Hecke eigenvalues and relations for degree 2 Siegel Eisenstein series

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1. Introduction

Modular forms are of central interest in number theory, particularly because Hecke theory tells us that their Fourier coefficients carry number theoretic information. Eisenstein series are fundamental examples of modular forms and play an important role in the theory of modular forms. In the case of elliptic modular forms (i.e. Siegel degree 1), the Eisenstein series are well-understood; for instance, we have explicit formulas for their Fourier coefficients, we know that the space of Eisenstein series can be simultaneously diagonalised with respect to the Hecke operators attached to primes not dividing the level, and when the level is square-free, the space can be simultaneously diagonalised with respect to the full Hecke algebra. In the case of Siegel degree $n > 1$, the situation is less well understood, but some parallel results have been established. Freitag [3] has shown that the space of Eisenstein series can be simultaneously diagonalised with respect to the Hecke operators attached to primes not dividing the level. Many authors have worked on computing Fourier coefficients of Siegel Eisenstein series with $n > 1$. We do not try to give a comprehensive list of all the work that has contributed...
to this, but rather give a sampling. For level 1, see [8,9] for degree 2; [5] for degree 3; [1,2,6,7] for arbitrary degree. For degree 2, level \( N \) and primitive character modulo \( N \). Fourier coefficients for 1 of the Eisenstein series \( \mathbb{E}(N,1) \) have been computed in [10] when \( N \) is odd and square-free, and in [11] for arbitrary \( N \).

In this work, without using any knowledge of Fourier coefficients, we evaluate the action of Hecke operators on the natural basis \( \{ \mathbb{E}_\rho \} \) for the space of degree 2 Siegel Eisenstein series of square-free level \( N \) and arbitrary character \( \chi \) (Propositions 3.3–3.10); here \( \rho = (N_0, N_1, N_2) \) varies so that \( N_0N_1N_2 = N \) and \( \chi_{N_1}^2 = 1 \). This evaluation reveals Hecke relations among these Eisenstein series in the case that \( \chi^2 \) is not primitive. Using these relations, we construct another basis \( \{ \tilde{\mathbb{E}}_\rho \} \) consisting of eigenforms for the full Hecke algebra (which is generated by \( \{ T(p), T_1(p^2): p \text{ prime} \} \); when \( \chi^2 \) is primitive, \( \tilde{\mathbb{E}}_\rho = \mathbb{E}_\rho \). Then for any prime \( p \), the eigenvalues of \( \tilde{\mathbb{E}}_\rho \) for \( T(p) \) and \( T_1(p^2) \) are

\[
\left( \chi_{N_0}(p)p^{k-1} + \chi_{N_1}N_2(p) \right) \left( \chi_{N_0}N_1(p)p^{k-2} + \chi_{N_2}(p) \right)
\]

and

\[
\left( p + \chi_{N_2}(p^2) \right) \left( \chi_{N_0}(p^2)p^{2k-3} + \chi(p)p^{k-3}(p-1) + \chi_{N_2}(p^2) \right)
\]

(Proposition 3.11). In the case that \( \chi^2 \) is not primitive, these Hecke relations also allow us to generate some of the other Eisenstein series from \( \mathbb{E}(N,1) \) by applying particular elements of the Hecke algebra; in particular, when \( \chi = 1 \), we can generate a basis from \( \mathbb{E}(N,1) \) (Theorem 3.12). In the remark following this theorem, we briefly discuss how we can use [4] and the Fourier coefficients of the degree 2, level 1 Eisenstein series \( E \) to generate the Fourier coefficients of all the degree 2, level \( N \) Eisenstein series in the case that \( N \) is square-free and the character \( \chi = 1 \).

2. Preliminaries

Here we set notation and define degree 2 Siegel Eisenstein series and Hecke operators. We begin by fixing square-free \( N \in \mathbb{Z}_+ \). With \( Sp_2(\mathbb{Z}) \) the group of \( 4 \times 4 \) integral symplectic matrices, we set

\[
\Gamma_\infty = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in Sp_2(\mathbb{Z}) \right\},
\]

\[
\Gamma_0(N) = \left\{ \gamma \in Sp_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (N) \right\}.
\]

The 0-dimensional cusps for \( \Gamma_0(N) \) correspond to the elements of the double coset \( \Gamma_\infty \setminus Sp_2(\mathbb{Z}) / \Gamma_0(N) \). For \( k \in \mathbb{Z}_+ \) and \( \chi \) a Dirichlet character modulo \( N \), we have one Siegel Eisenstein character series for each cusp, defined as follows. For \( \gamma_0 \in Sp_2(\mathbb{Z}) \), the Eisenstein series associated to the cusp \( \Gamma_\infty \gamma_0 \Gamma_0(N) \) is

\[
\mathbb{E}_{\gamma_0}(\tau) = \sum_{\chi} \chi(\det D_{\gamma_0}^{-1} \chi) 1|\gamma(\tau)
\]

where \( \Gamma_\infty \gamma \) varies over the \( \Gamma_0(N) \)-orbit of \( \Gamma_\infty \gamma_0 \)

\[
\tau \in \mathcal{H}(\mathbb{Z}) = \left\{ X + iY : X, Y \in \mathbb{R}^{2,2}_{\text{sym}}, Y > 0 \right\}
\]

where \( Y > 0 \) denotes that \( Y \) is the matrix for a positive definite quadratic form, and

\[
1\left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right| \tau = \det(C \tau + D)^{-k}.
\]
This sum is well-defined provided $\chi_q^2 = 1$ whenever $q$ is a prime dividing $\mathcal{N}$ and rank$_q M_0 = 1$ where $\gamma_0 = \left( \begin{smallmatrix} a & b \\ M_0 & N_0 \end{smallmatrix} \right)$ and rank$_q M_0$ denotes the rank of $M_0$ modulo $q$. When well-defined, the sum is non-zero provided $\chi(-1) = (-1)^k$, and it is absolutely uniformly convergent on compact regions provided $k \geq 4$ (and hence it is analytic, meaning analytic in each variable of $\tau$). For $\gamma' \in \Gamma_0(\mathcal{N})$, $\Gamma_\infty \gamma' \gamma$ varies over the $\Gamma_0(\mathcal{N})$-orbit of $\Gamma_\infty \gamma$ does, and hence $E_{\gamma_0} | \gamma' = \chi(\det D_{\gamma'}) E_{\gamma_0}$. As noted in [3], these Eisenstein series are linearly independent, and the 0th Fourier coefficient of $E_{\gamma_0}$ is 0 unless $\gamma_0 \in \Gamma_0(\mathcal{N})$, in which case it is 1.

A pair of $2 \times 2$ matrices $(M \ N)$ is called symmetric if $M^t N = N^t M$ with $^t N$ denoting the transpose of $N$; it is called a coprime pair if $M$, $N$ are integral and $(GM \ GN)$ is integral only if $G$ is. Note that $(M \ N)$ is a coprime pair if and only if, for each prime $p$, rank$_p (M \ N) = 2$. It is well known that for $\gamma', \gamma'' \in \text{Sp}_2(\mathbb{Z})$, $\gamma$ and $\gamma''$ lie in the same coset in $\Gamma_\infty \setminus \text{Sp}_2(\mathbb{Z})$ if and only if $\gamma = \left( \begin{smallmatrix} a & b \\ M & N \end{smallmatrix} \right)^p$, $\gamma'' = \left( \begin{smallmatrix} a & b \\ GM & GN \end{smallmatrix} \right)$ for some $G \in GL_2(\mathbb{Z})$. Thus these cosets can be parameterised by $GL_2(\mathbb{Z})$-equivalence classes of coprime symmetric pairs; so $E_{\gamma_0}$ is supported on a set of $GL_2(\mathbb{Z})$-equivalence class representatives for the $\Gamma_0(\mathcal{N})$-orbit of $GL_2(\mathbb{Z})(M_0 \ N_0)$.

For each prime $p$, we have Hecke operators $T(p)$ and $T_1(p^2)$ that act on degree 2 Siegel modular forms, and $\{T(p), T_1(p^2)\}$ generates the Hecke algebra. For $f$ a degree 2 Siegel modular form of weight $k$, level $\mathcal{N}$, and character $\chi$, and for $\gamma' = \left( \begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$, we set

$$\gamma' \circ \tau = (A \tau + B)(C \tau + D)^{-1},$$

and

$$f(\tau)|\gamma' = (\det \gamma')^{k/2} \det(C \tau + D)^{-k} f(\gamma' \circ \tau).$$

Then

$$f|T(p) = p^{k-3} \sum_{\gamma} \chi(\det D_{\gamma}) f|\delta^{-1} \gamma$$

where $\delta = \left( \begin{smallmatrix} p^{1/2} & 0 \\ 0 & 1 \end{smallmatrix} \right)$ and $\gamma$ varies over a set of coset representatives for

$$(\delta \Gamma_0(\mathcal{N}) \delta^{-1} \cap \Gamma_0(\mathcal{N})) \setminus \Gamma_0(\mathcal{N}).$$

Somewhat similarly,

$$f|T_1(p^2) = p^{k-3} \sum_{\gamma} \chi(\det D_{\gamma}) f|\delta_1^{-1} \gamma$$

where $\delta_1 = \left( \begin{smallmatrix} X & 0 \\ 0 & X^{-1} \end{smallmatrix} \right)$, $X = \left( \begin{smallmatrix} p \end{smallmatrix} \right)$, and $\gamma$ varies over a set of coset representatives for

$$(\delta_1 \Gamma_0(\mathcal{N}) \delta_1^{-1} \cap \Gamma_0(\mathcal{N})) \setminus \Gamma_0(\mathcal{N}).$$

In Propositions 2.1 and 3.1 of [4] we computed an explicit set of upper triangular block matrices giving the action of the Hecke operators, and we will use these here in evaluating the action of Hecke operators on Eisenstein series. (Note that in [4] we did not introduce the normalisation of $T_1(p^2)$ until we averaged the Hecke operators to produce an alternative basis for the Hecke algebra.)

Given $Q \in \mathbb{Z}_{\text{sym}}^2$, and $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$, $p$ prime, we can think of $Q$ as a quadratic form on $V = \mathbb{F}^2 \oplus \mathbb{F}^2$. We say a non-zero vector $v \in V$ is isotropic if $Q(v) = 0$ (in $\mathbb{F}$). Suppose $p$ is odd. Then $Q$ is a $GL_2(\mathbb{F})$ conjugate of $\mathbb{H} = (1, -1)$ or of $\mathbb{A} = (1, -\omega)$ where $(\frac{\omega}{p}) = -1$ and $(\ast, \ast)$ denotes a diagonal matrix; we write $V \simeq \mathbb{H}$ or $V \simeq \mathbb{A}$ accordingly. Note that when $V \simeq \mathbb{H}$, $V$ contains 2 isotropic lines, and when
$V \simeq A$, $V$ contains no isotropic lines. Now suppose $p = 2$; then either $Q$ is a $GL_2(\mathbb{F})$ conjugate of $I$ or (over $\mathbb{F}$) of $Q = (0^1_{10})$ (which is stabilised under conjugation by $GL_2(\mathbb{F})$). When $V \simeq I$, $V$ contains 1 isotropic line; when $V \simeq (0^1_{10})$, all 3 lines in $V$ are isotropic.

3. Action of Hecke operators on Eisenstein series of square-free level

Throughout this section, we assume $k \in \mathbb{Z}_+$ with $k \geq 4$, and that $\mathcal{N}$ is square-free. We first show that we can parameterise the $I_0(\mathcal{N})$ cusps by multiplicative partitions $\rho = (\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2)$ where $\mathcal{N}_0 \mathcal{N}_1 \mathcal{N}_2 = \mathcal{N}$: For such $\rho$, fix diagonal $M_\rho \in \mathbb{Z}^{2,2}$ so that

$$M_\rho = \begin{cases} 0 & (\mathcal{N}_0), \\ (1^1_{00}) & (\mathcal{N}_1), \\ (1^0_{01}) & (\mathcal{N}_2). \end{cases}$$

Then set $\gamma_\rho = (M_\rho)^t$. For $\gamma = (K L_{M N})$, $\gamma' = (K' L'_{M' N'}) \in Sp_2(\mathbb{Z})$, we know that $I_\infty \gamma = I_\infty \gamma'$ if and only if $G(M N) = (M' N')$ for some $G \in GL_2(\mathbb{Z})$. Thus it suffices to show that for a coprime symmetric pair $(M N)$, we have $GL_2(\mathbb{Z})(M N)$ in the $I_0(\mathcal{N})$-orbit of $GL_2(\mathbb{Z})(M N I)$ if and only if $\text{rank}_q M = \text{rank}_q M_\rho$ for all primes $q | \mathcal{N}$.

