$$\simeq q^2\langle 1,\dots,1,\eta\rangle \perp q^3\langle \eta'\rangle \pmod{q^t}.$$

Using Artin's formulas, we see there are $(q^{m-2}-1)/(q-1)$ choices for K', and $q+1$ choices for K'' in each K'.

If $x \in K - R$ with $q^2|Q(x)$, then $x \in K''$ for exactly one pair (K', K'').

If $x \in R - qK$, then $q^2 \nmid Q(x)$, so x is never in K'' but $qx \in K''$ for all K''.

Now suppose $x \in K - R$. Again, $\overline{qx} \in \text{rad } K'/qK'$ if and only if $\bar{w} \in \langle \bar{x}\rangle^\perp$ in K/qK. When $q|Q(x)$, $\langle \bar{x}\rangle^\perp$ has dimension $m-1$, radical $\langle \bar{x}\rangle \oplus \text{rad } K/qK$, and Witt index $m/2-2$; hence there are $(q^{m-3}-1)/(q-1)$ ways to choose $\langle \bar{w}\rangle \oplus \text{rad } K/qK$ with $\bar{w} \in \langle \bar{x}\rangle^\perp$. Say $q \nmid Q(x)$; then $\langle \bar{x}\rangle^\perp \simeq \langle 1,\dots,1,Q(x)\eta\rangle \perp \langle 0\rangle$ with Witt index $m/2-1$ if $(Q(x)/q) = ((-1)^{m/2-1}\eta/q)$, and $m/2-2$ otherwise. Thus the number of 2-dimensional totally isotropic subspaces $\langle \bar{w}\rangle \oplus \text{rad } K/qK$ with $\bar{w} \in \langle \bar{x}\rangle^\perp$ is

$$\begin{cases} \dfrac{(q^{m/2-1}-1)(q^{m/2-2}+1)}{q-1} & \text{if } \left(\dfrac{Q(x)}{q}\right) = \left(\dfrac{(-1)^{m/2-1}\eta}{q}\right), \\[2ex] \dfrac{(q^{m/2-1}+1)(q^{m/2-2}-1)}{q-1} & \text{if } \left(\dfrac{Q(x)}{q}\right) \neq \left(\dfrac{(-1)^{m/2-1}\eta}{q}\right). \end{cases}$$

Hence the number of pairs (K', K'') with $qx \in K''$ is

$$\begin{cases} \dfrac{2q^{m-2}-q-1}{q-1} & \text{if } q|Q(x), \\ \dfrac{2q^{m-2}-q-1}{q-1} + q^{m/2-1} & \text{if } \left(\dfrac{Q(x)}{q}\right) = \left(\dfrac{(-1)^{m/2-1}\eta}{q}\right), \\ \dfrac{2q^{m-2}-q-1}{q-1} - q^{m/2-1} & \text{if } \left(\dfrac{Q(x)}{q}\right) \neq \left(\dfrac{(-1)^{m/2-1}\eta}{q}\right). \end{cases}$$

Recall that the nth coefficient of $\theta(K;\tau)$ is $r(K,2n)$, so

$$\theta(K;\tau)|R_q = \sum_{n\geqslant 0}\left(\frac{n}{q}\right) r(K,2n)e\{2n\tau\} = \sum_{x\in K}\left(\frac{2Q(x)}{q}\right) e\{Q(x)\tau\}.$$

Thus

$$\theta(K;\tau)|\left[U_q^2 + \left(\frac{(-1)^{m/2-1}2\eta}{q}\right)q^{m/2-1}R_q + \frac{2q^{m-2}-2q}{q-1}\right]B_q^2 + \theta(R;\tau)|q^{m-2}B_q^2$$
$$= \sum_{(K',K'')} \theta(K'';\tau),$$

where the sum is over all pairs (K', K''). Averaging over gen K and applying Proposition 2.1 yields the desired formula. □

For q an odd prime dividing N, the level of K, let $\mathscr{C}_K(q)$ denote the subspace of Eisenstein series E of level N, weight $m/2$, character χ, such that $E|T_K(q) = \lambda_K(q)E$.

LEMMA 2.3. *For any prime* $q\|N$, $\operatorname{span}\{E_D\colon D|N/q\} \cap \mathscr{C}_K(q) = \{0\}$, *where the* E_D *are the Eisenstein series with character* χ, *level* N, *and weight* $m/2$ *(as defined in §1).*

Proof. We simply examine the action of $T_K(q)$ on the Fourier coefficients of E_D, $D|N/q$. Let $b_D(n)$ denote the nth coefficient of $E_D|T_K(q)$. Thus

$$b_D(n) = \begin{cases} \lambda_K(q)a_D(n) & \text{if } q|n, \\ q^{m-2}a_D(n) & \text{if } q\nmid n, q^2|dK, \\ \left(q^{m-2} + \left(\dfrac{(-1)^{m/2-1}\eta}{q}\right)\left(\dfrac{2n}{q}\right)q^{m/2-1}\right)a_D(n) & \text{if } q\nmid n, q\|dK. \end{cases}$$

Notice that $q^{m-2} \pm q^{m/2-1}$ is never $\lambda_K(q)$. Thus for each $n|N/q$,

$$0 = \sum_{D|N/q} \alpha_D a_D(n^2) = \sum_{D|N/q} \alpha_{N/qD}(n, D)^{m-2}.$$

We represent these equations with matrices as follows. Let $q_1, \ldots, q_h$ be the primes dividing N/q; order the divisors of N/q according to their order in the tensor product

$$(1 \quad q_1) \otimes (1 \quad q_2) \otimes \cdots \otimes (1 \quad q_h).$$

Let $\bar{\alpha}$ be the vector whose entries are indexed by the divisors D of N/q and whose D-entry is $\alpha_{N/qD}$. Then the above equation implies

$$\bar{\alpha}A = 0,$$

where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & q_1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & q_2 \end{pmatrix} \otimes \cdots \begin{pmatrix} 1 & 1 \\ 1 & q_h \end{pmatrix}.$$

Since each matrix in the tensor product defining A is invertible, A is invertible as well. Hence, $\bar{\alpha} = 0$ and $\sum_{D|N/q} \alpha_D E_D = 0$. □

Set

$$c_K(q) = \begin{cases} \dfrac{q^{m/2} - 1}{q^{m-2} - q^{m/2}} & \text{if } q^2|dK, \\ \left(\dfrac{(-1)^{m/2-1}\eta}{q}\right) q^{1-m/2} & \text{if } q\|dK, \end{cases}$$

and extend $c_K(*)$ multiplicatively.

Proposition 2.4. *Suppose $q\|N$. Then $\mathscr{C}_K(q) = \operatorname{span}\{E_D + \chi_q(2)c_K(q)E_{Dq}: D|N/q\}$.*

Proof. By looking at Fourier coefficients, one easily verifies that $E_D + \chi_q(2)c_K(q)E_{Dq} \in \mathscr{C}_K(q)$ for all $D|N/q$. The proposition now follows from the preceding lemma. □

Theorem 2.5. *Suppose N is square-free and odd. Let $E = \sum_{D|N} \chi_D(2)c_K(D)E_D$. Then $\cap_{q|N} \mathscr{C}_K(q) = \mathbb{C}E$.*

Proof. Write $N = q_1 \cdots q_\ell$. Using induction on $r \leqslant \ell$, we argue that

$$\cap_{1\leqslant i\leqslant r} \mathscr{C}_K(q_i) = \operatorname{span}\left\{ \sum_{d|q_1\cdots q_r} \chi_d(2)c_K(d)E_{Dd}: D|N/q_1 \cdots q_r \right\}.$$

This is clearly true for $r = 0$. Take $r \geqslant 0$ and $f \in \cap_{1 \leqslant i \leqslant r+1} \mathscr{C}_K(q_i)$. The induction hypothesis tells us that

$$f = \sum_{D|N/q_1 \cdots q_r} \alpha_D \left(\sum_{d|q_1 \cdots q_r} \chi_d(2) c_K(d) E_{Dd} \right)$$
$$= \sum_{\substack{D|N/q_1 \cdots q_{r+1} \\ d|q_1 \cdots q_r}} \chi_d(2) c_K(d) (\alpha_D E_{Dd} + \alpha_{Dq_{r+1}} E_{Ddq_{r+1}}) .$$

Since $f \in \mathscr{C}_K(q_{r+1})$, Proposition 2.4 implies that $\alpha_{Dq_{r+1}} = \chi_{q_{r+1}}(2) c_K(q_{r+1}) \alpha_D$. Hence

$$f = \sum_{D|N/q_1 \cdots q_{r+1}} \alpha_D \left(\sum_{d|q_1 \cdots q_{r+1}} \chi_d(2) c_K(d) E_{Dd} \right) . \qquad \square$$

Since the zeroth Fourier coefficient of $\theta(\text{gen } K; \tau)$ is 1, and the zeroth coefficient of E is $\chi(2) c_K(N) a_N(0)$ (where $a_N(0)$ is defined in §1), Proposition 2.2 and Theorem 2.5 immediately give us the following result.

COROLLARY 2.6. *Suppose K has minimal level and discriminant. Then $\theta(\text{gen } K; \tau) = (1/c_K(N) a_N(0)) \cdot E$. Thus for $n \in \mathbb{Z}_+$*

$$r(\text{gen } K, n) = \frac{\chi(2)}{c_K(N) a_N(0)} \sum_{\substack{D|N \\ d|n}} c_K(D) \chi_D(d) \chi_{N/D}(2n/d) d^{m/2-1},$$

where χ_D, $\chi_{N/D}$ are the unique characters modulo D, N/D (respectively) so that $\chi_D \chi_{N/D} = \chi$. As in §1,

$$a_N(0) = \frac{N^{m/2-1} \Gamma(m/2)}{(2\pi i)^m/2} G(\chi_{N'}, 1) \mu(N/N') L\left(\chi, \frac{m}{2}\right) \prod_{\substack{q|N/N' \\ q \text{ prime}}} \frac{1 - \chi_{N'}(q) q^{1-m/2}}{1 - \chi_{N'} q^{2-m/2}},$$

with $N' = \text{cond } \chi$.

