FEEDBACK: ASSESSED HW3

Overall, people did well on #1 and #6; there were various issues on #3 (as described below).

1. The only thing a bit perplexing on a few of these solutions is that people worked too hard. By showing $(i) \implies (ii)$, $(ii) \implies (iii)$, $(iii) \implies (iv)$, $(iv) \implies (i)$, one gets the equivalence of the 4 statements.
3. Major issues:
   - One needs to show that $\frac{t^7-1}{t-1}$ is irreducible over $\mathbb{Q}$ (note that t^7-1 is reducible over $\mathbb{Q}$ since $1^7-1=0$). Showing that a (monic) polynomial over $\mathbb{Z}$ has no root modulo 2 only shows that the polynomial has no linear factor in $\mathbb{Z}[t]$, not that it is irreducible in $\mathbb{Z}[t]$. The substitution $t \mapsto t+1$ allows one to show that $\frac{t^7-1}{t-1}$ is irreducible over $\mathbb{Q}$, using a result from Algebra 2.
   - We have seen that with $K \subseteq \overline{K}$, $f \in K[t] \setminus K$, and $\alpha_1, \ldots, \alpha_n \in \overline{K}$ all the roots of f, we have that $K(\alpha_1, \ldots, \alpha_n) : K$ is a splitting field extension for f (Proposition 5.1), so this does not need to be proved in this exercise.
   - We see that $\mathbb{Q}(\sqrt{2}, \sqrt{-3}) : \mathbb{Q}$ is a splitting field extension for t^4+t^2-6. We have that t^2-2 is irreducible over $\mathbb{Z}$ by Eisenstein's Criterion, and thus over $\mathbb{Q}$ by Gauss' Lemma. So $[Q(\sqrt{2}) : \mathbb{Q}] = 2$. Also, as $\sqrt{-3}$ is a root of t^2+3, we have$m_{\sqrt{-3}}(\mathbb{Q}(\sqrt{2}))|t^2+3$. Hence $\deg m_{\sqrt{-3}}(\mathbb{Q}(\sqrt{2})) = 1$ or 2; this degree is 1 exactly when $\sqrt{-3} \in \mathbb{Q}(\sqrt{2})$, but this is not the case since $\sqrt{-3} \notin \mathbb{R}$ and $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{R}$. So $[\mathbb{Q}(\sqrt{2}, \sqrt{-3}) : \mathbb{Q}(\sqrt{2})] = 2$.
   - $\mathbb{Q}(\sqrt{2}, \sqrt{-3}) \neq \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{3})$.
6. (a) To show $\tau(L) = L$, one simply needs to note that for $g \in \mathcal{K}[t] \setminus K$ and $\alpha \in L$ with $g(\alpha) = 0$, we have $0\tau(g(\alpha)) = g(\tau(\alpha)$ since τ is a K-homomorphism. Then to show that with $\alpha \in L$ we have $\tau(\alpha) \in L$, one notes that τ must permute the roots of $m_\alpha(K)$, and since all these roots lie in L, so $\alpha = \tau(\beta)$ for some $\beta \in L$ with β a root of $m_\alpha(K)$. (Alternatively, one can show $\tau(L) \subseteq L$ and then apply Theorem 3.4, somehow noting that we have proved this result.)

   (b) To show $L : M$ is normal, it suffices to consider **irreducible** $g \in M[t]$ so that g has a root $\alpha \in L$. Then since $m_\alpha(M)|m_\alpha(K)$ and $m_\alpha(K)$ must split over L (since $L : K$ is normal), we have that $m_\alpha(M)$ must split over L (since $L[t]$ is a UFD).