To do this, suppose first that $(M N)$ is a coprime symmetric pair so that $(M N) \equiv (M_\rho I) (\mathcal{N})$. So there is some $(K L_{M N}) \in Sp_2(\mathbb{Z})$. Since $N \equiv 1(\mathcal{N})$, $L$ is symmetric modulo $\mathcal{N}$ so we can choose symmetric $W \equiv -L(\mathcal{N})$. Then

$$\left( \begin{array}{cc} K' & L' \\ M & N \end{array} \right) = \left( \begin{array}{cc} I & W \\ 0 & I \end{array} \right) \left( \begin{array}{cc} K & L \\ M & N \end{array} \right) \in Sp_2(\mathbb{Z})$$

with $L' \equiv 0 (\mathcal{N})$; so $K' \equiv I (\mathcal{N})$. Thus

$$\left( \begin{array}{cc} I & 0 \\ -M_\rho & I \end{array} \right) \left( \begin{array}{cc} K' & L' \\ M & N \end{array} \right) \in I_0(\mathcal{N}),$$

and hence $(M N) \equiv (M_\rho I) I_0(\mathcal{N})$.

Now suppose $(M N)$ is a coprime symmetric pair so that for each prime $q | \mathcal{N}$, we have $\text{rank}_q M = \text{rank}_q M_\rho$. We want to show that for some $E \in GL_2(\mathbb{Z})$ and $\gamma \in I_0(\mathcal{N})$ we have $E(M N) \gamma \equiv (M_\rho I) (\mathcal{N})$. We know $SL_2(\mathbb{Z})$ projects onto $SL_2(\mathbb{Z}/\mathcal{N}' \mathbb{Z})$. Also,

$$SL_2(\mathbb{Z}/\mathcal{N}' \mathbb{Z}) \simeq SL_2(\mathbb{Z}/\mathcal{N}_0 \mathbb{Z}) \times SL_2(\mathbb{Z}/\mathcal{N}_1 \mathbb{Z}) \times SL_2(\mathbb{Z}/\mathcal{N}_2 \mathbb{Z})$$

via the map $G(\mathcal{N}) \mapsto (G(\mathcal{N}_0), G(\mathcal{N}_1), G(\mathcal{N}_2))$, which is well-defined with inverse map

$$(G(\mathcal{N}_0), G(\mathcal{N}_1), G(\mathcal{N}_2)) \mapsto r \mathcal{N}_1 \mathcal{N}_2 G_0 + s \mathcal{N}_0 \mathcal{N}_1 G_1 + t \mathcal{N}_1 \mathcal{N}_1 G_2 \quad (\mathcal{N})$$

where $r, s, t \in \mathbb{Z}$ so that $r \mathcal{N}_1 \mathcal{N}_2 + s \mathcal{N}_0 \mathcal{N}_1 + t \mathcal{N}_0 \mathcal{N}_1 = 1$. Thus we can choose $E, G \in SL_2(\mathbb{Z})$ so that

$$EMG = \begin{cases} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) & (\mathcal{N}_1), \\ \left( \begin{array}{cc} 1 & 0 \\ m & 1 \end{array} \right) & (\mathcal{N}_2). \end{cases}$$
where \( m \) is a unit modulo \( \mathcal{N}_2 \), and \( EN^{-1}G^{-1} \equiv I \pmod{\mathcal{N}_0} \). Write \( EN^{-1}G^{-1} = (n_1 \ n_2) \); then by symmetry, \( n_3 \equiv mn_2 \pmod{\mathcal{N}_2} \), \( n_3 \equiv 0 \pmod{\mathcal{N}_1} \), and since \( M, N \) are coprime, \( n_4 \) is a unit modulo \( \mathcal{N}_1 \). Now choose symmetric \( W \) so that

\[
W \equiv \begin{cases} 
(1-n_1 & -n_2) \\
-n_1 & 0 \\
1-n_1 & -n_2 \\
-m^2 & mn_4 \end{cases} \pmod{\mathcal{N}_1}, \quad \begin{cases} 
1-n_1 & -n_2 \\
-n_2 & 0 \\
1-n_1 & -n_2 \\
m^2 & -mn_4 \end{cases} \pmod{\mathcal{N}_2}
\]

and \((a \ b \ c \ d) \in \text{SL}_2(\mathbb{Z})\) so that

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{cases} 
1 & (\mathcal{N}_0), \\
n_4 & n_3 \\
m & m \end{cases} \pmod{\mathcal{N}_1}, \quad \begin{cases} 
1 & (\mathcal{N}_2) \end{cases}
\]

Set

\[
\gamma = \begin{pmatrix} G & -I_G \\ I & W \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 \end{pmatrix}.
\]

Then \( \gamma \in \Gamma_0(\mathcal{N}) \) and \( E(MN)\gamma \equiv (M_{\rho} \ 1) \pmod{\mathcal{N}} \). Hence with the previous paragraph, we have that \( \text{GL}_2(\mathbb{Z})(MN) \) is in the \( \Gamma_0(\mathcal{N})\)-orbit of \( \text{GL}_2(\mathbb{Z})(M_{\rho} \ 1) \), as claimed. (So the \( \text{GL}_2(\mathbb{Z})\)-equivalence classes of two coprime symmetric pairs \((MN), (M' \ N')\) are in the same \( \Gamma_0(\mathcal{N})\)-orbit if and only if \( \text{rank}_q M = \text{rank}_q M' \) for all primes \( q \mid \mathcal{N} \).)

Note that

\[
\text{GL}_2(\mathbb{Z})(M_{\rho} \ 1) = \text{SL}_2(\mathbb{Z})(M_{\rho} \ 1) \cup \text{SL}_2(\mathbb{Z})(M_{\rho} \ 1) \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix},
\]

so we can identify the cusp with \( \text{SL}_2(\mathbb{Z})(M_{\rho} \ 1) \Gamma_0(\mathcal{N}) \). We use \( E_{\gamma} \) to denote \( E_{\gamma_{\rho}} \). To ease the discussions during our computations we consider \( 2E_{\rho} \) to be supported on a set of representatives for the \( \text{SL}_2(\mathbb{Z})\)-equivalence classes in the \( \Gamma_0(\mathcal{N})\)-orbit of \((M_{\rho} \ 1)\).

For \( q \) prime, we say \((MN)\) has \( q \)-type \( i \) if \((MN)\) is a coprime symmetric pair with \( \text{rank}_q M = i \). For \((MN)\) of \( q \)-type \( i \), choose \( E \in \text{GL}_2(\mathbb{Z}) \) so that \( q \) divides the lower \( 2-i \) rows of \( EM \); then we say \((MN)\) (or simply \( M \)) is \( q^2 \)-type \( i, j \) where \( j = \text{rank}_q \begin{pmatrix} -1 \\
^0 \\
^1 \end{pmatrix} \) \( EM \). Given square-free \( \mathcal{N} \) and a partition \( \rho = (\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2) \) of \( \mathcal{N} \), we say \((MN)\) has \( \mathcal{N} \)-type \( \rho \) if \((MN)\) is a coprime symmetric pair and, for each prime \( q \mid \mathcal{N}_i \), \( \text{rank}_q M = i \).

Given a character \( \chi \) modulo \( \mathcal{N}_i \), and \((MN) = (M_{\rho} \ 1)\gamma \) where \( \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(\mathcal{N}) \), we can describe \( \chi(\text{det } D) \) in terms of \( M, N, \rho \) as follows. For each prime \( q \mid \mathcal{N}_i \), we have \( N \equiv D \pmod{q} \), so \( \chi_q(\text{det } D) = \chi_q(\text{det } N) \). For each prime \( q \mid \mathcal{N}_2 \), we have \( M \equiv A \equiv ^T D \pmod{q} \), so \( \chi_q(\text{det } D) = \chi_q(\text{det } M) \). Now take a prime \( q \mid \mathcal{N}_1 \); write \( D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \). Thus modulo \( q \) we have

\[
A \equiv \text{det } D \begin{pmatrix} d_4 & -d_3 \\ -d_2 & d_1 \end{pmatrix},
\]
so consequently modulo \( q \) we have
\[
M \equiv \det D \begin{pmatrix} d_4 & -d_3 \\ 0 & 0 \end{pmatrix}, \quad N \equiv \begin{pmatrix} * & * \\ d_3 & d_4 \end{pmatrix}.
\]

We know \( q \nmid (d_3 \ d_4) \), so \( \chi_q(\det D) = \chi_q(m_1)\chi_q(n_4) \) or \( \chi_q(-m_2)\chi_q(n_3) \), whichever is non-zero. Take \( E \in \text{SL}_2(\mathbb{Z}) \) so that \( q \) divides row 2 of \( EM \); thus \( E \equiv \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mod (q) \), and with
\[
EM = \begin{pmatrix} m'_1 & m'_2 \\ q m'_3 & q m'_4 \end{pmatrix}, \quad EN = \begin{pmatrix} n'_1 & n'_2 \\ n'_3 & n'_4 \end{pmatrix},
\]
we have
\[
\chi_q(m'_1)\chi_q(n'_4) = \chi_q(m_1)\chi_q(n_4), \quad \chi_q(-m'_2)\chi_q(n'_3) = \chi_q(-m_2)\chi_q(n_3)
\]
provided \( \chi_q^2 = 1 \). So when \( \chi_q^2 = 1 \) and \((M \ N)\) has \( q \)-type 1, we can choose \( E \in \text{SL}_2(\mathbb{Z}) \) so that
\[
EM = \begin{pmatrix} m_1 & m_2 \\ q m_3 & q m_4 \end{pmatrix}, \quad EN = \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix}; \quad \text{set } \chi_{(1,q,1)}(M, N) = \chi_q(m_1)\chi_q(n_4) \text{ or } \chi_q(-m_2)\chi_q(n_3) \text{ (whichever is non-zero). Then set}
\[
\chi_\rho(M, N) = \prod_{q \mid N_0} \chi_q(\det N) \prod_{q \mid N_1} \chi_{(1,q,1)}(M, N) \prod_{q \mid N_2} \chi_q(\det M).
\]

Hence \( 2\mathbb{E}_\rho(\tau) = \sum \chi_\rho(M, N) \det(M\tau + N)^{-k} \) where \((M \ N)\) varies over a set of \( \text{SL}_2(\mathbb{Z}) \)-equivalence class representatives for pairs of \( N \)-type \( \rho \). Also note that for \( G \in \text{SL}_2(\mathbb{Z}) \), \( \chi_\rho(G M, G N) = \chi_\rho(M, N) \), and since \( (G \ g^{-1}) \in \text{Sp}_2(\mathbb{Z}) \), we have \( \chi_\rho(MG, N^1G^{-1}) = \chi_\rho(M, N) \).

For the remainder of this section, fix a partition \( \rho = (N_0, N_1, N_2) \) of \( N \), and fix a character \( \chi \) modulo \( N \). We decompose \( \chi \) as \( \chi_{N_0}\chi_{N_1}\chi_{N_2} \) where \( \chi_{N_i} \) has modulus \( N_i \); we assume \( \chi \) has been chosen so that \( \chi_{N_1}^2 = 1 \). For \( p \) prime, let \( \mathcal{G}_1(p) \) be a set of representatives for
\[
\left\{ \gamma \in \text{SL}_2(\mathbb{Z}): \gamma \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \mod (p) \right\} \setminus \text{SL}_2(\mathbb{Z});
\]

note that for \( p \nmid N \), we can take the elements in \( \mathcal{G}_1(p) \) to be congruent modulo \( N \) to \( I \).

When evaluating the action of the Hecke operators, we often use the following simple lemmas.

**Lemma 3.1.** Say \( p \) is a prime and \((M \ N)\) is \( p \)-type 1, \( M = \begin{pmatrix} m_1 & m_2 \\ p m_3 & p m_4 \end{pmatrix} \), and \( N = \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix} \). Then \( p \mid m_1 \) if and only if \( p \nmid n_4 \), and \( p \mid m_2 \) if and only if \( p \nmid n_3 \).

**Proof.** Say \( p \mid m_1 \). Then \( p \nmid m_2 \) since \( \text{rank}_p M = 1 \); by symmetry, \( p \mid m_3 n_4 \) and hence \( p \mid n_4 \). The other arguments needed to prove the lemma are essentially identical to this. \( \square \)

**Lemma 3.2.** Let \( V = F x_1 \oplus F x_2 \) where \( F = \mathbb{Z}/p\mathbb{Z} \), \( p \) prime. For \( G \in \mathcal{G}_1(p) \), let \( (x'_1 \ x'_2) = (x_1 \ x_2)^t G \). Then as \( G \) varies over \( \mathcal{G}_1(p) \), \( F x'_1 \) varies over all lines in \( V \).

**Proof.** Representatives for \( \mathcal{G}_1(p) \) are \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \) where \( \alpha \) varies modulo \( p \). Thus \( F x'_1 \) varies as claimed. \( \square \)

**Proposition 3.3.** For \( p \) a prime not dividing \( N \), we have
\[
\mathbb{E}_\rho | T(p) = \chi_{N_0}(p^2) \chi_{N_1}(p) p^{2k-3} + \chi_{N_0 N_2}(p) p^{k-2} (p+1) + \chi_{N_1}(p) \chi_{N_2}(p^2) \mathbb{E}_\rho.
\]
Proof. Decompose $E_\rho$ as $E_0 + E_1 + E_2$ where $E_i$ is supported on pairs of $N'$-type $\rho$ and $p$-type $i$.