Suppose K has minimal level and discriminant. Say $q|N/N'$; then $K_{(q)} \simeq \langle 1, \ldots, 1, \eta \rangle \perp q \langle 1, \eta' \rangle$, where $((-1)^{m/2-1} \eta/q) = (-\eta'/q) = -1$. Thus

$$\chi_{N'}(q) = \left(\frac{(-1)^{m/2} N'}{q} \right) = \left(\frac{(-1)^{m/2}\, dK/q^2}{q} \right) = \left(\frac{(-1)^{m/2} \eta \eta'}{q} \right) = 1 .$$

Also the first Fourier coefficient of $\theta(\text{gen } K; \tau)$ is nonnegative, and (from Corollary 2.6) it is equal to

$$\frac{\chi(2)}{c_K(N)a_N(0)}\prod_{q|N}(1+c_K(q)).$$

Since $|c_K(q)|<1$ for all q, we must have $c_K(N)a_N(0)>0$. Thus

$$\frac{\chi(2)}{c_K(N)a_N(0)}=\frac{(2\pi)^{m/2}}{\Gamma(m/2)}\prod_{p,\text{prime}} f(p),$$

where

$$f(p)=\begin{cases} 1-\chi(p)p^{-m/2} & \text{if } p\nmid N,\\ \dfrac{p^{m/2-2}-1}{p^{m/2-1}-1} & \text{if } p|N/N',\\ p^{-1/2} & \text{if } p|N'.\end{cases}$$

To write $r(\text{gen } K, 2n)$ as a product, we set

$$\rho_{K,\infty}=\frac{(2\pi)^{m/2}}{\Gamma(m/2)},$$

and for $n\in\mathbb{Z}_+$ with $e=\operatorname{ord}_p(n)$,

$$\rho_{K,p}(n)=\begin{cases}\dfrac{p^{(m/2-1)(e+1)}-\chi(p^{e+1})}{p^{m/2-1}-\chi(p)}f(p) & \text{if } p\nmid N,\\ (p^{(m/2-1)e}+\chi_p(2n/p^e)\chi_{N/p}(p^e)c_K(p))f(p) & \text{if } p|N.\end{cases}$$

Then one easily verifies the next result.

COROLLARY 2.7. *Suppose K has minimal level and discriminant. With $\rho_{K,p}(n)$ as above,*

$$r(\text{gen } K, 2n)=\rho_{K,\infty}\prod_{p\text{ prime}}\rho_{K,p}(n).$$

COROLLARY 2.8. *The average theta series attached to the genera within* fam K *are linearly independent.*

Proof. Let $\{K_\alpha\colon \alpha|A\}$ be a set of representatives for the genera in fam K

where A is as in the remark following Lemma 1.3. First notice that for $\alpha|A$, $D|N$, $c_{K_\alpha}(D) = (\alpha/D)c_K(D)$. Take $d|N$; then

$$\sum_{\alpha|A}\left(\frac{\alpha}{d}\right)\theta(\text{gen } K_\alpha;\tau) = \frac{1}{a_N(0)c_K(N)}\sum_{D|N}\left(\sum_{\alpha|A}\left(\frac{\alpha}{Dd}\right)\right)c_K(D)E_D$$
$$= 2^h\frac{1}{a_N(0)c_K(N)}(c_K(d)E_d + c_K(N/d)E_{N/d}).$$

Note that $\{E_d\colon d|N, 0<d<N\}$ is a linearly independent set. □

§3. Lattices of descent. Now we fix an integral $\mathbb{Z}$-lattice L of even rank m and odd level N'. For convenience, assume L is scaled so that $Q(L)\subseteq 2\mathbb{Z}$, $Q(L)\not\subseteq 2n\mathbb{Z}$ for any $n>1$. We show that L descends from a lattice of minimal level and discriminant, then we construct chains of lattices from the minimal lattice to lattices K_0 in gen L; by counting how often an element of the minimal lattice lies in these lattices K_0, we obtain formulas for $r(\text{gen } L, 2n)$.

Notation. Fix a prime $q|N'$ and set $s=s(L,q)=[\text{ord}_q N'/2]$. Fix $t>2s+1$; then by Lemma 1.2,

$$L = L_0\oplus\cdots\oplus L_{2s+1}\simeq\langle 1,\dots,1,\varepsilon_0\rangle\perp\cdots\perp q^{2s+1}\langle 1,\dots,1,\varepsilon_{2s+1}\rangle \pmod{q^t},$$

where the $\varepsilon_i\in\mathbb{Z}-q\mathbb{Z}$, and the ith component, $L_i\simeq q^i\langle 1,\dots,1,\varepsilon_i\rangle$, has rank $m_i\geqslant 0$. Let $H_{2i}=q^{-i}L_{2i}$, $H_{2i+1}=q^{-i}L_{2i+1}$. Thus

$$L = H_0\oplus H_1\oplus qH_2\oplus qH_3\oplus\cdots\oplus q^sH_{2s}\oplus q^sH_{2s+1},$$

where the H_{2i} are unimodular (mod q^t), and the H_{2i+1} are q-modular (mod q^t). Let

$$\mathcal{H}_i = \bigoplus_{\substack{0\leqslant\ell\leqslant i\\ \ell\equiv i\ (\text{mod } 2)}} H_\ell,\qquad r_i=r_i(L,q)=\text{rank }\mathcal{H}_i,\qquad \eta_{2i}=\eta_{2i}(L,q)=\text{disc }\mathcal{H}_{2i},$$

$$q^{r_{2i+1}}\eta_{2i+1}=q^{r_{2i+1}}\eta_{2i+1}(L,q)=\text{disc }\mathcal{H}_{2i+1},\qquad \mu_i=\mu_i(L,q)=\left(\frac{(-1)^{\ell_i}\eta_i}{q}\right),$$

where $\ell_i=[r_i/2]$. (When $r_i=0$, set $\mu_i=1$.) Note that s, r_i, μ_i are invariants of gen L, and when r_i is even, $\mu_i=1$ exactly when $\mathcal{H}_i$ is hyperbolic modulo q. (Here $\mathcal{H}_i$ is scaled by $1/q$ when i is odd.)

LEMMA 3.1. *Fix a prime q dividing the level of L and let μ_j, r_j be as above.*

(a) *If r_{2s} is odd or $\mu_{r_s}=-1$, then there is a lattice K on V with $q^{s+1}K\subseteq L\subseteq K$, $K_{(p)}\simeq L_{(p)}$ for all primes $p\neq q$, and K has minimal level and discriminant at q.*

(b) *If* r_{2s} *is even and* $\mu_{2s} = 1$, *then there is a lattice* K^q *on* V *so that* $q^{s+1}K^q \subseteq L \subseteq K^q$, $K_{(p)} \simeq L^q_{(p)}$ *for all primes* $p \neq q$, *and* K^q *has minimal level and discriminant at* q.

Furthermore, if r_{2s} *is even,* $\mu_{2s} = -1$ *and* $\mu_{2s+1} = 1$, *then*

$$K_{(q)} \simeq \langle 1, \ldots, 1, \varepsilon_K \rangle, \qquad \left(\frac{(-1)^{m/2}\varepsilon_K}{q}\right) = -1.$$

If r_{2s} *is even,* $\mu_{2s} = -1 = \mu_{2s+1}$, *then*

$$K_{(q)} \simeq \langle 1, \ldots, 1, \varepsilon_K \rangle \perp q\langle 1, \varepsilon'_K \rangle, \qquad \left(\frac{(-1)^{m/2-1}\varepsilon_K}{q}\right) = -1 = \left(\frac{-\varepsilon'_K}{q}\right).$$

If r_{2s} *is even and* $\mu_{2s} = 1$, *then*

$$K^q_{(q)} \simeq \langle 1, \ldots, 1, \varepsilon_K \rangle, \qquad \left(\frac{(-1)^{m/2}\varepsilon_K}{q}\right) = \mu_{2s+1}.$$

Note that gen K *is determined by* gen L.

Remark. Since dL, dK, and dK^q differ by squares, their theta series are associated with the same character χ (although the modulus may differ between the theta series).

Proof. Set $L_0 = L$. For $0 \leqslant i \leqslant s$, set

$$L_{2i+1} = \text{preimage in } L_{2i} \text{ of rad } L_{2i}/qL_{2i}, \text{and}$$

$$qL_{2i+2} = \text{preimage in } L_{2i} \text{ of rad } L_{2i+1}/qL_{2i+1}.$$

One easily verifies that L_i is a $\mathbb{Z}$-lattice with $q^s L_{2s} \subseteq L$ and that

$$L_{2i} = \mathscr{H}_{2i} \oplus \mathscr{H}_{2i+1} \oplus \sum_{k=2}^{2s+1-2i} q^{[k/2]} H_{2i+k},$$

$$L_{2i+1} = \mathscr{H}_{2i+1} \oplus q\mathscr{H}_{2i+2} \oplus \sum_{k=3}^{2s+1-2i} q^{[k/2]} H_{2i+k}.$$

Notice that $L_{2s} = \mathscr{H}_{2s} \oplus \mathscr{H}_{2s+1} \simeq \langle 1, \ldots, 1, \eta_{2s} \rangle \perp q\langle 1, \ldots, 1, \eta_{2s+1} \rangle \pmod{q^t}$.

Constructing the lattice K is a bit more complicated.

Case (a). Say r_{2s} is odd or $\mu_{2s} = -1$. So

$$L_{2s+1} = \mathscr{H}_{2s+1} \oplus q\mathscr{H}_{2s},$$

where $\mathscr{H}_{2s}$ is unimodular (but not hyperbolic) modulo q^t, and $\mathscr{H}_{2s+1}$ is q-modular modulo q^t. Let $\bar{C}$ be a maximal totally isotropic subspace of the space L_{2s+1}/qL_{2s+1}, and set

$$qK = \text{preimage in } L_{2s+1} \text{ of } \bar{C}.$$

Then K is as described in the lemma.

