Analysis Workshop
Fibonacci Numbers and the Golden Ratio

Story. The Fibonacci numbers (defined below) form a famous sequence of numbers that capture patterns that occur abundantly in nature. These numbers are named after Leonardo of Pisa (circa 1180-1250), who was also known as Fibonacci. Raised primarily in North Africa and educated by the Moors, later Fibonacci traveled in the Orient as a merchant. Upon returning to Italy, Fibonacci used the knowledge gained through these experiences to popularise the decimal number system that we use today.

Perhaps less well-known than the Fibonacci sequence is the golden ratio (also called the golden number, the golden section, the golden mean). This number is purported to capture a ratio yielding sights and sounds that are very pleasing to the eye and to the ear, and it appears frequently in art, architecture, and music. Luca Pacioli (1445-1517), a Franciscan friar, mathematican and art lover, called the golden ratio the "divine proportion".

In this two-part workshop, you explored a curious relationship between the Fibonacci numbers and the golden ratio.

The Fibonacci numbers F_n are defined as follows.

$$F_0 = 1,\ F_1 = 1,\ F_n = F_{n-2} + F_{n-1} \text{ for all integers } n \geq 2.$$

The golden ratio is the real number $\frac{1}{x}$ so that

$$\frac{1}{x} = \frac{x}{1-x}.$$

Denoting the golden ratio by φ, an easy exercise in algebra shows that

$$\varphi = \frac{\sqrt{5}+1}{2}.$$

Here you first related the Fibonacci numbers to the numbers a_n, defined as follows.

$$a_1 = 1,\ a_n = 1 + \frac{1}{a_{n-1}} \text{ for all integers } n \geq 2.$$

In Part II, you used this to relate the Fibonacci numbers to the golden ratio, finding that

$$\varphi = \sup\left\{ \frac{F_n}{F_{n-1}} :\ n \in \mathbb{Z},\ n \text{ odd} \right\}$$

and

$$\varphi = \inf\left\{ \frac{F_n}{F_{n-1}} :\ n \in \mathbb{Z},\ n \text{ even} \right\}.$$

For further exploration: Upon returning from traveling the Orient, Fibonacci wrote "Liber Abaci" (1202), containing arithmetical and algebraic information he had collected, and then wrote "Practica Geometriae" (1220), containing geometric and trigonometric information. It is thought Fibonacci may also have included in these books some original work, such as a proof that the roots of $x^3+2x^2+10x = 20$ cannot be constructed by ruler and compass. (This information is taken from "A Concise History of Mathematics" by Dirk Struik.)

There are many articles in the literature about the Fibonacci sequence arising in nature, and about the golden ratio arising in art and music.