Using the matrices for $T(p)$ as described in Proposition 3.1 of [4], we can describe $E_\rho|T(p)$ as follows. First, let $G_1 = G_1(p)$. Then

$$2E_\rho(\tau)|T(p) = p^{2k-3} \sum_{0 \leq r \leq 2} \chi(p^{2-r}) H(x(M,N) \det(MD_r G_r^{-1} + pND_r^{-1} G_r + MY_r G_r)^{-k}$$

where $(M,N)$ varies over a set of $SL_2(\mathbb{Z})$-equivalence class representatives for pairs of $N'$-type $\rho$, $D_\ell = \left( b_{pM_n} \right)$, $G_0 = G_2 = I$, $G_1$ varies over $G_1$, $Y_0 = 0$, $Y_1 = (\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix})$ with $y$ varying modulo $p$, and $Y_2 \in \mathbb{Z}_{2\ Sym}$ varying modulo $p$. For convenience, we choose $Y_i \equiv 0 \pmod{N}$. 

Case I. Say $r = 0$. We take $(M',N') = D_\ell^{-1}(pM,N)$ where $\ell = \text{rank}_p N$, and the $SL_2(\mathbb{Z})$-equivalence class representative $(M,N)$ is chosen so that $p$ divides the lower $2 - \ell$ rows of $N$.

First suppose $\ell = 2$. Thus rank$_p M' = 0$, and

$$\chi(p^2) \chi(M',N') = \chi_{N_0}(p^2) \chi_{N_1}(p) \chi(M,N).$$

So the contribution from these terms is $\chi_{N_0}(p^2) \chi_{N_1}(p) p^{2k-3} E_0$. 

Next suppose $\ell = 1$. So $p$ divides row 2 of $N$ and hence does not divide row 2 of $M$, and hence rank$_p M' = 1$ with $p$ dividing row 1 of $M$, $p$ not dividing row 1 of $N'$ (and so $(M',N') = 1$). We have $\chi(p^2) \chi(M',N') = \chi_{N_0}(p^2) \chi_{N_2}(p) \chi(M,N)$. Reversing, take $(M',N')$ of $N'$-type $\rho$, $p$-type 1. We need to count the equivalence classes $SL_2(\mathbb{Z})(M,N)$ so that $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) p(M,N) \in SL_2(\mathbb{Z})(M',N')$. For any $E \in SL_2(\mathbb{Z})$, we have $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) E(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) E(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \in SL_2(\mathbb{Z})$ if and only if $E \equiv (\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}) \pmod{p}$; thus we need to count the integral, coprime pairs $(M,N) = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) E(M'/N')$ where $E$ varies over $G_1$. (Note that we automatically have $M'N$ symmetric since $M'^t N'$ is symmetric.) We can assume $p$ divides row 2 of $M'$. To have $M$ integral, we need $p$ dividing row 1 of $EM'$; there is 1 choice of $E$ so that this is the case. Thus $p$ does not divide row 2 of $M$ or row 1 of $N$, so $(M,N) = 1$. So the contribution from these terms is $\chi_{N_0}(p^2) p^{2k-3} E_1$.

Now suppose $\ell = 0$. Thus rank$_p M = 2 = \text{rank}_p M'$. We have

$$\chi(p^2) \chi(M',N') = \chi_{N_1}(p) \chi_{N_2}(p^2) \chi(M,N).$$

So the contribution from these terms is $\chi_{N_1}(p) \chi_{N_2}(p^2) p^{-3} E_2$.

Case II. $r = 1$. Here we take

$$(M'G N'^t G^{-1}) = D_\ell^{-1} \left( M \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) + MY \right),$$

where $\ell = \text{rank}_p \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) N(\begin{smallmatrix} p \\ 1 \end{smallmatrix}) \pmod{p}$, $G \in G_1$, $Y = (\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix})$, and the equivalence class representative $(M,N)$ is adjusted so that $(M',N')$ is integral.

Suppose $\ell = 2$. Then $(M',N') = 1$, rank$_p M' = 1$, and $\chi(p) \chi(M',N') = \chi_{N_0}(p^2) \chi_{N_1}(p) \chi(M,N)$. Reversing,

$$(M,N) = \left( M'G \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \left( N'^t G^{-1} - M'GY \right) \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) \right).$$

We can assume $p$ divides row 2 of $M'$; to have $M$ integral, we need to choose the unique $G$ so that $q/m_2$ where $M'G = \left( \begin{smallmatrix} m_1 & m_2 \\ pm_3 & pm_4 \end{smallmatrix} \right)$. Then $p \not| m_1$, and by symmetry, $p \not| n_4$ where $N'^t G^{-1} = \left( \begin{smallmatrix} n_1 & n_2 \\ n_3 & n_4 \end{smallmatrix} \right)$. To have
N integral, we need to choose the unique y modulo p so that \( n_1 \equiv m_1 y \pmod{p} \). So \( M, N \) are integral and coprime, and the contribution from these terms is \( \chi_{\mathcal{N}_0}(p^2) \chi_{\mathcal{N}_1}(p) p^{2k-3} \mathbb{E}_1 \).

Say \( \ell = 1 \). Then we must have \( M = (m_1 m_2 \mod{p m_3 m_4}), N = (n_1 n_2 \mod{n_3 n_4}) \) with \( p \nmid (m_1 n_2) \); since \( (M, N) = 1 \), we must also have \( p \nmid (m_4 n_3) \). Thus \( (M', N') = 1 \) with \( \text{rank}_p M' = 0, 1, \) or 2, and

\[
\chi(p) \chi_{\rho}(M', N') = \chi_{\mathcal{N}_b \mathcal{N}_c}(p) \chi_{\rho}(M, N).
\]

Reversing,

\[
(M N) = \left( \frac{1}{p} \right) E \left( M' G \left( \frac{1}{p} \right) (N' \ell G^{-1} - M' G Y) \left( \frac{1}{p} \right) 1 \right),
\]

\( E \in \mathcal{G}_1 \).

Still assuming \( \ell = 1 \), suppose \( \text{rank}_p M' = 0 \). Then to have \( N \) integral, for each \( E \) we must choose the unique \( q \) so that \( q | n_1 \) where \( EN' \ell G^{-1} = (n_1 n_2 \mod{n_3 n_4}) \). Thus \( p \nmid n_2 n_3 \), and hence \( \text{rank}_p N = 2 \) for all choices of \( Y \). So the contribution from these terms is \( \chi_{\mathcal{N}_b \mathcal{N}_c}(p) p^{2k-3} \cdot p(p + 1) \mathbb{E}_0 \).

Continuing with the assumption \( \ell = 1 \), suppose \( \text{rank}_p M' = 1 \); we can assume \( p \) divides row 2 of \( M' \). To have \( M, N \) integral, we need \( p | m_2 \), \( n_1 \equiv m_1 y \pmod{p} \) where \( EM' G = (m_1 m_2 \mod{m_3 m_4}), EN' \ell G^{-1} = (n_1 n_2 \mod{n_3 n_4}) \). Say \( p \) divides row 2 of \( EM' \) (this is the case for \( q \) choices of \( E \)); then we need to choose the unique \( G \) so that \( p | m_2 \). Then \( p \nmid m_1 \), and by symmetry, \( p | n_3 \). But then \( p \) divides row 2 of both \( M \) and \( N \), so \( (M, N) \neq 1 \). So take the unique \( E \) so that \( p \) does not divide row 2 of \( EM' \); thus, by our choice of representatives in \( \mathcal{G}_1 \), we have \( p \) dividing row 1 of \( EM' \). To have \( N \) integral, we need to choose the unique \( G \) so that \( p | n_1 \). Then \( p \nmid n_2 \), and by symmetry, \( p | m_4 \). Since \( \text{rank}_p M' = 1, p \nmid m_3 \).

To have \( (M, N) = 1 \), we need to choose \( y \) so that \( n_3 \nmid m_3 y \pmod{p} \); so we have \( p - 1 \) choices for \( Y \). The contributions from these terms is \( \chi_{\mathcal{N}_b \mathcal{N}_c}(p) p^{k-3} (p - 1) \mathbb{E}_1 \).

Now assume \( \ell = 1 \), \( \text{rank}_p M' = 2 \). Then for each \( E \), choose the unique \( G \) so that \( p | m_2 \) where \( EM' G = (m_1 m_2 \mod{m_3 m_4}) \); then \( M \) is integral. Also, \( p \nmid m_1 m_3 \) so \( \text{rank}_p M = 2 \). Choose the unique \( y \) so that \( n_1 \equiv m_1 y \pmod{p} \); so \( M, N \) are integral and coprime. The contributions from these terms is \( \chi_{\mathcal{N}_b \mathcal{N}_c}(p) p^{k-3} (p + 1) \mathbb{E}_2 \).

Now assume \( \ell = 0 \); since \( (M, N) = 1 \), we must have \( M = (pm_1 m_2 \mod{pm_3 m_4}), N = (n_1 m_2 \mod{n_3 n_4}) \) with \( p \nmid m_2 n_3 - n_1 m_4 \). Thus \( (M', N') = 1 \), with \( \text{rank}_p M' = 1, \) and

\[
\chi(p) \chi_{\rho}(M', N') = \chi_{\mathcal{N}_1}(p) \chi_{\mathcal{N}_b \mathcal{N}_c}(p^2) \chi_{\rho}(M, N).
\]

Reversing,

\[
(M N) = p \left( M' G \left( \frac{1}{p} \right) (N' \ell G^{-1} - M' G Y) \left( \frac{1}{p} \right) 1 \right).
\]

Say \( \text{rank}_p M' = 1 \); we can assume \( p \) divides row 2 of \( M' \). So to have \( (M, N) = 1 \) we need to choose \( G \) so that \( p \nmid m_2 \) where

\[
M' G = \left( \begin{array}{cc} m_1 & m_2 \\ pm_3 & pm_4 \end{array} \right);
\]

this gives us \( p \) choices for \( G \). Write

\[
N' \ell G^{-1} = \left( \begin{array}{cc} n_1 & n_2 \\ n_3 & n_4 \end{array} \right);
\]
by symmetry, if \( p \mid n_3 \) then we must have \( p \mid n_4 \), and consequently \( (M', N') \neq 1 \). So \( p \nmid n_3 \), and \( (M, N) = 1 \) for all choices of \( y \). The contribution from these terms is \( \chi_{\mathcal{N}_1}(p)\chi_{\mathcal{N}_2}(p^2)p^{-1}E_1 \).

Now say \( \text{rank}_p M' = 2 \); write

\[
M'G = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}, \quad N'^tG^{-1} = \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix}.
\]

Then \( (M, N) = 1 \) unless \( \left( \frac{n_1}{n_3} \right) = \frac{m_1}{m_3} \in \text{span}_p(m_4) \). This gives us \( p - 1 \) choices for \( y \), and the contribution is \( \chi_{\mathcal{N}_1}(p)\chi_{\mathcal{N}_2}(p^2)p^{-3}(p^2 - 1)E_2 \).

**Case III.** \( r = 2 \). So \( (M', N') = D^{-1}(M \ p \ N + M \ Y) \) where \( Y \in \mathbb{Z}_{\text{sym}}^{2,2} \) varies modulo \( p \), and \( \ell = \text{rank}_p M \) and we assume \( p \) divides the lower \( 2 - \ell \) rows of \( M \).

Suppose \( \ell = 2 \). So \( (M, N) = (M' (N' - M' Y)/p) \); there is a unique \( Y \) so that \( N' \equiv M' Y \pmod{p} \). The contribution is \( \chi_{\mathcal{N}_0}(p^2)\chi_{\mathcal{N}_1}(p)p^{2k - 3}E_2 \).

Now suppose \( \ell = 1 \); so \( \text{rank}_p M' \geq 1 \), and \( \chi_{\rho}(M', N') = \chi_{\mathcal{N}_0\mathcal{N}_2}(p)\chi_{\rho}(M, N) \). Reversing,

\[
(M N) = \begin{pmatrix} 1 \\ p \end{pmatrix} E(M' (N' - M' Y)/p)
\]

where \( E \) varies over \( \mathcal{G}_1 \). Say \( \text{rank}_p M' = 2 \); write

\[
EM' = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}, \quad EN' = \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix}.
\]

To have \( N \) integral, we need \( (n_1 n_2) \equiv (m_1 m_2)Y \pmod{p} \), and to have \( (M, N) = 1 \), we need \( (n_3 n_4) \not\equiv (m_3 m_4)Y \pmod{p} \). Thus we have \( p - 1 \) choices for \( Y \), and the contribution from these terms is \( \chi_{\mathcal{N}_0\mathcal{N}_2}(p)p^{k - 3}(p^2 - 1)E_2 \).

Say \( \ell = 1 \) and \( \text{rank}_p M' = 1 \); assume \( p \) divides row 2 of \( M' \) (so \( p \) does not divide row 2 of \( N' \)). To have \( N \) integral, we need \( p \) not dividing row 1 of \( EM' \) and \( (n_1 n_2) \equiv (m_1 m_2)Y \pmod{p} \); this gives us \( p \) choices for \( E \) and \( p \) choices for \( Y \). Then \( p \) does not divide row 1 of \( M \) or row 2 of \( N \), so \( (M, N) = 1 \). The contribution from these terms is \( \chi_{\mathcal{N}_0\mathcal{N}_2}(p)p^{k - 3}.p^2E_1 \).