Case (b). Say r_{2s} is odd and $\mu_{2s} = 1$. So

$$L_{2s} = \mathscr{H}_{2s} \oplus \mathscr{H}_{2s+1},$$

where $\mathscr{H}_{2s}$ is unimodular and hyperbolic modulo q^t, and $\mathscr{H}_{2s+1}$ is q-modular modulo q^t. Let $\bar{C}$ be a maximal totally isotropic subspace of L_{2s}/qL_{2s}, and set

$$qK = \text{preimage in } L_{2s+1} \text{ of } \bar{C}.$$

Then K^q is as described in the lemma. □

We construct descending chains of lattices $K, K_{2s}, \dots, K_0$ such that $K_i \in \text{gen } L_i$. We count how many K_0 contain a given vector $x \in K$, thereby obtaining formulas for $r(\text{gen } L, 2n)$.

More notation. For q fixed and r_i, μ_i as above, set $\mu = \mu_{2s}$, $\mu' = \mu_{2s+1}$,

$$d = \begin{cases} r_{2s}/2 & \text{if } 2|r_{2s},\ \mu = 1, \\ r_{2s+1}/2 & \text{if } 2|r_{2s},\ \mu_{2s} = -1,\ \mu' = 1, \\ (r_{2s+1}-1)/2 & \text{if } 2 \nmid r_{2s}, \\ r_{2s}/2 - 1 & \text{if } 2|r_{2s},\ \mu = -1 = \mu'; \end{cases}$$

$$\alpha = \alpha_{L,q} = \begin{cases} (q^d - 1)/[(q^{m/2} - 1 + 1)(q^{m/2-2} - 1)] & \text{if } 2|r_{2s},\ \mu = -1 = \mu', \\ (q^d - 1)/(q^{m-2} - 1) & \text{if } 2 \nmid r_{2s}, \\ (q^d - 1)/[(q^{m/2} - \mu\mu')(q^{m/2-1} + \mu\mu')] & \text{otherwise}; \end{cases}$$

$$\beta = \beta_{L,q} = \begin{cases} q^d(q^{m/2-d-1}+1)(q^{m/2-d-2}-1)/[(q^{m/2-1}+1)(q^{m/2-2}-1)] \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } 2|r_{2s}, \mu = -1 = \mu', \\ q^d(q^{m-2d-2}-1)/(q^{m-2}-1) \qquad\qquad \text{if } 2 \nmid r_{2s}, \\ q^d(q^{m/2-d}-\mu\mu')(q^{m/2-1}+\mu\mu')/[(q^{m/2}-\mu\mu')(q^{m/2-d-1}+\mu\mu')] \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise;} \end{cases}$$

and for $\omega = \pm 1$,

$$\gamma(\omega) = \gamma_{L,q}(\omega) = \begin{cases} (q^{m/2-d-1}+1)/(q^{m/2-1}+1) & \text{if } 2|r_{2s}, \mu = -1 = \mu', \\ (q^{m/2-d-1}+\omega\mu)/(q^{m/2-1}+\omega\mu) & \text{if } 2 \nmid r_{2s}, \\ (q^{m/2-d}-\mu\mu')/(q^{m/2}-\mu\mu') & \text{otherwise.} \end{cases}$$

LEMMA 3.2. *Let the notation be as above, and let χ denote the (primitive) character associated to $\theta(K;\tau)$ and to $\theta(L;\tau)$. We can construct sublattices K_{2s} of K such that $qK \subseteq K_{2s} \subseteq K$, and for all $t \in \mathbb{Z}_+$,*

$$K_{2s} \simeq \langle 1,\ldots,1,\eta_{2s}\rangle \perp q\langle 1,\ldots,1,\eta_{2s+1}\rangle \ (\mathrm{mod}\ q^t),$$

where the first component has rank r_{2s} and the second component has rank r_{2s+1}. (So $K_{2s} \in$ gen L_{2s}.) Set R = preimage in K of rad K/qK, *R_{2s} = preimage in K_{2s} of* rad K_{2s}/qK_{2s}. *(Here we scale R_{2s} by $1/q$ in the case $2|r_{2s}$, $\mu_{2s} = 1$.) Take $x \in K - R$.*

(a) *Say r_{2s} is odd or $\mu_{2s} = -1$. If $q \nmid Q(x)$, then $x \notin R_{2s}$, and the proportion of K_{2s} such that $x \in K_{2s} - R_{2s}$ is $\gamma(\omega)$, where $\omega = \chi_q(Q(x))$. If $q|Q(x)$, then the proportion of K_{2s} such that $x \in R_{2s}$ is α, and the proportion of K_{2s} such that $x \in K_{2s} - R_{2s}$ is β.*

(b) *Say r_{2s} is even and $\mu_{2s} = 1$. If $q \nmid Q(x)$, then x is in none of the lattices K_{2s}, and the proportion of K_{2s} such that $qx \in R_{2s} - qK_{2s}$ is $\gamma(\pm 1)$. If $q|Q(x)$, then the proportion of K_{2s} such that $x \in K_{2s} - R_{2s}$ is α.*

Notice that when $x \in K_{2s}$, we necessarily have $qx \in R_{2s}$.

Proof. Set $\mu = \mu_{2s}$, $\mu' = \mu_{2s+1}$.

(a) Say r_{2s} is odd or $\mu = -1$. Let $r = \dim$ (rad K/qK); so

$$r = \begin{cases} 0 & \text{if } r_{2s} \text{ is even, } \mu' = 1, \\ 1 & \text{if } r_{2s} \text{ is odd,} \\ 2 & \text{if } r_{2s} \text{ is even, } \mu' = -1, \end{cases}$$

and $d = (1/2)(r_{2s+1} - r)$. To construct K_{2s}, we take a totally isotropic subspace $\bar{C}$ of K/qK such that $\dim \bar{C} = d + r$ and $\text{rad } K/qK \subseteq \bar{C}$. Set

$$K' = \text{preimage in } K \text{ of } \bar{C},$$

$$qK_{2s} = \text{preimage in } K' \text{ of rad } K'/qK'.$$

Thus (using formulas from [1]), the number of choices we have for K_{2s} is

$$\begin{cases} \prod_{i=1}^{d}(q^{m/2-i+1}+1)(q^{m/2-i}-1)/(q^{d-i+1}-1) & \text{if } r_{2s} \text{ is even, } \mu' = 1, \\ \prod_{i=1}^{d}(q^{m-2i}-1)/(q^{d-i+1}-1) & \text{if } r_{2s} \text{ is odd,} \\ \prod_{i=1}^{d}(q^{m/2-i}+1)(q^{m/2-i-1}-1)/(q^{d-i+1}-1) & \text{if } r_{2s} \text{ is even, } \mu' = -1. \end{cases}$$

Take $x \in K - R$. We have $x \in R_{2s}$ if and only if $\bar{x} \in \bar{C}$, and $x \in K_{2s} - R_{2s}$ if and only if $\bar{x} \in \bar{C}^{\perp}$, $\bar{x} \notin \bar{C}$. When $\bar{x} \notin \bar{C}^{\perp}$ we have $qx \in K_{2s} - R_{2s}$. Also, $R \subseteq R_{2s}$. Thus the number of K_{2s} such that $x \in R_{2s}$ is

$$\begin{cases} \prod_{i=2}^{d}(q^{m/2-i+1}+1)(q^{m/2-i}-1)/(q^{d-i+1}-1) & \text{if } r_{2s} \text{ is even, } \mu' = 1, \\ \prod_{i=2}^{d}(q^{m-2i}-1)/(q^{d-i+1}-1) & \text{if } r_{2s} \text{ is odd,} \\ \prod_{i=2}^{d}(q^{m/2-i}+1)(q^{m/2-i-1}-1)/(q^{d-i+1}-1) & \text{if } r_{2s} \text{ is even, } \mu' = -1. \end{cases}$$

Now $\bar{x} \in \bar{C}^{\perp}$ if and only if $\bar{C} \subseteq \langle \bar{x} \rangle^{\perp}$ where $\langle \bar{x} \rangle$ denotes the space spanned by $\bar{x}$. When $q \nmid Q(x)$, $\langle \bar{x} \rangle^{\perp} = \bar{U} \perp \bar{R}$, where $\bar{U}$ is a regular space of dimension $m - r - 1$ and discriminant $Q(x)(-1)^{d}\eta_{2s}$. When $q|Q(x)$, $\langle \bar{x} \rangle^{\perp} = \langle \bar{x} \rangle \perp \bar{U} \perp \bar{R}$, where $\bar{U}$ is a regular space of dimension $m - r - 2$ and discriminant $(-1)^{d-1}\eta_{2s}$. Hence given our assumption that $\mu = -1$ when r_{2s} is even, $\bar{U}$ is never hyperbolic. So when $q \nmid Q(x)$ and $\omega = \chi_q(Q(x))$, the number of $\bar{C} \subseteq \langle \bar{x} \rangle^{\perp}$ is

$$\begin{cases} \prod_{i=1}^{d}(q^{m-2i}-1)/(q^{d-i+1}-1) & \text{if } r_{2s} \text{ is even, } \mu' = 1, \\ \prod_{i=1}^{d}(q^{m/2-i}-\omega\mu)(q^{m/2-i-1}+\omega\mu)/(q^{d-i+1}-1) & \text{if } r_{2s} \text{ is odd, } \mu = \mu, \\ \prod_{i=1}^{d}(q^{m-2i-2}-1)/(q^{d-i+1}-1) & \text{if } r_{2s} \text{ is even, } \mu = -1 = \mu'. \end{cases}$$

When $q|Q(x)$, the number of $\bar{C} \subseteq \langle \bar{x} \rangle^{\perp}$ such that $\bar{x} \notin \bar{C}^{\perp}$ is

$$\begin{cases} \prod_{i=1}^{d} q(q^{m/2-i}+1)(q^{m/2-i-1}-1)/(q^{d-i+1}-1) & \text{if } r_{2s} \text{ is even, } \mu' = 1, \\ \prod_{i=1}^{d} q(q^{m-2i-2}-1)/(q^{d-i+1}-1) & \text{if } r_{2s} \text{ is odd,} \\ \prod_{i=1}^{d} q(q^{m/2-i-1}+1)(q^{m/2-i-2}-1)/(q^{d-i+1}-1) & \text{if } r_{2s} \text{ is even, } \mu' = -1. \end{cases}$$

