FEEDBACK: ASSESSED HW4

Mostly, the work handed in was quite good. There was a nice variety of ways to prove $x^p - t$ is irreducible over J in #5. However, there were a few common oversights here and there, as detailed below.

1. We have $K, E, F \subseteq L$. Assume that $L \subseteq \overline{L}$ where $\overline{L}$ is an algebraic closure of L. [Note that we have not assumed that $L : K$ is algebraic, so we may not have $\overline{K} = \overline{L}$.]
   (i) Since $E : K$ and $F : K$ are finite extensions, this means that there are $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n \in L$ so that $E = K(\alpha_1, \dots, \alpha_m)$ and $F = K(\beta_1, \dots, \beta_n)$ **and** $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n$ **are ALGEBRAIC over** K.
   (ii)
   - An extension $M : K$ is normal if every **irreducible** $f \in K[t]$ either has no root on M or f splits over M.
   - To show $E \cap F : K$ is normal, begin by choosing **irreducible** $f \in (E \cap F)[t]$ so that f has a root in $E \cap F$. (There is no reason to agonise over cases of whether f splits over E or over F. Also, if f does not have a root in $E \cap F$, then there is nothing to show.)
   - Say $f \in (E \cap F)[t]$ is irreducible with a root in $E \cap F$. Thus f splits over E and f splits over F. Why are these splittings the same? That is, why are the roots of f that lie in E the same as the roots of f that lie in F? (Answer: $E, F \subseteq \overline{L}$ and $\overline{L}[t]$ is a UFD.)

   (iii) Here some people worked too hard. In the notes, it was proved in Theorem 6.7 that $EF : K$ is a normal extension when $E : K$ and $F : K$ are finite, normal extensions. Then Proposition 6.3 implies that $EK : E \cap F$ is normal; but as the notes say this is left as an exercise, we can prove it. As $EF : K$ is a finite, normal extension, it is a splitting field extension for some polynomial $f \in K[t]$. Thus $EF : E \cap F$ is also a splitting field extension for f, and hence $EF : E \cap F$ is a normal extension.

5. Here people neglected to use Theorem 8.7, which makes the problem much easier to solve. To use Theorem 8.7 in (a), one only need show that t is not a pth power in $J = \mathbb{F}_p(t, s)$. (Recall that $\mathbb{F}_p(t, s)$ is the field of fractions for $\mathbb{F}_p[t, s]$; here t is transcendental over $\mathbb{F}_p$, and s is transcendental over $\mathbb{F}_p(t)$.) For the sake of contradiction, suppose that $t = \alpha^p$ for some $\alpha \in \mathbb{F}_p(t, s)$. Thus $\alpha = \beta/\gamma$ for some $\beta, \gamma \in \mathbb{F}_p[t, s]$. If $\beta \notin \mathbb{F}_p[t]$ or $\gamma \notin \mathbb{F}_p[t]$, then $t\gamma^p - \beta^p = 0$ so s satisfies a nontrivial algebraic relation over $\mathbb{F}_p(t)$, contradicting that s is transcendental over $\mathbb{F}_p(t)$. So suppose that $\beta, \gamma \in \mathbb{F}_p[t]$; then $1 = \deg_t(\beta^p) - \deg_t(\gamma^p) = p(\deg_t \beta - \deg_t \gamma)$, which is impossible. [Note that for $\beta, \gamma \in \mathbb{F}_p[t]$,

$\deg_t \beta, \deg_t \gamma$ are well-defined as t is transcendental over $\mathbb{F}_p$.] Hence $x^p - t = m_t(J)$ and so $[E : J] = [J(\xi) : J] = p$.

To use Theorem 8.7 in (b), one needs to argue that s is not a pth power in $E = \mathbb{F}_p(\xi, s)$. For the sake of contradiction, suppose that $s = (\beta/\gamma)^p$ where $\beta, \gamma \in \mathbb{F}_p[\xi, s]$. Since $\xi^p = t$, we have $\beta^p, \gamma^p \in \mathbb{F}_p[t, s]$, and as polynomials in s, β^p, γ^p have degrees divisible by p; hence $s \neq (\beta/\gamma)^p$.

Alternatively, in (a) one could show that $x^p - t$ is irreducible over $\mathbb{F}_p[t, s]$ using Eisenstein's Criterion (and then by Gauss' Lemma, one has that $x^p - t$ is irreducible over J). To show t is irreducible in $\mathbb{F}_p[t, s]$, suppose that $t = gh$ for some (nonzero) $g, h \in \mathbb{F}_p[t, s]$. [Note: since t and s are transcendental over $\mathbb{F}_p$, g is a unit in $\mathbb{F}_p[t, s]$ if and only if $g \in \mathbb{F}_p^\times$.] So $0 = \deg_s t = \deg_s g + \deg_s h$. [Since s is transcendental over $\mathbb{F}_p(t)$, $\deg_s f$ makes sense for any $f \in \mathbb{F}_p[t, s]$.] Thus $g, h \in \mathbb{F}_p[t]$, and $1 = \deg_t g + \deg_t h$. [Since t is transcendental over $\mathbb{F}_p$, $\deg_t f$ makes sense for any $f \in \mathbb{F}_p[t]$.] Hence either g or h is a unit in $\mathbb{F}_p[t]$, meaning that g or h lies in $\mathbb{F}_p^\times$. Thus t is irreducible in $\mathbb{F}_p[t, s]$ and so by Eisenstein's Criterion, $x^p - t$ is irreducible over $\mathbb{F}_p[t, s]$.

To use Eisenstein's Criterion to show that s is irreducible in $\mathbb{F}_p[t, s]$, one needs to be a little careful; to argue as in (a), we need to first show that ξ is transcendental over $\mathbb{F}_p(s)$. For the sake of contradiction, suppose that ξ is the root of some $g \in \mathbb{F}_p(s)[x]$. If $g \in \mathbb{F}_p[x]$ then ξ is algebraic over $\mathbb{F}_p$ and hence $t = \xi^p$ is algebraic over $\mathbb{F}_p$, a contradiction. So suppose $g \notin \mathbb{F}_p[x]$. Write $g = (a_0/b_0) + (a_1/b_1)x + \cdots + (a_n/b_n)x^n$ where $a_i, b_i \in \mathbb{F}_p[s]$, $b_i \neq 0$ $(0 \leq i \leq n)$; also, $g \notin \mathbb{F}_p(s)$ so $a_n \neq 0$ with $n > 0$. Set $h = (b_0 \cdots b_n)g$; so h is a nonzero element of $\mathbb{F}_p[s, x]$ with $h(\xi) = 0$. Also,

$$0 = (h(\xi))^p = h^p(t) \in \mathbb{F}_p[t, s].$$

If $\deg_s h^p(t) > 0$ then s is algebraic over $\mathbb{F}_p(t)$, a contradiction. If $\deg_s h^p(t) = 0$ then t is algebraic over $\mathbb{F}_p$, a contradiction. Thus ξ must be transcendental over $\mathbb{F}_p(s)$, and hence arguing as in (a) [with ξ playing the role of s and s playing the role of t], we see that s is irreducible in $\mathbb{F}_p[\xi, s]$.

6. A few people had difficulty with an expansion of the sort $(a + b + c)^p$ working over a field of characteristic p. We have seen that

$$(a + b + c)^p = a^p + (b + c)^p,$$

and then $(b + c)^p = b^p + c^p$. More generally, by induction one has $(a_1 + \cdots + a_n)^p = a_1^p + \cdots + a_n^p$.

Another slight issue here was that some people didn't go a bit further in (b) to get $[J(\gamma) : J] = 1$ or p.