Say \( \ell = 0 \). So \( \text{rank}_p M' = 0, 1, \) or 2, and

\[
\chi_{\rho}(M', N') = \chi_{\mathcal{N}_1}(p)\chi_{\mathcal{N}_2}(p^2)\chi_{\rho}(M, N).
\]

Reversing, we have

\[
(M N) = p(M' (N' - M' Y)/p).
\]

Say \( \text{rank}_p M' = 2 \). We need to choose \( Y \) so that \( \text{rank}_p(M'M' Y - Y) = 2 \); as \( Y \) varies over \( \mathbb{Z}_{\text{sym}}^{2,2} \) modulo \( p \), so does \( M'M' Y - Y \), and \( p^2(p - 1) \) of these matrices have \( p \)-rank 2. Thus the contribution from these terms is \( \chi_{\mathcal{N}_1}(p)\chi_{\mathcal{N}_2}(p^2)p^{-1}(p - 1)E_2 \).

Now say \( \ell = 0 \) and \( \text{rank}_p M' = 1 \); we can assume \( p \) divides row 2 of \( M' \). To have \( (M, N) = 1 \), we need \( (n_1 n_2) - (m_1 m_2)Y \notin \text{span}_p(n_3 n_4) \). For each \( \alpha \) modulo \( p \), we have \( p \) choices of \( Y \) so that \( (n_1 n_2) - (m_1 m_2)Y \equiv \alpha(n_3 n_4) \pmod{p} \). Thus we have \( p^2(p - 1) \) choices for \( Y \) so that \( (M, N) = 1 \). The contribution from these terms is \( \chi_{\mathcal{N}_1}(p)\chi_{\mathcal{N}_2}(p^2)p^{-1}(p - 1)E_1 \).

Finally, say \( \ell = 0 \) and \( \text{rank}_p M' = 0 \). Thus \( \text{rank}_p N' = 2 = \text{rank}_p N \) for all choices of \( Y \). So the contribution from these terms is \( \chi_{\mathcal{N}_1}(p)\chi_{\mathcal{N}_2}(p^2)E_0 \).

Combining all the terms yields the result. \( \square \)
Proposition 3.4. For a prime not dividing \( N \),

\[
\mathbb{E}_\rho |T_1(p^2) = (p + 1)(\chi_{N_0}(p^2)p^{2k-3} + \chi(p)p^{k-3}(p - 1) + \chi_{N_2}(p^2))\mathbb{E}_\rho.
\]

Proof. Let \( G_1 = G_1(p) \), and let \( \mathbb{E}_0, \mathbb{E}_1, \mathbb{E}_2 \) be defined as in the proof of Proposition 3.3. Decompose \( \mathbb{E}_1, \mathbb{E}_2 \) as \( \mathbb{E}_1 = \mathbb{E}_{1,1} + \mathbb{E}_{1,2} \), \( \mathbb{E}_0 = \mathbb{E}_{0,0} + \mathbb{E}_{0,1} + \mathbb{E}_{0,2} \) where \( \mathbb{E}_{i,j} \) is supported on pairs \((M N)\) of \( p^2 \)-type \( i, j \). Further, we split \( \mathbb{E}_{0,2} \) as \( \mathbb{E}_{0,2,+} + \mathbb{E}_{0,2,-} \) where, for \( p \) odd, \( \mathbb{E}_{0,2,v} \) is supported on pairs \((M N)\) so that

\[
v \left( \frac{\det(MN/p)}{p} \right) = 1,
\]

and for \( p = 2 \), \( \mathbb{E}_{0,2,+} \) is supported on pairs \((M N)\) where \( \frac{1}{2}M^t N \equiv I (2) \) and \( \mathbb{E}_{0,2,-} \) on pairs \((M N)\) where \( \frac{1}{2}M^t N \equiv (0 1) (2) \). When \( p \) is odd, set \( \epsilon = (\frac{-1}{p}) \).

Using Proposition 2.1 of [4], the action of \( T_1(p^2) \) is given by matrices parameterised by \( r_0, r_1, r_2 \in \mathbb{Z}_{\geq 0} \) where \( r_0 + r_1 + r_2 = 1 \).

Case I. \( r_0 = 1 \). Here the summands are

\[
p^{k-3} \chi_{\rho}(M, N) \det \left( M \left( \begin{array}{c} 1 \\ \frac{1}{p} \end{array} \right) G^{-1} \right) + N \left( \begin{array}{c} p \\ 1 \end{array} \right) tG + M \left( \begin{array}{c} 1 \\ \frac{1}{p} \end{array} \right) Y tG \nabla^{1-k}
\]

where \((M N)\) varies over pairs of \( N \)-type \( \rho \), \( G \) varies over \( G_1 \), and \( Y = \left( \begin{array}{c} y_1 \\ y_2 \\ 0 \end{array} \right) \) with \( y_1 \) varying modulo \( p^2 \) and \( y_2 \) varying modulo \( p \).

Case Ia. Say \( M \left( \begin{array}{c} 1 \\ \frac{1}{p} \end{array} \right) \) is not integral. Then we can adjust the equivalence class representative for \((M' N')\) so that

\[
(M' G N'^t G^{-1}) = \left( \begin{array}{c} p \\ 1 \end{array} \right) M \left( \begin{array}{c} 1 \\ \frac{1}{p} \end{array} \right) N \left( \begin{array}{c} p \\ 1 \end{array} \right) + M \left( \begin{array}{c} 1 \\ \frac{1}{p} \end{array} \right) Y
\]

is coprime with \( \text{rank}_p M' = 1 \) or \( 2 \). So \( \chi_{\rho}(M', N') = \chi_{N_0}(p^2)\chi_{\rho}(M, N) \). Using the techniques used in the proof of Proposition 3.2, we find that in the case \( \text{rank}_p M' = 2 \) the contribution is \( \chi_{N_0}(p^2)p^{k-3}p^k(p + 1)E_2 \), and in the case \( \text{rank}_p M' = 1 \) the contribution is \( \chi_{N_0}(p^2)p^{k-3}p^{k+1}E_1 \).

Case Ib. Suppose \( M \left( \begin{array}{c} 1 \\ \frac{1}{p} \end{array} \right) \) is integral and that \((M', N') = 1 \) where

\[
(M' G N'^t G^{-1}) = M \left( \begin{array}{c} 1 \\ \frac{1}{p} \end{array} \right) N \left( \begin{array}{c} p \\ 1 \end{array} \right) + M \left( \begin{array}{c} 1 \\ \frac{1}{p} \end{array} \right) Y.
\]

Note that \( \text{rank}_p M' \geq 1 \), else \( \text{rank}_p N' \leq 1 \) and hence \((M', N') \neq 1 \). Also, note that \( \chi_{\rho}(M', N') = \chi(p)\chi_{\rho}(M, N) \).

Reversing, when \( \text{rank}_p M' = 2 \), we find that for each \( G \), there are \( p - 1 \) choices for \( Y \) so that \((M N)\) is an integral, coprime pair; the contribution from these terms is \( \chi(p)p^{k-3}(p^2 - 1)E_2 \). When \( \text{rank}_p M' = 1 \), we can assume \( p \) divides row 2 of \( M' \); to have \( N \) integral we need to choose the unique \( G \in G_1 \) so that \( p \) divides the 2,1-entry of \( N^t G^{-1} \), and then we have \( p(p - 1) \) choices for \( Y \) so that \((M, N) = 1 \). So the contribution from these terms is \( \chi(p)p^{k-3}p(p - 1)E_1 \).
Case Ic. Suppose $M\left(\frac{1}{p} \mid 1\right)$ is integral and that

$$\text{rank}_p\left(M\left(\frac{1}{p} \mid 1\right) N\left(\frac{p}{1}\right)\right) < 2;$$

adjusting the equivalence class representative, we have $(M', N') = 1$ where

$$(M' G N'^{-1} G) = \left(\frac{1}{p} \mid 1\right) \left(\frac{1}{p} \mid 1\right) N\left(\frac{p}{1}\right) + M\left(\frac{1}{p} \mid 1\right) Y.$$ 

Also, $\chi_\rho(M', N') = \chi_{\mathcal{N}_2}(p^2) \chi_\rho(M, N)$.

Reversing, take a coprime pair $(M' N')$. Then arguing as above, we find that when rank$_p M' = 2$, the contribution is $\chi_{\mathcal{N}_2}(p^2)p^{-2}(p^2 - 1)(p + 1)E_2$. When rank$_p M' = 1$, the contribution is $\chi_{\mathcal{N}_2}(p^2)p^{-1}(p^2 + p - 1)E_1$. When rank$_p M' = 0$, the contribution is $\chi_{\mathcal{N}_2}(p^2)p^{k-3} \cdot p^{3-k}(p + 1)E_0$.

Case II. $r_1 = 1$. Here the summands are

$$p^{k-3} \chi(p) \chi_\rho(M, N) \det(MG^{-1} \tau + N^\tau G + M\left(\frac{p}{1} \mid 0\right) \tau G)^{-k}$$

where $(M N)$ varies over pairs of $\mathcal{N}$-type $\rho$, $y$ varies modulo $p$ with $p \nmid y$, $G$ varies over $G_1$.

Case IIa. Say $M\left(\frac{1}{p} \mid 1\right)$ is not integral. Adjust the representative $(M N)$

$$(M' G N'^{-1} G) = \left(\frac{p}{1}\right) N + M\left(\frac{y}{p} \mid 0\right)$$

is integral. Then $(M', N') = 1$ with $(M' N')$ of $p^2$-type $1, 2$ or $0, 1$ or $0, 2$. Also, $\chi(p) \chi_\rho(M', N') = \chi_{\mathcal{N}_0}(p^2) \chi_\rho(M, N)$.

Reversing, when $(M' N')$ is $p^2$-type $1, 2$, we can assume $p$ divides row 2 of $M'$; we find there are unique choices for $G, Y$ so that $N$ is integral, and then we get $(M, N) = 1$. Thus the contribution from these terms is $\chi_{\mathcal{N}_0}(p^2)p^{k-3} \cdot p^{k}E_{1,2}$.

Next suppose $M' N'$ is $p^2$-type $0, 1$; we can assume $p^2$ divides row 2 of $M'$. For 1 choice of $E$ we have $p^2|(m_1 m_2)$; but then we cannot have $N$ integral and coprime to $M$. For the other $p$ choices of $E$ we have $p^2|(m_3 m_4)$. Choose the unique $G$ so that $p|n_2$; so $p \nmid n_1 n_4$ and $p^2 \nmid m_1$, contradicting that $M'$ is $p^2$-type $(0, 1)$. Then choose the unique $y \neq 0 (p)$ so that $n_1 \equiv m_1 Y (p)$; we get $p$ not dividing row 1 of $M$ or row 2 of $N$, so $(M, N) = 1$. The contribution from these terms is $\chi_{\mathcal{N}_0}(p^2)p^{k-3} \cdot p^{k}E_{0,1}$.

Now suppose $(M' N')$ is $p^2$-type $0, 2$. For each choice of $E \in G_1$, we choose the unique $G \in \mathcal{N}_1^{-1} G$ so that $E N' G = (n_1 n_4)$ (thus $p \nmid n_1 n_4$). To have $N$ integral, we need to choose $y \neq 0 (p)$ so that $n_1 \equiv m_1 Y (p)$; this is possible if and only if $p^2 \nmid m_1$. Let $V = \mathbb{F}x_1 \oplus \mathbb{F}x_2$ be equipped with the quadratic form $\frac{1}{p} M'^{-1} N'^{-1}$ relative to the basis $(x_1 x_2)$. Then with $(x'_1 x'_2) = (x_1 x_2)^{-1} E, \mathbb{F}x'_1$ varies over all lines in $V$, and $\frac{1}{p} EM'^{-1} N'^{-1}$ represents the quadratic form relative to $(x'_1 x'_2)$. When $p$ is odd and $V \cong \mathbb{H}$, there are 2 choices of $E$ so that $\mathbb{F}x'_1$ is isotropic; equivalently, when $V \cong \mathbb{H}$ there are 2 choices of $E$ so that $p^2 \nmid m_1$ (since the value of the quadratic form on $x'_1$ is $m_1 n_1/p \in \mathbb{F}$). When $p$ is odd and $V \not\cong \mathbb{H}$, we have $p^2 \nmid m_1$ for all $p + 1$ choices of $E$. So the contribution when $p$ is odd is

$$\chi_{\mathcal{N}_0}(p^2)p^{k-3} \cdot p^{k}(p - 1)E_{0,2, \epsilon} + \chi_{\mathcal{N}_0}(p^2)p^{k-3} \cdot p^{k}(p + 1)E_{0,2, -\epsilon}.$$
When \( p = 2 \) and \( V \simeq (0, 1) \) we have \( \mathbb{F} \chi'_1 \) isotropic for all \( E \); when \( V \simeq I \) we have \( \mathbb{F} \chi'_1 \) isotropic for 1 choice of \( E \). So the contribution when \( p = 2 \) is \( 2 \chi_{\mathcal{N}_0}(p^2)p^{2k-3}E_{0,2,+} \).