(b) Say r_{2s} is even, $\mu = 1$. So $K^q_{(q)} \simeq \langle 1, \ldots, 1, \varepsilon \rangle$, and K^q/qK^q is hyperbolic if and only if $\mu' = 1$. To construct K_{2s} from K, we take a totally isotropic subspace $\bar{C}$ of K^q/qK^q of dimension d where $d = r_{2s}/2$, and set

$$K^q_{2s} = \text{preimage in } K^q \text{ of } \bar{C}.$$

Thus the number of choices we have for K_{2s} is

$$\prod_{i=1}^{d} (q^{m/2-i+1}-\mu')(q^{m/2-i}+\mu')(q^{d-i+1}-1).$$

Take $x \in K - R$. We have $x \in K_{2s} - R_{2s}$ if and only if $\bar{x} \in \bar{C}$, and $qx \in R_{2s} - qK_{2s}$ if and only if $\bar{x} \in \bar{C}^{\perp}$, $\bar{x} \notin \bar{C}$. When $\bar{x} \notin \bar{C}^{\perp}$, we have $x \notin K_{2s}$, $qx \in K_{2s} - R_{2s}$. Thus the number of K_{2s} such that $x \in K_{2s} - R_{2s}$ is

$$\begin{cases} \prod_{i=2}^{d} (q^{m/2-i+1}-\mu')(q^{m/2-i}+\mu')(q^{d-i+1}-1) & \text{if } q|Q(x), \\ 0 & \text{if } q \nmid Q(x). \end{cases}$$

The number of K_{2s} such that $qx \in R_{2s} - qK_{2s}$ is

$$\begin{cases} \prod_{i=1}^{d} q(q^{m/2-i}-\mu')(q^{m/2-i-1}+\mu')/(q^{d-i+1}-1) & \text{if } q|Q(x), \\ \prod_{i=1}^{d} (q^{m-2i}-1)/(q^{d-i+1}-1) & \text{if } q \nmid Q(x). \end{cases}$$

The lemma now follows. □

LEMMA 3.3. *Let K_{2s} be as in Lemma 3.2. We can construct descending chains of lattices $K_{2s}, \dots, K_0$ so that $K_0 \in \operatorname{gen} L$, and $q^s K_{2s} \subseteq K_0$. Fix such a chain and let*

$$R_{2i} = \text{preimage in } K_{2i} \text{ of rad } K_{2i}/qK_{2i},$$

$$qR_{2i+1} = \text{preimage in } K_{2i+1} \text{ of rad } K_{2i+1}/qK_{2i+1}.$$

Take $x \in K_{2s} - qK_{2s}$ and fix ℓ, $0 \leqslant \ell \leqslant s$. The chain of lattices has the following properties.

(a) *First suppose $x \in R_{2s}$. Then $q^\ell x \in K_0$ if and only if $x \in K_{2i-1}$ for all i, $\ell < i \leqslant s$. Also, when $x \in R_{2i} \cap K_{2i-1}$, we necessarily have $x \in R_{2i-2} \subseteq K_{2i-2}$.*

(b) *Suppose now $x \notin R_{2s}$. Then $q^\ell x \in K_0$ if and only if $x \in K_{2i-2}$ for all i, $\ell < i \leqslant s$. Also, when $x \in K_{2i}$ but $x \notin R_{2i}$, we necessarily have $x \in R_{2i-1}$.*

Proof. With K_{2s} as above and $s > 0$, we inductively define lattices K_j, R_j, $0 \leqslant j < 2s$, as follows. Suppose that for $i \leqslant s$,

$$K_{2i} = \tilde{J}_{2i} \oplus \tilde{J}_{2i+1} \oplus J'$$

$$\simeq \langle 1, \dots, 1, \eta_{2i} \rangle \perp q\langle 1, \dots, 1, \eta_{2i+1} \rangle \perp q^2 \langle \alpha_1, \alpha_2, \dots \rangle \pmod{q^t},$$

with $\eta_{2i}, \eta_{2i+1} \in \mathbb{Z} - q\mathbb{Z}$, $\alpha_j \in \mathbb{Z}$. Set

$$R_{2i} = \text{preimage in } K_{2i} \text{ of rad } (K_{2i}/qK_{2i}) = \tilde{J}_{2i+1} \oplus q\tilde{J}_{2i} \oplus J'.$$

Let $\overline{M}$ be an (r_{2i-1}, μ_{2i-1})-subspace of R_{2i}/qR_{2i}, and let

$$K_{2i-1} = \text{preimage in } R_{2i} \text{ of } \overline{M} + \overline{qK_{2i}} = \tilde{J}_{2i-1} \oplus q\tilde{J}_{2i} \oplus qJ_{2i+1} \oplus qJ',$$

where $\tilde{J}_{2i-1} \oplus J_{2i+1} = \tilde{J}_{2i+1}$, and $\tilde{J}_{2i-1} \simeq \langle 1, \dots, 1, \eta_{2i-1} \rangle \pmod{q}$. Since we know that $(\eta_{2i+1}/q) = (\eta_{2i-1}\varepsilon_{2i+1}/q)$, $\tilde{J}_{2i+1} \simeq \langle 1, \dots, 1, \varepsilon_{2i+1} \rangle \pmod{q}$. So by Lemma 1.1, we have

$$K_{2i-1} = \tilde{J}_{2i-1} \oplus q\tilde{J}_{2i} \oplus qJ_{2i+1} \oplus qJ'$$

$$\simeq q\langle 1, \dots, 1, \eta_{2i-1} \rangle \perp q^2 \langle 1, \dots, 1, \eta_{2i} \rangle$$

$$\perp q^3 \langle 1, \dots, 1, \varepsilon_{2i+1} \rangle \perp q^4 \langle \alpha_1, \alpha_2, \dots \rangle \pmod{q^t}$$

(where $t > 2s + 1$). Now set

$$qR_{2i-1} = \text{preimage in } K_{2i-1} \text{ of rad } (K_{2i-1}/qK_{2i-1})$$

$$= q\tilde{J}_{2i} \oplus q^2\tilde{J}_{2i-1} \oplus q^2 J_{2i+1} \oplus q^2 J'.$$

Let $\overline{M'}$ be an (r_{2i-2}, μ_{2i-2})-subspace of R_{2i-1}/qR_{2i-1}, and let

K_{2i-2} = preimage in R_{2i-1} of $\overline{M'} + \overline{K_{2i-1}} = \tilde{J}_{2i-2} \oplus \tilde{J}_{2i-1} \oplus qJ_{2i} \oplus J_{2i+1} \oplus qJ'$, where

$$\tilde{J}_{2i-2} \oplus J_{2i} = \tilde{J}_{2i}, \quad \tilde{J}_{2i-2} \simeq \langle 1, \dots, 1, \eta_{2i-2} \rangle \pmod q,$$
$$J_{2i} \simeq \langle 1, \dots, 1, \varepsilon_{2i} \rangle \pmod q.$$

Thus, by Lemma 1.1,

$$K_{2i-2} \simeq \langle 1, \dots, 1, \eta_{2i-2} \rangle \perp q\langle 1, \dots, 1, \eta_{2i-1} \rangle \perp q^2\langle 1, \dots, 1, \varepsilon_{2i} \rangle$$
$$\perp q^3\langle 1, \dots, 1, \varepsilon_{2i+1} \rangle \perp q^4\langle \alpha_1, \alpha_2, \dots \rangle \pmod{q^t}.$$

One easily verifies that the choices of K_{2i-1} and K_{2i-2} are uniquely determined by the subspaces $\overline{M} + \overline{qK_{2i}} \subseteq R_{2i}/qR_{2i}$ and $\overline{M'} + \overline{K_{2i-1}} \subseteq R_{2i-1}/qR_{2i-1}$. In fact, given a Jordan decomposition of $(K_{2i})_{(q)}$ (and hence of $(R_{2i})_{(q)}$) as above, K_{2i-1} is uniquely determined by the subspace $\overline{M}$ of $\tilde{J}_{2s+1}/q\tilde{J}_{2s+1}$; similarly, K_{2i-2} is uniquely determined by the subspace $\overline{M'}$ of $\tilde{J}_{2s}/q\tilde{J}_{2s}$.

To summarize, we have

$$K_{2i} = \tilde{J}_{2i} \oplus \tilde{J}_{2i+1} \oplus \sum_{\ell=1}^{s-i} q^{\ell}(J_{2i+2\ell} \oplus J_{2i+2\ell+1}),$$

$$R_{2i} = \tilde{J}_{2i+1} \oplus q\tilde{J}_{2i} \oplus \sum_{\ell=1}^{s-i} q^{\ell}(J_{2i+2\ell} \oplus J_{2i+2\ell+1}),$$

$$K_{2i-1} = \tilde{J}_{2i-1} \oplus q\tilde{J}_{2i} \oplus qJ_{2i+1} \oplus \sum_{\ell=1}^{s-i} q^{\ell+1}(J_{2i+2\ell} \oplus J_{2i+2\ell+1}),$$

$$R_{2i-1} = \tilde{J}_{2i} \oplus \tilde{J}_{2i-1} \oplus J_{2i+1} \oplus \sum_{\ell=1}^{s-i} q^{\ell}(J_{2i+2\ell} \oplus J_{2i+2\ell+1}),$$

$$K_{2i-2} = \tilde{J}_{2i-2} \oplus \tilde{J}_{2i-1} \oplus \sum_{\ell=1}^{s-i+1} q^{\ell}(J_{2i+2\ell-2} \oplus J_{2i+2\ell-1}).$$

Notice that $q^{\ell}K_{2s} \cap K_0 = q^{\ell}K_{2\ell}$ and $K_{2\ell} \subseteq K_{2\ell+2} \subseteq \cdots \subseteq K_{2s}$. Hence $q^{\ell}x \in K_0$ if and only if K_{2i} for all i, $\ell < i \leqslant s$. Also $R_{2i} \cap K_{2i-2} = K_{2i-1} = R_{2i-2}$ (proving (a)). When $x \notin R_{2i}$, we necessarily have $x \notin K_{2i-1}$, but $x \in R_{2i-1}$ and $r_{2i-1} = R_{2i} \supseteq K_{2i-2}$. □

To help us more easily count how many choices of K_{j-1} contain a given vector $x \in K_j - qK_j$, we introduce an auxiliary counting function and establish some basic identities.

Definition. Let $\overline{M}$ be a regular $\mathbb{Z}/q\mathbb{Z}$-quadratic space of type (c, ε). An ordered basis $(\bar{x}_1, \dots, \bar{x}_c)$ for $\overline{M}$ is an alternating basis if

(i) $\bar{x}_{2i}$ is isotropic for $1 \leqslant i < c/2$ and, if $2|c$ and $\varepsilon = 1$, for $i = c/2$,

(ii) $\bar{x}_{2i-1}$ is anisotropic $(1 \leqslant i \leqslant c/2)$, and

(iii) relative to this basis,

$$\overline{M} \simeq \begin{pmatrix} * & * \\ * & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} * & * \\ * & 0 \end{pmatrix} \perp \bar{A}$$

where $\bar{A}$ is anisotropic and diagonal of dimension 0, 1, or 2.

Let $\Psi[(c, \varepsilon) \colon (d, \mu)]$ be the number of ways to choose $\bar{x}_1, \dots, \bar{x}_c$ such that $(\bar{x}_1, \dots, \bar{x}_c)$ is an alternating basis for some (c, ε) subspace of a fixed (d, μ) space. Let $\Psi_{\bar{x}}[(c, \varepsilon) \colon (d, \mu)]$ be the number of such subspace bases $(\bar{x}_1, \dots, \bar{x}_c)$ with $\bar{x}_1 = \bar{x}$ if $\bar{x}$ is anisotropic, and $\bar{x}_2 = \bar{x}$ if $\bar{x}$ is isotropic. Note that if $\bar{x}$ is a basis vector for a (c, ε) subspace of an (d, μ) space $\overline{W}$, we have $\bar{x} \notin \operatorname{rad} \overline{W}$.

LEMMA 3.4. *Suppose $\overline{W}$ is a (d, μ)-space, $d > 1$. Given $\omega \neq 0$, the number of solutions to $Q(\overline{w}) = \omega$ with $\overline{w} \in \overline{W}$ is*

$$\begin{cases} q^{d/2-1}(q^{d/2} - \mu) & \text{if } 2|d, \\ q^{(d-1)/2}\left(q^{(d-1)/2} + \left(\dfrac{\omega}{q}\right)\mu\right) & \text{otherwise}. \end{cases}$$

Proof. First suppose $\overline{W}$ is a $(2, \mu)$-space. As described in [1], there are $q - \mu$ symmetries of $\overline{W}$. Given anisotropic $\bar{x} \in \overline{W}$, one easily verifies that only the trivial symmetry fixes $\bar{x}$, and the action of a symmetry on $\bar{x}$ determines its action on $\overline{W}$. Thus, if $Q(\overline{w}) = \omega$ has any solutions, it has exactly $q - \mu$ solutions. We know there are $(q - \mu)(1 + \mu)$ (nonzero) isotropic vectors in $\overline{W}$; consequently, there must be $q - \mu$ solutions to $Q(\overline{w}) = \omega$ for any $\omega \neq 0$.

Next suppose $d > 2$; set $\ell = [(d-1)/2]$. Write $\overline{W} = \bar{U} \perp \bar{A}$, where $\bar{U}$ is a $(2\ell, 1)$-space and $\bar{A}$ is a $(d - 2\ell, \mu)$-space. Using induction on ℓ, we count the number of solutions to $Q(\bar{u}) + Q(\bar{a}) = \omega$ with $\bar{u} \in \bar{U}$, $\bar{a} \in \bar{A}$. $\square$

LEMMA 3.5. (1) *Suppose that $d > c > 2$, or that $d > c = 2$ and $\mu = \varepsilon$. Letting Ψ_* denote Ψ or $\Psi_{\bar{x}}$, we have*

$$\Psi_*[(c, \varepsilon) \colon (d, \mu)] = \Psi_*[(2, 1) \colon (d, \mu)] \cdot \Psi[(c-2, \varepsilon) \colon (d-2, \mu)].$$

(2) *With* $\varepsilon = \pm 1$,

$$\Psi[(1,\varepsilon)\colon (d,\mu)] = \begin{cases} q^{d/2-1}(q-1)(q^{d/2}-\mu) & \text{if } 2|d, \\ q^{(d-1)/2}(q-1)(q^{(d-1)/2}+\varepsilon\mu) & \text{otherwise;} \end{cases}$$

$$\Psi_{\bar{x}}[(1,\varepsilon)\colon (d,\mu)] = \begin{cases} 1 & \text{if } \left(\frac{Q(x)}{q}\right) = \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

(3) *Suppose* $d > 2$, *or* $d = 2$, $\mu = 1$. *Then*

$$\Psi[(2,1)\colon (d,\mu)] = \begin{cases} q^{d-2}(q-1)^2(q^{d/2}-\mu)(q^{d/2-1}+\mu) & \text{if } 2|d, \\ q^{d-2}(q-1)^2(q^{d-1}-1) & \text{otherwise;} \end{cases}$$

$$\Psi_{\bar{x}}[(2,1)\colon (d,\mu)] = \begin{cases} q^{d/2-1}(q-1)(q^{d/2-1}+\mu) & \text{if } 2|d,\ Q(\bar{x}) \neq 0, \\ q^{(d-3)/2}(q-1)\left(q^{(d-1)/2} - \left(\frac{Q(x)}{q}\right)\mu\right) & \text{if } 2\nmid d,\ Q(\bar{x}) \neq 0, \\ q^{d-2}(q-1)^2 & \text{if } Q(\bar{x}) = 0. \end{cases}$$

(4) *Suppose* $d > 2$, *or* $d = 2$, $\mu = -1$. *Then assuming* $Q(\bar{x}) \neq 0$,

$$\Psi[(2,-1)\colon (d,\mu)] = \begin{cases} \frac{1}{2}q^{d-1}(q-1)^2(q^{d/2}-\mu)(q^{d/2-1}-\mu) & \text{if } 2|d, \\ \frac{1}{2}q^{d-1}(q-1)^2(q^{d-1}-1) & \text{otherwise;} \end{cases}$$

$$\Psi_{\bar{x}}[(2,-1)\colon (d,\mu)] = \begin{cases} \frac{1}{2}q^{d/2}(q-1)(q^{d/2-1}-\mu) & \text{if } 2|d, \\ \frac{1}{2}q^{(d-1)/2}(q-1)\left(q^{(d-1)/2} - \left(\frac{Q(x)}{q}\right)\mu\right) & \text{otherwise.} \end{cases}$$

Proof. The proof of (1) follows from the observation that $\bar{U}^\perp$ is a $(d-2,\mu)$-space whenever $\bar{U}$ is a $(2,1)$-subspace of a (d,μ)-space.

(2) follows immediately from the preceding lemma.

(3) Choosing an alternating basis $\{\bar{x},\bar{y}\}$ for a $(2,1)$-subspace, we have

$$\Psi[(1,1)\colon (d,\mu)] + \Psi[(1,-1)\colon (d,\mu)]$$

choices for $\bar{x}$. Set $\mu' = (-1/q)\mu$ if $2|d$, and $\mu' = \mu$ otherwise. Then $\langle \bar{x} \rangle^{\perp}$ is a

$$\left(d-1, \left(\frac{Q(x)}{q}\right)\mu'\right)$$

space, so we can use the formulas of [1] to count isotropic $\bar{y} \notin \langle \bar{x} \rangle^{\perp}$. Given isotropic $\bar{y}$ (not in the radical), $\langle \bar{y} \rangle^{\perp}$ is a $(d-1; d-2, \mu)$-space, so we can count anisotropic $\bar{x} \notin \langle \bar{y} \rangle^{\perp}$.

(4) Choosing an orthogonal basis $\{\bar{x}, \bar{y}\}$ for a $(2, -1)$-subspace, we have

$$\Psi[(1,1):(d,\mu)] + \Psi[(1,-1):(d,\mu)]$$

choices for $\bar{x}$. We choose $\bar{y} \in \langle \bar{x} \rangle^{\perp}$ such that $(-Q(x)Q(y)/q) = -1$; thus we have

$$\Psi_{\bar{x}}[(2,-1):(d,\mu)] = \Psi\left[\left(1, -\left(\frac{-Q(x)}{q}\right)\right) : \left(d-1, \left(\frac{Q(x)}{q}\right)\mu'\right)\right]$$

choices for $\bar{y}$ (where μ' is as in (3)). □

LEMMA 3.6. *For $0 \leqslant j \leqslant 2s+1$, $\omega = 0, \pm 1$, set*

$$r_j' = \begin{cases} m + r_{2s} & \text{if } j \text{ is even,} \\ m + r_{2s+1} & \text{if } j \text{ is odd,} \end{cases}$$

and set

$$v_j = v_j(\omega; L, q) = \begin{cases} q^{(r_j - r_j')/2}(q^{r_j/2} - \mu_j) & \text{if } 2|r_j,\ \omega \neq 0, \\ q^{(r_j - r_j' + 1)/2}(q^{(r_j-1)/2} + \omega\mu_j) & \text{if } 2 \nmid r_j,\ \omega \neq 0, \\ q^{1 - r_j'/2}(q^{r_j/2} - \mu_j)(q^{r_j/2 - 1} + \mu_j) & \text{if } 2|r_j,\ \omega = 0, \\ q^{1 - r_j'/2}(q^{r_j - 1} - 1) & \text{if } 2 \nmid r_j,\ \omega = 0. \end{cases}$$

For $j > s$, let $v_{2j}(\omega) = v_{2s}(\omega)$, $v_{2j+1}(\omega) = v_{2s+1}(\omega)$. Choose $x \in K_{2s} - qK_{2s}$; fix $\ell \geqslant 0$.