**Case IIb.** Suppose \( M \left( \begin{array}{c} \frac{1}{2} \\ 1 \end{array} \right) \) is integral, and

\[
(M'G N'^tG^{-1}) = \left( MN + M \begin{pmatrix} \frac{\chi}{p} \\ 0 \end{pmatrix} \right)
\]

is a coprime symmetric pair with \( \text{rank}_p M' \leq 1 \). Note that \( \chi(p)\chi_{\rho}(M', N') = \chi(p)\chi_{\rho}(M, N) \).

Reversing, first suppose that \( \text{rank}_p M' = 1 \), and assume \( p \) divides row 2 of \( M' \). To have \( N \) integral, choose the unique \( G \) so that \( \det G \) \( \equiv 0 \mod p \). Then \( (M, N, 1) \), we need \( p \) not dividing row 2 of \( N \). If \( M' \) is \( p^2 \)-type \((1, 1)\), we can assume \( p^2 \) divides row 2 of \( M' \) and then for any choice of \( y \neq 0 \) \((p)\) we have \( p \) not dividing row 2 of \( N \). If \( M' \) is \( p^2 \)-type \((1, 2)\) then \( p^2 \) \( \equiv 0 \mod m_3 \) so there are \( p-2 \) choices for \( y \neq 0 \) \((p)\) so that \( p \) does not divide row 2 of \( N \). Thus the contribution from these terms is \( \chi(p)p^{k-3}(p-1)\mathbb{E}_{1,1} + \chi(p)p^{k-3}(p-2)\mathbb{E}_{1,2} \).

Now suppose \( M' \) is \( p^2 \)-type \((0, 0)\); then \( N \) is invertible modulo \( p \) for all choices of \( G \) and \( y \). Hence the contribution is \( \chi(p)p^{k-3}(p^2 - 1)\mathbb{E}_{0,0} \).

Suppose \( M' \) is \( p^2 \)-type \((0, 1)\); then we can assume \( p^2 \) divides row 2 of \( M' \). For 1 choice of \( G \) we have \( p^2 \) \( \equiv 0 \mod m_1 \) and then for any \( y \) we have \( \text{rank}_p N = 2 \). Say we take any of the other \( p \) choices for \( G \) so that \( p \not\mid m_1 \). By symmetry, \( p \not\mid n_4 \). So there are \( p-2 \) choices of \( y \neq 0 \) \((p)\) so that \( \text{rank}_p N = 2 \). Thus the contribution from these terms is \( \chi(p)p^{k-3}(p^2 - p - 1)\mathbb{E}_{0,1} \).

Suppose \( M' \) is \( p^2 \)-type \((0, 2)\). Let \( V = \mathbb{F}x_1 \oplus \mathbb{F}x_2 \) be equipped with the quadratic form \( \frac{1}{p}N'M' \) relative to \( (x_1, x_2) \). Then with \( (x'_1, x'_2) = (x_1, x_2)G \), \( \mathbb{F}x'_1 \) \( \equiv \mathbb{F}x'_2 \) \( \not\equiv \mathbb{F}x'_2 \). Then we have \( \text{rank}_p = 2 \) if and only if \( \det N' \equiv y(m_1n_4 - m_3n_2) \mod p \). We have \( \text{rank}_p N = 2 \) if and only if \( \det N' \equiv y(m_1n_4 - m_3n_2) \mod p \). When \( p \) is odd and \( V \simeq \mathbb{H} \), there are 2 choices of \( G \) so that \( x'_1 \) is isotropic, and then \( \text{rank}_p N = 2 \) for all \( y \neq 0 \) \((p)\); for a choice of \( G \) so that \( x'_1 \) is anisotropic, there are \( p-2 \) choices of \( y \neq 0 \) \((p)\) so that \( \text{rank}_p N = 2 \). Hence the contribution from these terms when \( p \) is odd is \( \chi(p)p^{k-3} \cdot p(p-1)\mathbb{E}_{0,2,+} + \chi(p)p^{k-3} \cdot (p+1)(p-2)\mathbb{E}_{0,2,-} \). When \( p = 2 \), we have \( y = 1 \) \((p)\); when \( V \simeq (1, 0) \), \( \mathbb{F}x'_1 \) is isotropic for all 3 \( G \), and when \( V \simeq I \) there is 1 choice of \( G \) so that \( \mathbb{F}x'_1 \) is isotropic. So the contribution when \( p = 2 \) is \( \chi(p)p^{k-3}E_{0,2,+} + 3\chi(p)p^{k-3}E_{0,2,-} \).

**Case IIc.** Suppose \( M \left( \begin{array}{c} \frac{1}{2} \\ 1 \end{array} \right) \) is integral and

\[
\text{rank}_p \left( MN + M \begin{pmatrix} \frac{\chi}{p} \\ 0 \end{pmatrix} \right) = 1;
\]

we can adjust the representative so that

\[
(M'G N'^tG^{-1}) = \begin{pmatrix} 1 \\ \frac{1}{p} \end{pmatrix} \left( MN + M \begin{pmatrix} \frac{\chi}{p} \\ 0 \end{pmatrix} \right)
\]

is an integral pair. Then \( (M', N') = 1 \) with \( \text{rank}_p M' \geq 1 \), and

\[
\chi(p)\chi_{\rho}(M', N') = \chi_{\mathcal{N}_0}(p^2)\chi_{\rho}(M, N).
\]

Reversing, we need to count the equivalence classes \( \text{SL}_2(\mathbb{Z})(M, N) \) so that \( \left( \begin{array}{c} 1 \\ \frac{1}{p} \end{array} \right)(MN + M \begin{pmatrix} \frac{\chi}{p} \\ 0 \end{pmatrix}) \in \text{SL}_2(\mathbb{Z})(M', N') \). Thus we need to count the integral, coprime pairs.
\[(M \; N) = \left( \begin{array}{c} 1 \\ p \end{array} \right) E \left( t \begin{array}{cc} M'G & N' \cdot G^{-1} - M'G \left( \begin{array}{c} \frac{E}{\alpha} \\ 0 \end{array} \right) \right), \]

where \( E \in G_1 \). Write \( EM'G = (m_1, m_2, \bar{m}_1, \bar{m}_2) \), \( EN'G^{-1} = (\bar{n}_1, \bar{n}_2, n_1, n_2) \).

Say \( \text{rank}_p M' = 2 \). To have \( N \) integral, choose the unique \( G \) so that \( p | m_1 \) (and hence \( p \nmid m_2 m_3 \)). Then for any choice of \( y \), \( (M, N) = 1 \). So the contribution from these terms is \( \chi_{\mathcal{N}_2}(p^2) p^{k-3} \cdot p^{-k} (p^2 - 1) \).

Now say \( \text{rank}_p M' = 1 \); we can assume \( p \) divides row 2 of \( M' \). For \( p \) choices of \( E \), we have \( p \) dividing row 2 of \( EM' \), and then \( p \) divides row 2 of \( (M, N) \) (meaning \( (M, N) \neq 1 \)). So take the unique \( E \) so that \( p \) divides row 1 of \( EM' \) (and hence \( p \) does not divide row 1 of \( EN' \)). So to have \( (M, N) = 1 \), we need to choose \( G \) so that \( p \nmid m_3 \); this gives us \( p \) choices for \( G \). Then for every \( y \equiv 0 \pmod{p} \), we have \( (M, N) = 1 \). So the contribution from these terms is \( \chi_{\mathcal{N}_2}(p^2) p^{k-3} \cdot p^{-k} \cdot p(p - 1) \).

**Case III.** \( r_2 = 1 \). Here the summands are

\[ \chi(p^2) \chi_\rho((M, N) \; p) \cdot p^{k-3} \cdot \text{det} \left( M \left( \begin{array}{c} p \\ 1 \end{array} \right) G^{-1} + N \left( \begin{array}{c} 1 \\ \frac{E}{\alpha} \end{array} \right) G \right) ^{-k}, \]

where \( (M, N) \) varies over pairs of \( \mathcal{N} \)-type \( \rho \), and \( G \) varies over \( \mathcal{G}_1 \).

**Case IIIa.** Suppose \( N = (m_1, m_2) \) is not integral. We can adjust the representative so that \( N = (\frac{n_1}{m_1}, \frac{n_2}{m_2}) \) (so \( p \nmid n_1 \)). Write \( M = (m_3, m_4) \). Then \( p \nmid (m_4, n_4) \), else by symmetry, \( p | m_3 \) and \( (M, N) \neq 1 \). Thus with

\[ (M'G \; N' \cdot G^{-1}) = \left( \begin{array}{c} p \\ 1 \end{array} \right) \left( M \left( \begin{array}{c} p \\ 1 \end{array} \right) N \left( \begin{array}{c} 1 \\ \frac{E}{\alpha} \end{array} \right) \right), \]

we have \( (M', N') = 1 \) and \( \text{rank}_p M' \leq 1 \). When \( \text{rank}_p M' = 1 \), we must have \( M' \) of \( p^2 \)-type 1, 1 with \( p \) dividing row 1 of \( M' \), and

\[ \chi(p^2) \chi_\rho(M', N') = \chi_{\mathcal{N}_2}(p^2) \chi_\rho(M, N). \]

Reversing,

\[ (M, N) = \left( \begin{array}{c} 1 \\ \frac{E}{\alpha} \end{array} \right) \left( \begin{array}{c} p \\ 1 \end{array} \right) E \left( M'G \left( \begin{array}{c} 1 \\ \frac{E}{\alpha} \end{array} \right) N^{-1} \cdot G^{-1} \left( \begin{array}{c} p \\ 1 \end{array} \right) \right). \]

We know \( \left( \begin{array}{c} 1 \\ \frac{E}{\alpha} \end{array} \right) E(p, 1) \in SL_2(\mathbb{Z}) \) if and only if \( E \equiv (\frac{n_1}{n_2}, \frac{n_2}{n_2}) \pmod{p} \). Write \( EM'G = (m_1, m_2, \bar{m}_1, \bar{m}_2) \), \( EN'G^{-1} = (\bar{n}_1, \bar{n}_2, n_1, n_2) \).

First suppose \( M' \) is \( p^2 \)-type 1, 1 with \( p \) dividing row 1 of \( M' \). We know \( p \) divides row 1 of \( EM' \) if and only if \( E \equiv (\frac{n_1}{n_2}, \frac{n_2}{n_2}) \pmod{p} \); thus we only need to consider \( E = I \), and we can assume \( p^2 \) divides row 1 of \( M' \). To ensure \( N \) is integral, we need to choose the unique \( G \in \mathcal{G}_1 \) so that \( p | n_2 \); then by symmetry, \( p | m_3 \), and since \( \text{rank}_p M' = 1 \), \( p \nmid m_4 \). Then \( M, N \) are integral and coprime. So the contribution from these terms is \( \chi_{\mathcal{N}_2}(p^2) p^{k-3} \cdot p^{-k} p(p + 1) \).

Now suppose \( M' \) is \( p^2 \)-type 0, 0. Then for each \( E \in \mathcal{G}_1 \), there is a unique \( G \in \mathcal{G}_1 \) so that \( p | n_2 \) (and hence \( p \nmid n_1 n_4 \)). Thus \( \text{rank}_p N = 1 \) so \( (M, N) = 1 \), and the contribution from these terms is \( \chi_{\mathcal{N}_2}(p^2) p^{k-3} \cdot p^{-k}(p + 1) \).

Suppose \( M' \) is \( p^2 \)-type 0, 1; we can assume \( p^2 \) divides row 2 of \( EM' \). Suppose \( p^2 \) divides row 2 of \( EM' \); to have \( M \) integral, we choose the unique \( G \) so that \( p^2 | m_1 \). Then \( p^2 | m_2 \), and by symmetry, \( p | n_4 \). Hence \( p \nmid n_2 n_3 \), and \( N \) is not integral. So choose \( E \) so that \( p^2 \) does not divide row 2 of \( EM' \); we have
1 choice for $E$, and then $p^2$ divides row 1 of $EM'$. Choose the unique $G$ so that $p|n_2$; then $p \nmid n_1n_4$, and $N$ is integral with rank$_p N = 2$. So the contribution from these terms is $\chi_{N_0}(p^2)p^{k-3} \cdot p^k E_{0,1}$.