(a) *Say $x \in R_{2s}$; set $\omega = \chi_q(Q(x)/q)$. The proportion of chains $K_{2s}, \ldots, K_0$ such that $q^{\ell}x \in K_0$ is $(v_{2\ell+1}(\omega))/(v_{2s+1}(\omega))$.*

(b) *Say $x \notin R_{2s}$; set $\omega = \chi_q(Q(x))$. The proportion of chains $K_{2s}, \ldots, K_0$ such that $q^{\ell}x \in K_0$ is $v_{2\ell}(\omega)/v_{2s}(\omega)$.*

Proof. First notice that if $r_{2s+1} = 1$, or if $r_{2s+1} = 2$ and $\mu_{2s+1} = -1$, then $q^2 \nmid Q(x)$ for $x \in R_{2s} - qK_{2s}$; similarly, if $r_{2s} = 1$, or if $r_2 = 2$ and $\mu_{2s} = -1$, then

$q \nmid Q(x)$ for $x \in K_{2s} - R_{2s}$. Thus $\nu_{2s+1}(\omega)$, $\nu_{2s}(\omega)$ are never zero, where

$$\omega = \begin{cases} \chi_q(Q(x)/q) & \text{if } x \in R_{2s} - qK_{2s}, \\ \chi_q(Q(x)) & \text{if } x \in K_{2s} - r_{2s}. \end{cases}$$

As argued in the proof of Lemma 3.3, K_j is determined by the choice of the (r_j, μ_j)-subspace $\overline{M}$ of the (r_{j+2}, μ_{j+2})-space R_{j+1}/qR_{j+1}. Now the number of alternative bases for an (r, μ)-space is $\Psi[(r, \mu)\colon (r, \mu)]$. Hence, having chosen K_i for $j < i \leqslant 2s$, the number of choices for K_j is

$$\frac{\Psi[(r_j, \mu_j)\colon (r_{j+2}, \mu_{j+2})]}{\Psi[(r_j, \mu_j)\colon (r_j, \mu_j)]},$$

and the number of choices of K_j containing a given vector x is

$$\frac{\Psi_{\bar{x}}[(r_j, \mu_j)\colon (r_{j+2}, \mu_{j+2})]}{\Psi_{\bar{x}}[(r_j, \mu_j)\colon (r_j, \mu_j)]}.$$

By Lemma 3.5,

$$\frac{\Psi_{\bar{x}}[(r_j, \mu_j)\colon (r_{j+2}, \mu_{j+2})]\Psi[(r_j, \mu_j)\colon (r_j, \mu_j)]}{\Psi_{\bar{x}}[(r_j, \mu_j)\colon (r_j, \mu_j)]\Psi[(r_j, \mu_j)\colon (r_{j+2}, \mu_{j+2})]} = \frac{\nu_j(\omega)}{\nu_{j+2}(\omega)}.$$

Thus, for $x \in K_{2s}$, the proportion of chains with $q^\ell x \in K_0$ is

$$\begin{cases} \displaystyle\prod_{\ell < i \leqslant s} \frac{\nu_{2i-1}(\omega)}{\nu_{2i+1}(\omega)} & \text{if } x \in R_{2s} - qK_{2s}, \\ \displaystyle\prod_{\ell < i \leqslant s} \frac{\nu_{2i-2}(\omega)}{\nu_{2i}(\omega)} & \text{if } x \in K_{2s} - R_{2s}. \end{cases}$$

Thus, by Lemma 3.3, if $x \in R_{2s} - qK_{2s}$, then the proportion of chains with $q^\ell x \in K_0$ is

$$\prod_{\ell \leqslant i < s} \frac{\nu_{2i+1}(\omega)}{\nu_{2i+3}(\omega)} = \frac{\nu_{2\ell+1}(\omega)}{\nu_{2s+1}(\omega)}.$$

Similarly, if $x \in K_{2s} - R_{2s}$, then the proportion of chains with $q^\ell x \in K_0$ is

$$\prod_{\ell \leqslant i < s} \frac{\nu_{2i}(\omega)}{\nu_{2i+2}(\omega)} = \frac{\nu_{2\ell}(\omega)}{\nu_{2s}(\omega)}. \quad \square$$

THEOREM 3.7. *Suppose* $m = \operatorname{rank} L$ *is even,* $m \geqslant 6$, *and* $(-1)^{m/2}\, dL \equiv 1 \pmod 4$. *For* $n \in \mathbb{Z}$, *the average representation number is*

$$r(\text{gen } L, 2n) = \rho_{L,\infty} \prod_{q'} \rho_{L,q}(n),$$

where $\rho_{L,\infty} = (2\pi)^{m/2}/\Gamma(m/2)$, *and for fixed* q, $\rho_{L,q}(n)$ *is defined as follows.*
Write $dL = NN_1^2$, *where* N *is square-free; define* χ *by*

$$\chi(d) = \operatorname{sgn}(d)^{m/2}\left(\frac{(-1)^{m/2}\, dL}{d}\right),$$

where $(*)$ *denotes the Kronecker symbol. Thus* χ *is a character with conductor* N. *Let* $v_j(\varepsilon) = v_j(\varepsilon; L, q)$ *be as defined in Lemma 3.6. Then for* $e = \operatorname{ord}_q(n)$, $\varepsilon = ((n/q^e)/q)$,

$$\rho_{L,q}(n) = v_e(\varepsilon; L, q) + \sum_{0 \leqslant \ell \leqslant e-1} q^{(m/2-1)(e-\ell)} v_\ell(0; L, q).$$

Proof. If L has minimal level and discriminant, then the theorem follows immediately from Corollary 2.7. Thus, we argue by induction on the number of primes q at which L does not have minimal level and discriminant at q.

The induction hypothesis is that the theorem holds for all lattices which have fewer than h primes at which the lattice does not have minimal level and discriminant. Let L be a lattice with h such primes. Fix such a prime q, and let K be as in Lemma 3.1. If r_{2s} is odd or $\mu_{2s} = -1$, then K has minimal level and discriminant at q, and the induction hypothesis (and the fact that the local structures of K and L agree for primes $p \neq q$) implies that

$$r(\text{gen } K, 2n) = \rho_{K,q}(n)\rho_{L,\infty} \cdot \prod_{p \neq q} \rho_{L,p}(n),$$

where $\rho_{K,q}(n)$ is as in Corollary 2.7, and $\rho_{L,p}(n)$ is as in the statement of the theorem to be proved. If r_{2s} is even and $\mu_{2s} = 1$, then K^q has minimal level and discriminant at q, and the induction hypothesis again implies that

$$r(\text{gen } K^q, 2n) = \rho_{K^q,q}(n)\rho_{L,\infty} \cdot \prod_{p \neq q} \rho_{L,p}(qn) = \chi_N(q)\rho_{K^q,q}(n) \cdot \prod_{p \neq q} \rho_{L,p}(n),$$

where $\rho_{K^q,q}(n)$ is as in Corollary 2.7, and $\rho_{L,p}(n)$ is as in the theorem to be proved.

Case 1. First consider the case that either $2\nmid r_{2s}$ or $\mu_{2s}=-1$; so K, L lie in the same quadratic space. Assume $s\geqslant 1$. For $\varepsilon=\pm 1$, $\omega=0,\ \pm 1$, set

$$B_\ell(\varepsilon)=\gamma(\varepsilon)\frac{\nu_{2\ell}(\varepsilon)}{\nu_{2s}(\varepsilon)}+(1-\gamma(\varepsilon))\frac{\nu_{2\ell-1}(0)}{\nu_{2s+1}(0)},$$

$$A_\ell(\omega)=\alpha\frac{\nu_{2\ell+1}(\omega)}{\nu_{2s+1}(\omega)}+\beta\frac{\nu_{2\ell}(0)}{\nu_{2s}(0)}+(1-\alpha-\beta)\frac{\nu_{2\ell-1}(0)}{\nu_{2s+1}(0)}.$$

(Here the notation is as in Lemmas 3.2 and 3.6; we take $\nu_{-1}(*)=0$.) Fix $\ell\geqslant 0$, and let $R=$preimage in K of rad K/qK. Then from Lemmas 3.2 and 3.6, we have the following.

(a) For $x\in K-R$ and $q|Q(x)$, the proportion of K_0 in K containing $q^\ell x$ is $A_\ell(\omega)$, where $\omega=((Q(x)/q)/q)$.

(b) For $x\in K-R$ and $q\nmid Q(x)$, the proportion of K_0 in K containing $q^\ell x$ is $B_\ell(\varepsilon)$, where $\varepsilon=(Q(x)/q)$.

(c) For $x\in R-qK$ and $q|Q(x)$, the proportion of K_0 in K containing $q^\ell x$ is $\nu_{2\ell+1}(\varepsilon)/\nu_{2s+1}(\varepsilon)$, where $\varepsilon=((Q(x)/q)/q)$.

If $q\nmid N$, $R=qK$. So suppose $q|N$. Let q' be a prime associated to q as in Proposition 2.1. Then, as discussed in the proof of Proposition 2.1, we know that

$$\theta(\text{gen }K;\tau)|\frac{1}{(q')^{m/2-1}+1}T_{q'}=\theta(\text{gen }M;\tau),$$

where $M_{(p)}\simeq K^{q'}_{(p)}$ for all primes $p\neq q'$, and $M_{(q')}\simeq K_{(q')}$. Thus, the nth Fourier coefficient of $\theta(\text{gen }M;\tau)$ is $r(\text{gen }M,2n)=\rho_{K,\infty}\prod_p\rho_{M,p}(n)$, where our conditions on q' give us

$$\rho_{M,p}(n)=\begin{cases}\rho_{K,p}(n) & \text{for } p\nmid N\\ \rho_{K,p}(q'n) & \text{for } p|N\end{cases}=\begin{cases}\rho_{K,p}(qn) & \text{for } p\neq q,\\ \rho_{K,q}(q'n) & \text{for } p=q.\end{cases}$$

Then, Proposition 2.1 implies that the nth Fourier coefficient of $\theta(\text{gen }R;\tau)$ is

$$r(\text{gen }R,2n)=\begin{cases}0 & \text{if } q\nmid n,\\ \dfrac{q^{m/2-1}+1}{q^{m/2-1}}r(\text{gen }M,2n/q)-\dfrac{1}{q^{m/2-1}}r(\text{gen }K,2n) & \text{if } q|n,\end{cases}$$

so

$$r(\text{gen }R,2n)=\frac{r(\text{gen }K,2n)}{\rho_{K,q}(n)}\rho_{R,q}(n),$$

where $\rho_{R,q}(n)=0$ if $q\nmid n$, and for $e=\text{ord}_q(n)\geqslant 1$,

$$\rho_{R,q}(n)/f(q)=\left(\frac{q^{m/2-1}+1}{q^{m/2-1}}\rho_{K,q}(q'n/q)-\frac{1}{q^{m/2-1}}\rho_{K,q}(n)\right)/f(q)$$

$$=c_K(q)+\frac{q^{m/2-1}+1}{q^{m/2-1}}\chi_q(2q'n/q^e)\chi_{N/q}(q^{e-1})q^{(m/2-1)(e-1)}$$

$$-\frac{1}{q^{m/2-1}}\chi_q(2n/q^e)\chi_{N/q}(q^e)q^{(m/2-1)e}$$

$$=c_K(q)+\chi_q(2n/q^e)\chi_{N/q}(q^e)q^{(m/2-1)(e-1)}$$

$$\times\left(\frac{q^{m/2-1}+1}{q^{m/2-1}}\chi_q(q')\chi_{N/q}(q)-\frac{1}{q^{m/2-1}}\right).$$

Now, by our conditions on q', we have

$$\chi_q(q')\chi_{N/q}(q)=\chi_q(q')\chi_{N/q}(q')=\chi(q')=1;$$

so for $e\geqslant 1$,

$$\rho_{R,q}(n)=(c_K(q)+\chi_q(2n/q^e)\chi_{N/q}(q^e)q^{(m/2-1)(e-2)})f(q).$$

Notice that for $e\geqslant 2$, $\rho_{R,q}(n)=\rho_{K,q}(n/q^2)=\rho_{qK,q}(n)$.

Let δ denote the number of K_0 in K; we have

$$\frac{1}{\delta}\sum_{K_0\subseteq K}\theta(K_0;\tau)=\theta(q^{s+1}K;\tau)+\sum_{\substack{x\in K-R\\ q\mid Q(x)}}\sum_{0\leqslant\ell\leqslant s}A_\ell\left(\left(\frac{Q(x)/q}{q}\right)\right)e\{Q(q^\ell x)\tau\}$$

$$+\sum_{\substack{x\in K\\ q\nmid Q(x)}}\sum_{0\leqslant\ell\leqslant s}B_\ell\left(\left(\frac{Q(x)}{q}\right)\right)e\{Q(q^\ell x)\tau\}$$

$$+\sum_{x\in R-qK}\sum_{0\leqslant\ell\leqslant s}\frac{\nu_{2\ell+1}\left(\left(\frac{Q(x)/q}{q}\right)\right)}{\nu_{2s+1}\left(\left(\frac{Q(x)/q}{q}\right)\right)}e\{Q(q^\ell x)\tau\}$$

$$=\theta(q^tK;\tau)+\sum_{\substack{x\in K-R\\ q\mid Q(x)}}\sum_{0\leqslant\ell\leqslant t-1}A_\ell\left(\left(\frac{Q(x)/q}{q}\right)\right)e\{Q(q^\ell x)\tau\}$$

$$+\sum_{\substack{x\in K\\ q\nmid Q(x)}}\ \sum_{0\leqslant\ell\leqslant t-1} B_\ell\left(\left(\frac{Q(x)}{q}\right)\right)e\{Q(q^\ell x)\tau\}$$

$$+\sum_{x\in R-qK}\ \sum_{0\leqslant\ell\leqslant t-1}\frac{\nu_{2\ell+1}\left(\left(\frac{Q(x)/q}{q}\right)\right)}{\nu_{2s+1}\left(\left(\frac{Q(x)/q}{q}\right)\right)}e\{Q(q^\ell x)\tau\}$$

for any $t>s$. So with $e=\mathrm{ord}_q(n)$ and $\varepsilon=((2n/q^e)/q)$, the nth coefficient of $1/\delta\sum_{K_0\subseteq K}\theta(K_0;\tau)$ is

$$\begin{cases} A_t(\varepsilon)(a(K,n/q^{2t})-a(R,n/q^{2t})) \\ \quad +\dfrac{\nu_{2t+1}(\varepsilon)}{\nu_{2s+1}(\varepsilon)}(a(R,n/q^{2t})-a(qK,n/q^{2t})) \\ \quad +\displaystyle\sum_{0\leqslant\ell\leqslant t-1} A_\ell(0)(a(K,n/q^{2\ell})-a(R,n/q^{2\ell})) \\ \quad +\displaystyle\sum_{0\leqslant\ell\leqslant t-1}\frac{\nu_{2\ell+1}(0)}{\nu_{2s+1}(0)}(a(R,n/q^{2\ell})-a(qK,n/q^{2\ell})) & \text{if } e=2t+1,\\ B_t(\varepsilon)a(K,n/q^{2t}) \\ \quad +\displaystyle\sum_{0\leqslant\ell\leqslant t-1} A_\ell(0)(a(K,n/q^{2\ell})-a(R,n/q^{2\ell})) \\ \quad +\displaystyle\sum_{0\leqslant\ell\leqslant t-1}\frac{\nu_{2\ell+1}(0)}{\nu_{2s+1}(0)}(a(R,n/q^{2\ell})-a(qK,n/q^{2\ell})) & \text{if } e=2t. \end{cases}$$

Note that $a(qK,n/q^{2\ell})=a(K,n/q^{2\ell+2})$; so $a(qK,n/q^{2t})=0$ when $q^{2t+1}\|n$. Now

$$\frac{1}{\delta}\sum_{K_0\subseteq K}\theta(K_0;\tau)=\frac{1}{\delta}\sum_{L'\in\mathrm{gen}\,L}\frac{\#\{\sigma\in O(V)\colon\{K\colon\sigma L'\}=\{K\colon L\}\}}{o(L')}\theta(L;\tau),$$

where $\{K\colon L\}$ denotes the invariant factors of L in K (see [4]), and $o(L')$ denotes the order of the orthogonal group of L'; thus

$$\sum_{\substack{M\in\mathrm{gen}\,K\\ K_0\subseteq M}}\frac{1}{o(M)}\theta(K_0;\tau)$$

$$=\sum_{L'\,\mathrm{gen}\,L}\ \sum_{M\in\mathrm{gen}\,K}\frac{\#\{\sigma\in O(V)\colon\{L'\colon\sigma q^{s+1}M\}=\{L\colon q^{s+1}K\}\}}{o(q^{s+1}M)}\frac{1}{o(L')}\theta(L';\tau).$$

Since the invariant factors $\{L\colon q^{s+1}K\}$ are all powers of q,

$$\delta' = \sum_{M\in \operatorname{gen} K} \frac{\#\{\sigma \in O(V)\colon \{L'\colon \sigma q^{s+1}M\} = \{L\colon q^{s+1}K\}\}}{o(q^{s+1}M)}$$

is determined by the structure of $L_{(q)}$; hence δ' is independent of the choice of $L' \in \operatorname{gen} L$. So

$$\frac{\delta' \operatorname{mass} L}{\delta \operatorname{mass} K}\theta(\operatorname{gen} L;\tau) = \frac{1}{\delta \operatorname{mass} K}\sum_{\substack{M\in \operatorname{gen} K\\ K_0\subseteq M}} \frac{1}{o(M)}\theta(K_0;\tau),$$

and for $n\neq 0$, the nth Fourier coefficient of $1/(\delta \operatorname{mass} K)\sum_{\substack{M\in \operatorname{gen} K\\ K_0\subseteq M}}(1/o(M))\theta(K_0;\tau)$ is

$$\begin{cases} A_t(\varepsilon)(r(\operatorname{gen} K, 2n/q^{2t}) - r(\operatorname{gen} R, 2n/q^{2t})) \\ \quad +\dfrac{\nu_{2t+1}(\varepsilon)}{\nu_{2s+1}(\varepsilon)} r(\operatorname{gen} R, 2n/q^{2t}) \\ \quad + \sum_{0\leqslant \ell\leqslant t-1} A_\ell(0)(r(\operatorname{gen} K, 2n/q^{2\ell}) - r(\operatorname{gen} R, 2n/q^{2\ell})) \\ \quad + \sum_{0\leqslant \ell\leqslant t-1} \dfrac{\nu_{2\ell+1}(0)}{\nu_{2s+1}(0)}(r(\operatorname{gen} R, 2n/q^{2\ell}) - r(\operatorname{gen} K, 2n/q^{2\ell+2})) & \text{if } e = 2t+1, \\ B_t(\varepsilon) r(\operatorname{gen} K, 2n/q^{2t}) \\ \quad + \sum_{0\leqslant \ell\leqslant t-1} A_\ell(0)(r(\operatorname{gen} K, 2n/q^{2\ell}) - r(\operatorname{gen} R, 2n/q^{2\ell})) \\ \quad + \sum_{0\leqslant \ell\leqslant t-1} \dfrac{\nu_{2\ell+1}(0)}{\nu_{2s+1}(0)}(r(\operatorname{gen} R, 2n/q^{2\ell}) - r(\operatorname{gen} K, 2n/q^{2\ell+2})) & \text{if } e = 2t. \end{cases}$$

Note that the zeroth coefficients of $\theta(\operatorname{gen} L;\tau)$ and $1/(\delta \operatorname{mass} K)\sum_{\substack{M\in \operatorname{gen} K\\ K_0\subseteq M}}(1/o(M))\times \theta(K_0;\tau)$ are both 1, so $\delta' \operatorname{mass} L/(\delta \operatorname{mass} K) = 1$. Note also that when $q^2|n'$, $\rho_{R,q}(n') = \rho_{K,q}(n'/q^2)$. Our induction hypothesis now gives us

$$r(\operatorname{gen} L, 2n) = \rho_{K,\infty}\prod_p \rho_{K,p}(n),$$

where

$$\rho_{L,q}(n)=\begin{cases} A_t(\varepsilon)(\rho_{K,q}(n/q^{2t})-\rho_{R,q}(n/q^{2t})) \\ \quad +\dfrac{\nu_{2t+1}(\varepsilon)}{\nu_{2s+1}(\varepsilon)}\rho_{R,q}(n/q^{2t}) \\ \quad +\sum\limits_{0\leqslant \ell\leqslant t-1} A_\ell(0)(\rho_{K,q}(n/q^{2\ell})-\rho_{K,q}(n/q^{2\ell+2})) & \text{if } e=2t+1, \\ B_t(\varepsilon)\rho_{K,q}(n/q^{2t}) \\ \quad +\sum\limits_{0\leqslant \ell\leqslant t-1} A_\ell(0)(\rho_{K,q}(n/q^{2\ell})-\rho_{K,q}(n/q^{2\ell+2})) & \text{if } e=2t. \end{cases}$$