Suppose $M'$ is $p^2$-type 0,2. For each $E \in G_1$, choose the unique $G \in G_1$ so that $q|n_2$. To have $M$ integral, we need $p^2|m_1$. Let $V = \mathbb{F}x_1 \oplus \mathbb{F}x_2$ be equipped with the quadratic form given by $\frac{1}{p} EM'N'$ relative to $(x_1, x_2)$. Then with $(x_1', x_2') = (x_1, x_2)E$, the quadratic form on $V$ is given by $\frac{1}{p} EM'N'E$ relative to $(x_1', x_2')$. As $E$ varies over $G_1$, $x_1'$ varies over all lines in $V$. Suppose $p$ is odd; then to have $p^2|m_1$, we need $V \simeq \mathbb{H}$, and then $p|m_1$ for 2 choices of $E$. Hence the contribution from these terms when $p$ is odd is $2 \chi_{N_0}(p^2)p^{k-3} \cdot p^k E_{0,2,\epsilon}$. In the case that $p = 2$, the contribution is $\chi_{N_0}(p^2)p^{2k-3}E_{0,2,+} + 3 \chi_{N_0}(p^2)p^{2k-3}E_{0,2,-}.$

**Case IIIb.** Suppose $N(\frac{1}{p})$ is integral, and rank$_p (M'N') = 2$ where

$$\left( M'G N'^t G^{-1} \right) = \left( M \left( \begin{array}{c} p \\ 1 \end{array} \right) N \left( \begin{array}{c} \frac{1}{p} \\ 1 \end{array} \right) \right).$$

So rank$_p M \geq 1$, rank$_p M' \leq 1$, and $M'$ cannot be $p^2$-type 0,0. Also, when $M'$ is $p$-type 1, we can assume $p$ divides row 2 of $M(\frac{1}{p} 1)$; then using symmetry, we see $p$ divides row 2 of $N$, so we must have $M'$ of $p^2$-type 1,2. Note that $\chi(p^2)\chi_{\rho}(M', N') = \chi(p)\chi_{\rho}(M, N)$.

Reversing, $(M, N) = (M'G(\frac{1}{p} 1)N'^t G^{-1}(\frac{1}{p} 1))$. Write

$$M'G = \left( \begin{array}{ccc} m_1 \\ m_2 \\ m_3 \\ m_4 \end{array} \right), \quad N'^t G^{-1} = \left( \begin{array}{ccc} n_1 & n_2 \\ n_3 & n_4 \end{array} \right).$$

Suppose $M'$ is $p^2$-type 1,2; assume $p$ divides row 2 of $M'$. Choose the unique $G$ so that $p|m_1$; thus $p \nmid m_2$, $p^2 \nmid m_3$. So $(M, N) = 1$ and the contribution from these terms is $\chi(p)p^{k-3}E_{1,2}$.

Suppose $M'$ is $p^2$-type 0,1; assume $p^2$ divides row 2 of $M'$. We have $(M, N) = 1$ if and only if $p^2 \nmid m_1n_4$; by symmetry, $p^2|m_1$ if and only if $p|m_4$. So choose $G$ so that $p|m_1$; we have $p$ choices for $G$, and hence the contribution from these terms is $\chi(p)p^{k-3} \cdot p^k E_{0,1}$.

Now suppose that $(M', N')$ is $p^2$-type 0,2. Let $V = \mathbb{F}x_1 \oplus \mathbb{F}x_2$ be equipped with the quadratic form given by $\frac{1}{p} GN'M'G = \tilde{d}(n^m_{m_3})$ relative to $(x_1, x_2)$. So relative to $(x_1', x_2') = (x_1, x_2)G$, the quadratic form is given by $\frac{1}{p} GN'M'G = \tilde{d}(n^m_{m_3}) \cdot (p)$ where $d = \det N'$ (recall that $p^2|m_3$). We know $\mathbb{F}x_1'$ varies over all lines in $V$ as $G$ varies over $G_1$. When $p$ is odd, $p^2 \nmid m_1n_4$ for $p - 1$ choices of $G$ if $V \simeq \mathbb{H}$, and $p^2 \nmid m_1n_4$ for $p + 1$ choices of $G$ otherwise. So the contribution from these terms when $p$ is odd is $\chi(p)p^{k-3}(p - 1)E_{0,2,\epsilon} + \chi(p)p^{k-3}(p + 1)E_{0,2,-\epsilon}$. When $p = 2$ the contribution is $2\chi(p)p^{6-3}E_{0,2,\epsilon}.$

**Case IIIc.** Suppose $N(\frac{1}{p})$ is integral with

$$\text{rank}_p \left( M \left( \begin{array}{c} p \\ 1 \end{array} \right) N \left( \begin{array}{c} \frac{1}{p} \\ 1 \end{array} \right) \right) = 1.$$  

Adjust the equivalence class representative of $(M, N)$ so that $p$ divides row 2 of $(M(\frac{1}{p} 1)N(\frac{1}{p} 1)) = 1$. Set

$$\left( M'G N'^t G^{-1} \right) = \left( \begin{array}{c} 1 \\ \frac{1}{p} \end{array} \right) \left( M \left( \begin{array}{c} p \\ 1 \end{array} \right) N \left( \begin{array}{c} \frac{1}{p} \\ 1 \end{array} \right) \right) = 1.$$  

Since $M'$, $N'$ are integral and $(M, N) = 1$, we have $(M', N') = 1$. Also, $\text{rank}_p M' \geq 1$ and $\chi(p^2)\chi_{\rho}(M', N') = \chi_{N_0}(p^2)\chi_{\rho}(M, N).$
Reversing, take \((M' N')\) of \(p\)-type 1 or 2, and set

\[
(M N) = \left( \begin{array}{c} 1 \\ p \end{array} \right) E \left( \begin{array}{c} \frac{1}{p} \\ 1 \end{array} \right) M' G \left( \begin{array}{c} \frac{1}{p} \\ 1 \end{array} \right) N' \tau G^{-1} \left( \begin{array}{c} p \\ 1 \end{array} \right)
\]

where \(E \in \mathcal{G}_1\). Write \(EM'G = (m_1 m_2, m_3 m_4)\), \(EN'\tau G^{-1} = (n_1 n_2, n_3 n_4)\).

First suppose \(\text{rank}_p M' = 2\). For each \(E\), choose the unique \(G\) so that \(p \nmid m_1\). Thus \(p \nmid m_2m_3\), so \((M, N) = 1\). The contribution from these terms is \(\chi_{N_2}(p^2) p^{k-3} \cdot p^{-k} \cdot (p + 1) \mathbb{E}_2\).

Finally, suppose \(\text{rank}_p M' = 1\); assume \(p\) divides row 2 of \(M'\). Suppose \(p\) divides row 2 of \(EM'\); choosing \(G\) so that \(p \nmid m_1\), by symmetry \(p \nmid n_4\) and hence \((M, N) \neq 1\). So choose the unique \(E\) so that \(p\) does not divide row 2 of \(EM'\); then \(p\) divides row 1 of \(EM'\). To have \((M, N) = 1\), choose \(G\) so that \(p \nmid m_3\); we have \(p\) choices for \(G\), and the contribution from these terms is \(\chi_{N_2}(p^2) p^{k-3} \cdot p^{1-k} \mathbb{E}_1\).

Combining all the contributions yields the result. \(\square\)

Now we determine the action on Eisenstein series of \(T(q), T_1(q^2)\) where \(q\) is a prime dividing \(N\). We let \(\mathbb{E}\) denote \(\mathbb{E} / q \mathbb{E}\); when \(q\) is odd, we let \(\omega = \left( -\frac{3}{q} \right)\), and we fix \(q\) so that \(\left( \frac{\omega}{q} \right) = -1\). Let \(\mathcal{G}_1 = \mathcal{G}_1(q)\); note that we can choose the elements of \(\mathcal{G}_1\) to be congruent modulo \(N/q\) to 1.

**Proposition 3.5.** Suppose \(q\) is a prime dividing \(N_2\); then

\[
\mathbb{E}_\rho | T(q) = \chi_{N_0}(q^2) \chi_{N_1}(q) q^{2k-3} \mathbb{E}_\rho.
\]

**Proof.** From Proposition 3.1 of [4], we know that

\[
2 \mathbb{E}_\rho(\tau)| T(q) = q^{2k-3} \sum \overline{X}(N_0, N_1, N_2/q)(M, N) \chi_q(\det M) \det(M \tau + qN + MY)^{-k}
\]

where \((M N)\) varies over a set of \(\text{SL}_2(\mathbb{Z})\)-equivalence class representatives for pairs of \(N\)-type \(\rho\), and \(Y\) varies over all symmetric matrices modulo \(q\); recall that we can choose \(Y \equiv 0 \ (N'/q)\). Take \((M N)\) of \(N\)-type \(\rho\); set \((M' N') = (M qN + MY)\). Thus \((M' N')\) is \(N\)-type \(\rho\); also, \(\chi_{(N_0, N_1, N_2/q)}(M' N') = \chi_{(N_0, N_1, N_2/q)}(M, N)\). Reversing, take \((M' N')\) of \(N\)-type \(\rho\); set \((M N) = (M' q^{-1}(N' - M') Y)\). So we need to choose \(Y \equiv M' N' (q)\) to have \(N\) integral. Thus \(\overline{X}(N_0, N_1, N_2/q)(M', N') = \overline{X}(N_0(q^2) \chi_{N_1}(q) \overline{X}(N_0, N_1, N_2/q)(M, N)\). \(\square\)

**Proposition 3.6.** For \(q\) a prime dividing \(N_1\), let \(\rho' = (N_0, N_1/q, q N_2)\); then

\[
\mathbb{E}_\rho | T(q) = \begin{cases} 
\chi_{N_0 N_2}(q) (q^{k-1} \mathbb{E}_\rho + q^{k-3}(q^2 - 1) \mathbb{E}_{\rho'}) & \text{if } \chi_q = 1, \\
\chi_{N_0 N_2}(q) q^{k-1} \mathbb{E}_\rho & \text{if } \chi_q \neq 1.
\end{cases}
\]

**Proof.** Recall that we must have \(\chi_q^2 = 1\) since \(q \nmid N_1\). Take \((M N)\) of \(N\)-type \(\rho\) so that \(q\) divides row 2 of \(M\). Set

\[
(M' N') = \left( \begin{array}{c} 1 \\ \frac{1}{q} \end{array} \right) (M qN + MY);
\]

since \(q\) does not divide row 1 of \(M\) or row 2 of \(N\), \((M', N') = 1\). Also, \(\text{rank}_q M' \geq 1\), and \(\chi_{(N_0, N_1/q, q N_2)}(M', N') = \chi_{N_0 N_2}(q) \chi_{(N_0, N_1/q, q N_2)}(M, N)\).
Reversing, take \((M' N')\) of \(\mathcal{N}'\)-type 1 or 2, \(\mathcal{N}'/q\)-type \((\mathcal{N}_0, \mathcal{N}_1/q, \mathcal{N}_2)\); set
\[
\begin{pmatrix}
1 \\
q
\end{pmatrix} E(M' (N' - M'Y)/q),
\]
\(E \in \mathcal{G}_1\). Write \(EM' = (m_1 m_2)^2\), \(EN' = (n_1 n_2)^2\).

First suppose \(\text{rank}_q M' = 2\). To have \(N\) integral, we need to choose \(Y\) so that \((n_1 n_2) \equiv (m_1 m_2)Y (q)\); then to have \((M, N) = 1\), we need \((n_3 n_4) \neq (m_3 m_4)Y (q)\). If \(q \nmid m_1\) then \(y_4\) can be chosen freely; if \(q|m_1\) then \(y_1\) can be chosen freely. This gives us \(q - 1\) choices for \(Y\). Summing over these \(Y\), we have
\[
\sum_Y \chi_{(t, q, 1)}(M, N) = \begin{cases} 
q - 1 & \text{if } \chi_q = 1, \\
0 & \text{if } \chi_q \neq 1.
\end{cases}
\]

So the contribution from these terms is
\[
\begin{pmatrix} q^{2k-3} \cdot q^{-k(q^2 - 1)} & \text{if } \chi_q = 1, \\
0 & \text{if } \chi_q \neq 1.
\end{pmatrix}
\]

Now suppose \(\text{rank}_q M' = 1\); assume \(q\) divides row 2 of \(M'\). To have \(\text{rank}_q M = 1\), we cannot have \(q\) dividing row 1 of \(EM'\); this leaves us \(q\) choices for \(E\), and with these choices we have \(q\) dividing row 2 of \(EM'\). Then choose \(Y\) so that \((n_1 n_2) \equiv (m_1 m_2)Y (q)\); we have \(q\) choices for \(Y\). Then row 2 of \(N\) is congruent modulo \(q\) to \((n_3 n_4)\), so \((M, N) = 1\) and
\[
\sum_Y \chi_{(t, q, 1)}(M, N) = (q - 1) \chi_{(t, q, 1)}(M', N').
\]

So the contribution from these terms is \(\chi_{\mathcal{N}_0, \mathcal{N}_2}(q)q^{k-1}E_{\rho'}\). \(\square\)

**Proposition 3.7.** For \(q\) a prime dividing \(\mathcal{N}_0\), set
\[
\rho' = (\mathcal{N}_0/q, q\mathcal{N}_1, \mathcal{N}_2), \quad \rho'' = (\mathcal{N}_0/q, \mathcal{N}_1, q\mathcal{N}_2).
\]

Then
\[
E_{\rho'} T(q) = \begin{cases} 
\chi_{\mathcal{N}_1}(q)\chi_{\mathcal{N}_1}(q^2)(E_{\rho'} + q^{-1}(q - 1)E_{\rho'} + q^{-1}(q - 1)E_{\rho''}) & \text{if } \chi_q = 1, \\
\chi_{\mathcal{N}_1}(q)\chi_{\mathcal{N}_1}(q^2)(E_{\rho'} + q^{-2}(q - 1)E_{\rho''}) & \text{if } \chi_q^2 = 1, \chi_q \neq 1, \\
\chi_{\mathcal{N}_1}(q)\chi_{\mathcal{N}_1}(q^2)E_{\rho} & \text{if } \chi_q^2 \neq 1.
\end{cases}
\]

**Proof.** Take \((M N)\) of \(\mathcal{N}\)-type \(\rho\), and set
\[
(M' N') = \frac{1}{q}(M qN + MY).
\]

Since \(\text{rank}_q N = 2\), we have \(\text{rank}_q(M' N') = \text{rank}_q(M' N) = 2\); hence \((M', N') = 1\). Also,
\[
\chi_{(\mathcal{N}_0/q, \mathcal{N}_1, \mathcal{N}_2)}(M', N') = \chi_{\mathcal{N}_1}(q)\chi_{\mathcal{N}_1}(q^2)\chi_{(\mathcal{N}_0/q, \mathcal{N}_1, \mathcal{N}_2)}(M, N).
\]

Reversing, take \((M' N')\) of \(\mathcal{N}/q\)-type \((\mathcal{N}_0/q, \mathcal{N}_1, \mathcal{N}_2)\). Set
\[
(M N) = (qM' N' - M'Y).
\]
Write \( M' = \begin{pmatrix} m_1 & m_2 \\ n_3 & n_4 \end{pmatrix} \), \( N' = \begin{pmatrix} m_3 \\ n_4 \end{pmatrix} \). We need to choose \( Y \) so that \( \text{rank}_q N = 2 \). Then we sum 
\[ \sum_Y \chi_q(\det N) ; \] since \( \chi_0(\det N) = 0 \) when \( \text{rank}_q N < 2 \), we can simply sum over all \( Y \).

First suppose \( \text{rank}_q M' = 2 \). Then \( M'N' - Y \) varies over all symmetric matrices modulo \( q \) as \( Y \) does, so the contribution from these terms is \( \chi_q(\det M') \). Now, \( \text{rank}_q N = 2 \), \( H \). Hence we have \( \chi_q(\det M') \) when \( q \) is odd. Thus the contribution from these terms when \( q \) is odd is \( \chi_q(\det M') \).

So the contribution from these terms when \( q \) is odd is 
\[
\sum_Y \chi_q(\det N) = \begin{cases} 
q^2(q - 1) & \text{if } \chi_q = 1, \\
\epsilon q(q - 1)\chi_q(\det M') & \text{if } \chi_q^2 = 1, \chi_q \neq 1, \\
0 & \text{if } \chi_q^2 \neq 1.
\end{cases}
\]

When \( q = 2 \) we have \( \sum_Y \chi_q(\det N) = 4 \) so the contribution is \( \chi_{N_1}(q)q^{-1}(q - 1)\epsilon\rho' \).

Now say \( \text{rank}_q M' = 1 \); we can assume \( q \) divides row 2 of \( M' \). Choose \( E \in SL_2(\mathbb{Z}) \) so that \( M'E \equiv \begin{pmatrix} m_1 & 0 \\ 0 & 0 \end{pmatrix}(q) \); by symmetry, \( N'E^{-1} \equiv \begin{pmatrix} n_1 & n_2 \\ 0 & 0 \end{pmatrix}(q) \). Clearly \( E^{-1}Y'E^{-1} \) varies over all symmetric matrices as \( Y \) does, so 
\[
\sum_Y \chi_q(\det N) = q^2 \sum_u \chi_q((n'_1 - m'_1 u)n'_4)
\]

Thus the contribution from these terms is 
\[
\begin{cases} 
q^{-1}(q - 1)\epsilon\rho' & \text{if } \chi_q = 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Finally, suppose \( \text{rank}_q M' = 0 \). Then \( \sum_Y \chi_q(\det N) = q^3 \chi_q(\det N') \), so the contribution is \( \chi_{N_1}(q)\chi_{N_2}(q^2)\epsilon\rho'. \)

**Proposition 3.8.** For \( q \) a prime dividing \( N_2 \), we have 
\[ \epsilon(\rho')T_1(q^2) = \chi_{N_0}(q^2)(q + 1)q^{2k-3}\epsilon\rho'. \]
**Proof.** From Proposition 2.1 of [4], we know $2E_{\rho}(\tau)|T_1(q^2)$ is the sum of terms

$$p^{k-3} \chi_{\rho}(M, N) \det \left( M \left( \begin{array}{c} \frac{1}{q} \\ 1 \end{array} \right) G^{-1} \tau + N \left( \begin{array}{c} q \\ 1 \end{array} \right) G + M \left( \begin{array}{c} \frac{1}{q} \\ 1 \end{array} \right) Y G^{-1} \right)^{-k}$$

where $(M, N)$ varies over $SL_2(\mathbb{Z})$-equivalence class representatives of pairs of $\mathcal{N}$-type $\rho$, $G \in \mathcal{G}_1$, $Y = \left( \begin{array}{cc} y_1 & \rho_k \\ 0 & 1 \end{array} \right)$ with $y_1$ varying modulo $q^2$, $y_2$ modulo $q$. Recall that we can choose $G \equiv I \pmod{\mathcal{N}/q}$ and $Y \equiv 0 \pmod{\mathcal{N}/q}$.

We know $M(\begin{array}{c} \frac{1}{q} \\ 1 \end{array})$ is never integral. Adjust the equivalence class representative for $(M, N)$ so that

$$(M'G N' G^{-1}) = \left( \begin{array}{c} 1 \\ q \end{array} \right) M' \left( \begin{array}{c} 1 \\ q \end{array} \right) \left( \begin{array}{c} 1 \\ q \end{array} \right) N + M \left( \begin{array}{c} \frac{1}{q} \\ 1 \end{array} \right) Y$$

is an integral pair. Since rank$_q M' = 2$, we have $(M', N') = 1$. Also,

$$\chi_{\rho}(\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2/q)(M', N') = \chi_{\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2/q}(M, N) \chi_{\mathcal{N}_0}(q^2),$$

and we know $\chi_{\mathcal{X}_1} = 1$.

Reversing, choose $(M', N')$ of $\mathcal{N}$-type $\rho$, and set

$$(M, N) = \left( \begin{array}{c} 1 \\ q \end{array} \right) E M' G \left( \begin{array}{c} 1 \\ q \end{array} \right) \left( N' G^{-1} - M' G Y \right) \left( \begin{array}{c} 1 \\ q \end{array} \right),$$

where $E \in \mathcal{G}_1$. Write $E M' G = \left( \begin{array}{cc} m_1 & m_2 \\ m_3 & m_4 \end{array} \right)$, $E N' G^{-1} = \left( \begin{array}{cc} n_1 & n_2 \\ n_3 & n_4 \end{array} \right)$.

For each choice of $E$, we need to choose the unique $G$ so that $q|m_4$ to ensure $M$ is integral. Thus $q \nmid m_2 m_3$. Choose $y_2$ so that $n_4 \equiv m_3 y_2 \pmod{q}$; then choose $y_q$ so that $n_3 \equiv m_3 y_1 + m_4 y_2 \pmod{q}$. By symmetry, $m_1 n_3 + m_2 n_4 \equiv m_3 n_1 \pmod{q}$. Thus

$$m_3 n_1 \equiv m_3 m_3 y_1 + m_2 m_3 y_2 \pmod{q},$$

so $n_1 \equiv m_1 y_1 + m_2 y_2 \pmod{q}$, and hence $N$ is integral. Also, $\det M = \det M'$, so $\chi_{\rho}(M', N') = \chi_{\mathcal{N}_0}(q^2) \chi_{\rho}(M, N)$. □

**Proposition 3.9.** For $q$ a prime dividing $\mathcal{N}_1$, set $\rho' = (\mathcal{N}_0, \mathcal{N}_1/q, \mathcal{N}_2/q)$. Then

$$\mathbb{E}_{\rho'}|T_1(q^2) = \left\{ \begin{array}{ll} (\chi_{\mathcal{N}_0}(q^2)q^{2k-2} + \chi_{\mathcal{N}_2}(q^2)q) \mathbb{E}_{\rho} + q^{-1}(\chi_{\mathcal{N}_0/q}(q)q^{k-2} + \chi_{\mathcal{N}_2/q}(q^2))(q^2 - 1) \mathbb{E}_{\rho'} & \text{if } \chi_q = 1, \\
(\chi_{\mathcal{N}_0}(q^2)q^{2k-2} + \chi_{\mathcal{N}_2}(q^2)q) \mathbb{E}_{\rho} & \text{if } \chi_q \neq 1. \end{array} \right.$$ 

**Proof.** Recall that we must have $\chi_4^2 = 1$ since $q|\mathcal{N}_1$. Take $(M, N)$ of $\mathcal{N}$-type $\rho$; assume $q$ divides row 2 of $M'$.

First suppose $M(\begin{array}{c} \frac{1}{q} \\ 1 \end{array})$ is not integral; thus by symmetry,

$$N = \left( \begin{array}{cc} * & * \\ * & n_4 \end{array} \right)$$

where $q \nmid n_4$. Thus $(M', N') = 1$ where
\[(M'G N'^t G^{-1}) = \left( \frac{q}{1} \right) \left( M \left( \frac{1}{q} \frac{1}{1} \right) N \left( \frac{q}{1} \frac{1}{1} \right) + M \left( \frac{1}{q} \frac{1}{1} \right) Y \right).\]

We know \( q \nmid M' \) and \( q \mid \det M' \), so \( \text{rank}_q M' = 1 \). Also,

\[
\chi_{(N_0, N_1/q, N_2)}(M', N') = \chi_{N_0}(q^2) \chi(N_0, N_1/q, N_2)(M, N).
\]

Reversing, take \((M', N')\) of \( N'\)-type \( \rho \); assume \( q \) divides row 2 of \( M' \). Set

\[
(MN) = \left( \frac{1}{q} \right) \left( 1 \right) E(M'G) \left( \frac{q}{1} \right) \left( N'^t G^{-1} - M'GY \right) \left( \frac{1}{q} \right) 1
\]

where \( E \in G_1 \). Write \( EM'G = (m_{12} m_{4}) \), \( EN'^t G^{-1} = (n_{13} n_{4}) \). For 1 choice of \( E \), \( q \) divides row 1 of \( EM' \); then to have \( N \) integral we need \( q | (n_1 n_2) \), which is impossible since \((M', N') = 1\). So choose \( E \) so that \( q \) does not divide row 1 of \( EM' \) (\( q \) choices for \( E \)). Thus \( q \) divides row 2 of \( EM' \); to have \( M \) integral, choose the unique \( G \) so that \( q | m_{23} \); so \( q \nmid m_1 \), and by symmetry, \( q | n_3 \) (so \( q \nmid n_4 \)). Choose the unique \( y_2 \) (so that \( n_2 \equiv m_1 y_2 \) (\( q \)) and choose the unique \( y_1 \) (\( q^2 \)) so that \( n_1 \equiv m_1 y_1 + m_2 y_2 \) (\( q^2 \)). Thus \((M, N) = 1\), and \( \chi_{\rho}(M', N') = \chi_{\rho}(M, N) \). So the contribution from these terms is \( \chi_{N_0}(q^2)q^{2k-2}E_{\rho} \).

Now suppose \( M \left( \frac{1}{q} \right) \) is integral with

\[\text{rank}_q \left( M \left( \frac{1}{q} \right) \frac{1}{1} N \left( \frac{q}{1} \right) \right) = 2.\]

Thus \( M = (m_{12} m_{4}) \) (\( q \)) where \( q | m_1, m_3, m_4, q \nmid m_2 \). By symmetry, \( q \) divides row 2 of \( N \left( \frac{q}{1} \right) \). So \( \text{rank}_q \left( M \left( \frac{1}{q} \right) \frac{1}{1} N \left( \frac{q}{1} \right) \right) \geq 1 \). In the case this rank is 2, we have \( q^2 \nmid m_3 \), and \( \text{rank}_q M' = 2 \) where

\[
(M'G N'^t G^{-1}) = \left( M \left( \frac{1}{q} \frac{1}{1} \right) N \left( \frac{q}{1} \frac{1}{1} \right) + M \left( \frac{1}{q} \frac{1}{1} \right) Y \right).
\]

Also, \( \chi_{(N_0, N_1/q, N_2)}(M', N') = \chi_{N'/q}(q) \chi(N_0, N_1/q, N_2)(M, N) \).