For $q^{2\ell+2}|n$,

$$\rho_{K,q}(n/q^{2\ell})-\rho_{K,q}(n/q^{2\ell+2})=\begin{cases} (\chi(q)q^{m/2-1})^{e-1-2\ell}(1+\chi(q)q^{m/2-1})f(q) \\ \qquad\qquad\qquad\qquad \text{if } q\nmid dK, \\ \chi_q(2n/q^e)\chi_{N/q}(q^e)q^{(m/2-1)(e-2-2\ell)}(q^{m-2}1)f(q) \\ \qquad\qquad\qquad\qquad \text{if } q|dK. \end{cases}$$

Also if $q\nmid dK$, then $R=qK$; so for $e=2t$ or $2t+1$,

$$\rho_{K,q}(n/q^{2t})-\rho_{R,q}(n/q^{2t})=\rho_{K,q}(n/q^{2t})=\begin{cases} f(q) & \text{if } e=2t, \\ (1+\chi(q)q^{m/2-1})f(q) & \text{if } e=2t+1. \end{cases}$$

Next, for $e=2t+1$ and $q|dK$,

$$\alpha\rho_{K,q}(n/q^{2t})+(1-\alpha)\rho_{R,q}(n/q^{2t})=\begin{cases} (\mu_{2s}q^{1-m/2}+q^{d+1-m/2}\varepsilon(\mu_{2s}\mu_{2s+1})^e)f(q) \\ \qquad\qquad\qquad\qquad \text{if } q\|dK, \\ \dfrac{q^{-m/2}(q^{d+1}+1)(q^{m/2-1}-1)}{q^{m/2-2}-1}f(q) \\ \qquad\qquad\qquad\qquad \text{if } q^2|dK. \end{cases}$$

Note that when $q\nmid dK$, we must have r_{2s} even, $\mu_{2s}=-1$, $\mu_{2s+1}=1$, and $\chi_q=1$; hence $\chi(q)=\mu_{2s}\mu_{2s+1}=-1$. When $q\|dK$, we have $\chi_q(2n/q^e)=((2n/q^e)/q)$ and

$$\chi_{N/q}(q)=\left(\frac{q}{N_0/q}\right)=\left(\frac{(-1)^{m/2-1}N_0/q}{q}\right)=\left(\frac{(-1)^{m/2-1}\,dK/q}{q}\right)=\mu_{2s}\mu_{2s+1},$$

where $N = N_0 N_1^2$ with N_0 square-free (recall that $(-1)^{m/2}N \equiv 1 \pmod 4$). Similarly, when $q^2|dK$, $\chi_q = 1$ and

$$\chi_{N/q}(q) = \left(\frac{q}{N_0}\right) = \left(\frac{(-1)^{m/2}N_0}{q}\right) = \left(\frac{(-1)^{m/2}\, dK/q^2}{q}\right) = \mu_{2s}\mu_{2s+1} = 1.$$

Then straightforward computations show that $\rho_{L,q}(n)$ is as claimed.

Case 2. Now, suppose r_{2s} is even and $\mu_{2s} = 1$; so K^q is an integral lattice on V^q. Set

$$A_\ell(\omega) = \alpha\frac{v_{2\ell}(\omega)}{v_{2s}(\omega)} + \beta\frac{v_{2\ell-1}(0)}{v_{2s+1}(0)} + (1-\alpha-\beta)\frac{v_{2\ell-2}(0)}{v_{2s}(0)},$$

and

$$B_\ell(\varepsilon) = \gamma(\varepsilon)\frac{v_{2\ell-1}(\varepsilon)}{v_{2s+1}(\varepsilon)} + (1-\gamma(\varepsilon))\frac{v_{2\ell-2}(0)}{v_{2s}(0)}.$$

As in the preceding case, for $\ell \geqslant 0$, Lemmas 3.2 and 3.6 give us the following.

(a) For $x \in K - qK$, $q|qQ(x)$, the proportion of K_0 in K containing $q^\ell x$ ($\ell \geqslant 0$) is $A_\ell(\omega)$, where $\omega = (Q(x)/q)$.

(b) For $x \in K - qK$, $q \nmid qQ(x)$, the proportion of K_0 in K containing $q^\ell x$ ($\ell \geqslant 1$) is $B_\ell(\varepsilon)$, where $\varepsilon = (qQ(x)/q)$. Thus

$$\frac{1}{\delta}\sum_{K_0 \subseteq K}\theta(K_0^q;\tau) = \theta(q^{s+1}K^q;\tau) + \sum_{\substack{x\in K^q/qK^q\\ q|qQ(x)}}\sum_{0\leqslant\ell\leqslant s} A_\ell\left(\left(\frac{Q(x)}{q}\right)\right)e\{qQ(q^\ell x)\tau\}$$

$$+ \sum_{\substack{x\in K^q\\ q\nmid qQ(x)}}\sum_{1\leqslant\ell\leqslant s} B_\ell\left(\left(\frac{qQ(x)}{q}\right)\right)e\{qQ(q^\ell x)\tau\},$$

where δ is the number of K_0 in K. We have $L \in \operatorname{gen} K_0$, so an argument similar to that when r_{2s} is odd or $\mu_{2s} = -1$ gives us

$$r(\operatorname{gen} L_0^q, n) = \begin{cases} A_t(\varepsilon)r(\operatorname{gen} K^q, 2n/q^{2t}) \\ \quad + \sum\limits_{0\leqslant\ell<t} A_\ell(0)\left(r(\operatorname{gen} K^q, 2n/q^{2\ell}) - r(\operatorname{gen} K^q, 2n/q^{2\ell+2})\right) \\ \qquad\qquad \text{if } q^{2t+1}\|n, \\ B_t(\varepsilon)r(\operatorname{gen} K^q, 2n/q^{2t}) \\ \quad + \sum\limits_{0\leqslant\ell<t} A_\ell(0)\left(r(\operatorname{gen} K^q, 2n/q^{2\ell}) - r(\operatorname{gen} K^q, 2n/q^{2\ell+2})\right) \\ \qquad\qquad \text{if } q^{2t}\|n, t\geqslant 1, 0 \quad \text{otherwise}, \end{cases}$$

where $e = \mathrm{ord}_q(n)$ and $\varepsilon = ((n/q^e)/q)$. By hypothesis,

$$r(\text{gen } K^q, 2n) = \chi_N(q)\rho_{K^q,q}(n)\rho_{L,\infty}\prod_{p\neq q}\rho_{L,p}(n/q).$$

So, as in Case 1,

$$r(\text{gen } L^q, 2qn) = r(\text{gen } K_0^q, 2qn) = \rho_{L,\infty}\cdot\prod_p \rho_{L^q,p}(qn),$$

where

$$\rho_{L^q,q}(qn) = \begin{cases} A_t(\varepsilon)\rho_{K^q,q}(qn/q^{2t}) \\ \qquad + \sum\limits_{0\leqslant \ell<t} A_\ell(0)\left(\rho_{K^q,q}(qn/q^{2\ell}) - \rho_{K^q,q}(qn/q^{2\ell+2})\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } e = 2t, \\ B_t(\varepsilon)\rho_{K^q,q}(qn/q^{2t}) \\ \qquad + \sum\limits_{0\leqslant \ell<t} A_\ell(0)\left(\rho_{K^q,q}(qn/q^{2\ell}) - \rho_{K^q,q}(qn/q^{2\ell+2})\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } e = 2t-1, t\geqslant 1, \\ 0 \qquad \text{otherwise}. \end{cases}$$

We know that

$$\rho_{K^q,q}(n) = \frac{1-(\chi(q)q^{m/2-1})^{e+1}}{1-\chi(q)q^{m/2-1}}(1-\chi(q)q^{-m/2})$$

and

$$\begin{aligned}&\rho_{K^q,q}(n/q^{2\ell}) - \rho_{K^q,q}(n/q^{2\ell+2}) \\ &\qquad = \chi(q)^e q^{(m/2-1)(e-1-2\ell)}(q^{m/2-1}+\chi(q))(1-\chi(q)q^{-m/2}),\end{aligned}$$

where $\chi(q) = \mu_{2s}\mu_{2s+1} = \mu_{2s+1}$. Also, one easily verifies that $r(\text{gen } L^q, 2qn) = r(\text{gen } L, 2n)$. Thus

$$\begin{aligned} r(\text{gen } L, 2n) &= r(\text{gen } L^q, 2qn) = \rho_{L^q,q}(qn)\cdot\rho_{L,\infty}\prod_{p\neq q}\rho_{L,p}(qn) \\ &= \chi(q)\rho_{L^q,q}(qn)\cdot\rho_{L,\infty}\prod_{p\neq q}\rho_{L,p}(n) = \rho_{L,\infty}\prod_p\rho_{L,p}(n),\end{aligned}$$

where $\rho_{L,q}(n)$ is as claimed in the theorem. □

References

[1] E. Artin, *Geometric Algebra*, Wiley-Intersci. Publ., Wiley & Sons, New York, 1988.
[2] B. Jones, *The Arithmetic Theory of Quadratic Forms*, Carcus Monograph Ser. **10**, Math. Assoc. Amer., Buffalo, N.Y., 1950.
[3] A. Ogg, *Modular Forms and Dirichlet Series*, W. A. Benjamin, New York, 1969.
[4] O. T. O'Meara, *Introduction to Quadratic Forms*, Grundlehren Math. Wiss. **117**, Springer-Verlag, Berlin, 1971.
[5] C. L. Siegel, "Über die analytische Theorie der quadratischen Formen" in *Gesammelte Abhandlungen*, Springer-Verlag, Berlin, 1966, 326–405.
[6] L. H. Walling, *Hecke operators on theta series attached to lattices of arbitrary rank*, Acta Arith. **54** (1990), 213-240.
[7] ———, *Hecke eigenforms and representation numbers of quadratic forms*, Pacific J. Math. **151** (1991), 179–200.

Department of Mathematics, University of Colorado, Boulder, Colorado 80309-0395, USA; walling@boulder.colorado.edu