Reversing, first choose \((M', N')\) of \( N'\)-type \((N_0, N_1/q, qN_2)\). Set

\[
(MN) = \left( \frac{1}{q} \right) \left( 1 \right) E(M'G) \left( \frac{q}{1} \right) \left( N'^t G^{-1} - M'GY \right) \left( \frac{1}{q} \right) 1
\]

Write \( M'G = (m_{12} m_{4}) \), \( N'^t G^{-1} = (n_{13} n_{4}) \). Suppose \( \text{rank}_q M' = 2 \). Then for each \( G \), adjust the equivalence class representative so that \( q | m_4 \) (so \( q \nmid m_2 m_3 \)). Take \( u, y_2 \) so that

\[
\frac{M'}{n_3} = \begin{pmatrix} u \\ y_2 \end{pmatrix} \mod (q);
\]

set \( y_1 = u + qu' \) where \( u' \) varies modulo \( q \). Then summing over corresponding \( Y \), with \( n_3' = (n_3 - m_3 u - m_4 y_2) / q \), we have

\[
\sum_Y \chi_{(1, q, 1)}(M, N) = \sum_{u'} \chi_q(-m_{22}) \chi_q(n_3' - m_3 u')
\]

\[
= \begin{cases} (q - 1) & \text{if } \chi_q = 1, \\ 0 & \text{otherwise.} \end{cases}
\]

Thus the contribution from these terms is \( \chi_{N'/q}(q)q^{k-3}(q^2 - 1)E_{\rho} \), if \( \chi_q = 1 \), and 0 otherwise.
Suppose $M\left(\frac{1}{q} \right)$ is integral, $\text{rank}_q\left(M\left(\frac{1}{q} \right) N_{\text{q}}\right) = 1$. Since $q \nmid m_2n_3$, $(M', N')$ is an integral coprime pair where

$$(M'G N'^{-1} G^{-1}) = \left(\frac{1}{q} \right) M \left(\frac{1}{q} \right) N \left(\frac{q}{1} \right) + M \left(\frac{1}{q} \right) Y.$$ 

Note that $\text{rank}_q M' \geq 1$. So

$$X_{(\mathcal{N}_0, \mathcal{N}_1/q, \mathcal{N}_2)}(M', N') = X_{\mathcal{N}_2}(q^2) X_{(\mathcal{N}_0, \mathcal{N}_1/q, \mathcal{N}_2)}(M, N).$$

Reversing, take $(M', N')$ of $\mathcal{N}/q$-type $(\mathcal{N}_0, \mathcal{N}_1/q, \mathcal{N}_2)$, $\text{rank}_q M' \geq 1$. Set

$$(M N) = \left(\frac{1}{q} \right) E \left(\frac{q}{1} \right) M'G \left(\frac{q}{1} \right) (N'^{-1} G^{-1} - M'G Y) \left(\frac{1}{q} \right).$$

$E \in G_1$. Write $EM'G = (m_1 m_2, m_3 n_2), EN'^{-1} G^{-1} = (n_1 n_2, m_4 n_4)$.

To have $\text{rank}_q M = 1$, we need to choose $E, G$ so that $q \nmid m_2$, and to have $N$ integral with $(M, N) = 1$, we need to choose $Y$ so that $n_1 \equiv m_1 y_1 + m_2 y_2 (q)$, $n_3 \equiv m_3 y_1 + m_4 y_2 (q)$.

First suppose $\text{rank}_q M' = 2$. To have rank $M = 1$, for each $E$ we need to choose $G$ so that $q \nmid m_2$ (for each $E$, this gives us $q$ choices for $G$). If $q \mid m_3$ then we choose $y_1$ freely; if $q \nmid m_3$ then $q \mid m_1 m_4$, so we can choose $y_2$ freely (subject to $n_3 \nmid m_4 y_2 (q)$). In either case, we get

$$\sum_Y X_{(1, 1)}(M, N) = \chi_q(-m_2) \sum_Y \chi_q(n_3 - m_3 y_1 - m_4 y_2)$$

$$= \begin{cases} q(q - 1) & \text{if } \chi_q = 1, \\ 0 & \text{otherwise.} \end{cases}$$

So the contribution from these terms is $\chi_{\mathcal{N}_2}(q^2)q^{-1}(q^2 - 1)E_{\rho'}$ if $\chi_q = 1$, 0 otherwise.

Finally, suppose $\text{rank}_q M' = 1$; assume $q$ divides row 2 of $M'$. We have $q$ choices for $E$ so that $q$ does not divide row 1 of $EM'$ (and then $q$ divides row 2 of $EM'$); then we have $q$ choices for $G$ so that $q \nmid m_2$. By symmetry, $q \nmid n_3$. Choose $y_1$ freely, then choose $y_2$ so that $n_1 \equiv m_1 y_1 + m_2 y_2 (q)$. So the contribution from these terms is $\chi_{\mathcal{N}_2}(q^2)q^2E_{\rho'}$. □

**Proposition 3.10.** Let $q$ be a prime dividing $\mathcal{N}_0$. Set

$$\rho' = (\mathcal{N}_0/q, \mathcal{N}_1, \mathcal{N}_2), \quad \rho'' = (\mathcal{N}_0/q, \mathcal{N}_1, q\mathcal{N}_2).$$

Then

$$E_{\rho'} | T_1(q^2) = \begin{cases} \chi_{\mathcal{N}_2}(q^2)(q + 1)E_{\rho'} & \text{if } \chi_q = 1, \\ q^{-1} \chi_{\mathcal{N}_2}(q)q^{k-1} + \chi_{\mathcal{N}_2}(q^2)(q - 1)E_{\rho''} & \text{if } \chi_q^2 = 1, \chi_q \neq 1, \\ \chi_{\mathcal{N}_2}(q^2)(q + 1)E_{\rho'} & \text{if } \chi_q^2 
eq 1. \end{cases}$$

**Proof.** Take $(M N)$ of $\mathcal{N}$-type $\rho$. So $\text{rank}_q M = 0$, $\text{rank}_q M\left(\frac{1}{q} \right) \leq 1$. 

First suppose $(M', N') = 1$ where
\[
(M' \Gamma N'^{-1} G^{-1}) = \left( M \left( \frac{1}{q} \ 1 \right) N \left( \begin{array}{c} q \\ 1 \end{array} \right) + M \left( \frac{1}{q} \ 1 \right) Y \right),
\]
Since $\text{rank}_q N \left( \begin{array}{c} q \\ 1 \end{array} \right) = 1$, we must have $\text{rank}_q M' = 1$. Also,
\[
\chi(N_0/q, N_1, N_2) (M', N') = \chi(N_0/q) \chi(N_0/q, N_1, N_2) (M, N).
\]
Reversing, take $(M' N')$ of $\mathcal{N}$-type $\rho'$; set
\[
(M N) = \left( M' \Gamma \left( \begin{array}{c} q \\ 1 \end{array} \right) (N'^{-1} G^{-1} - M' \Gamma Y) \left( \begin{array}{c} \frac{1}{q} \\ 1 \end{array} \right) \right).
\]
Write $M' \Gamma = \left( \frac{m_1}{m_2} \frac{m_3}{m_4} \right)$, $N'^{-1} G^{-1} = \left( \frac{n_1}{n_2} \frac{n_3}{n_4} \right)$; we can assume that $q|(m_3, m_4)$. To have $q|\Gamma M$, choose the unique $G$ so that $q|m_2$; then $q \nmid m_1$, and by symmetry $q|n_3$ (so $q \nmid n_4$). To have $N$ integral, choose $u$ so that $n_1 \equiv m_1 u \ (q)$ and set $y_1 = u + q u'$. For each choice of $u'$, $y_2$, we have $\det N \equiv x - m_1 n_4 u' \ (q)$ where $x$ does not depend on $u'$. Hence, fixing $y_2$,
\[
\sum_{u'} \chi_q(x - m_1 n_4 u') = \begin{cases} \chi_q = 1, & (q - 1) \\ 0, & \text{otherwise}. \end{cases}
\]
So, letting $y_2$ vary modulo $q$, we see the contribution from these terms is
\[
\chi(N_0/q) q^{k-3} \cdot q(q - 1) \cdot \chi_{\rho'} \quad \text{if } \chi_q = 1,
\]
\[
0 \quad \text{otherwise}.
\]
Now say $\text{rank}_q \left( M \left( \begin{array}{c} \frac{1}{q} \\ 1 \end{array} \right) N \left( \begin{array}{c} q \\ 1 \end{array} \right) \right) = 1$; adjust the equivalence class representative $(M N)$ as necessary so that
\[
(M' \Gamma N'^{-1} G^{-1}) = \left( \frac{1}{q} \ 1 \right) \left( M \left( \begin{array}{c} \frac{1}{q} \\ 1 \end{array} \right) N \left( \begin{array}{c} q \\ 1 \end{array} \right) + M \left( \frac{1}{q} \ 1 \right) Y \right)
\]
is integral. Then $(M', N') = 1$ since $\text{rank}_q N' = 2$. Also,
\[
\chi(N_0/q) \chi(N_0/q, N_1, N_2) (M', N') = \chi(N_0/q, 2) \chi(N_0/q, N_1, N_2) (M, N).
\]
Reversing, take $(M' N')$ of $\mathcal{N}/q$-type $(N_0/q, N_1, N_2)$, and set
\[
(M N) = \left( \begin{array}{c} 1 \\ q \end{array} \right) \left( E M' \Gamma \left( \begin{array}{c} q \\ 1 \end{array} \right) (N'^{-1} G^{-1} - M' \Gamma Y) \left( \begin{array}{c} \frac{1}{q} \\ 1 \end{array} \right) \right).
\]
$E \in \mathcal{G}_1$. Write $E M' \Gamma = \left( \frac{m_1}{m_2} \frac{m_3}{m_4} \right)$, $E N'^{-1} G^{-1} = \left( \frac{n_1}{n_2} \frac{n_3}{n_4} \right)$.
Say $\text{rank}_q M' = 2$. To have $q|\Gamma M$, for each $E$ we need to choose the unique $G$ so that $q|m_2$; thus $q \nmid m_1 m_4$. To have $N$ integral, choose $u$ so that $n_1 \equiv m_1 u \ (q)$; set $y_1 = u + q u'$, $u'$ varying modulo $q$. By symmetry, $m_1 n_3 \equiv m_3 n_1 + m_4 n_2 \ (q)$, so $n_3 \equiv m_1 n_3 \equiv m_3 n_1 + m_1 m_4 n_2 \ (q)$; hence $\det N \equiv -m_1 m_4 (m_1 n_2 - y_2)^2 \ (q)$. Thus summing over $Y$ where $y_1 = u + q u'$,
\[
\sum_Y \chi_q(\det N) = q \chi_q(-m_1 m_4) \sum_{y_2(q)} \chi^2_q(m_1 n_2 - y_2)
\]
\[
= \begin{cases} 
q(q - 1) \chi_q(-m_1 m_4) & \text{if } \chi_q^2 = 1, \\
0 & \text{otherwise}.
\end{cases}
\]

So the contribution from these terms is
\[
\chi_{\mathcal{N}_2}(q^2)q^{-2}(q^2 - 1)\mathbb{E}_{\rho''} \quad \text{if } \chi_q = 1,
\]
\[
\chi_{\mathcal{N}_2}(q^2)q^{k-3} \cdot q^{-k} \cdot q(q - 1)\mathbb{E}_{\rho''} \quad \text{if } \chi_q \neq 1, \chi_q^2 = 1,
\]
otherwise.

Now say \(\text{rank}_q M' = 1\); we can assume \(q\) divides row 2 of \(M'\). For \(q\) choices of \(E\) we have \(q\) dividing row 2 of \(EM'\). Then to have \(q\mid M\), choose the unique \(G\) so that \(q\mid m_2\) and \(q \nmid m_1\) and by symmetry, \(q\mid n_3\). But then \(q\) divides row 2 of \((M, N)\), so \((M, N) \neq 1\). With the 1 other choice of \(E\), we have \(q\) dividing row 1 of \(EM'\). Then to have \(N\) integral, choose the unique \(G\) so that \(q\mid n_1\). Then \(q \nmid n_2\) (since \((M', N') = 1\)), so by symmetry, \(q\mid m_4\) (and hence \(q \nmid m_3\)). Thus
\[
\sum_Y \chi_q(\det N) = \sum_Y \chi_q(-n_2(n_3 - m_3 y_1))
\]
\[
= \begin{cases} 
q^3 & \text{if } \chi_q = 1, \\
0 & \text{otherwise}.
\end{cases}
\]

So the contribution from these terms is
\[
\chi_{\mathcal{N}_2}(q^2)q^{k-3} \cdot q^{-k} \cdot q^3(q - 1)\mathbb{E}_{\rho''} \quad \text{if } \chi_q = 1,
\]
otherwise.

Finally, say \(\text{rank}_q M' = 0\). So to have \(N\) integral, for each \(E\) we need to choose the unique \(G\) so that \(q\mid n_1\). Then \(q \nmid n_2 n_3\), and for each choice of \(Y\) we have \(N\) integral with \(\det N \equiv \det N' \ (q)\). Hence the contribution from these terms is \(\chi_{\mathcal{N}_2}(q^2)q^{k-3} \cdot q^{-k} \cdot q^3(q + 1)\mathbb{E}_{\rho''}\). \(\square\)

With these results, we now construct a basis of simultaneous Hecke eigenforms.

**Theorem 3.11.** Take square-free \(\mathcal{N} \in \mathbb{Z}_+\) and a Dirichlet character \(\chi\) modulo \(\mathcal{N}\) so that \(\chi(-1) = (-1)^k\). There is a basis
\[
\{\overline{\mathcal{E}}_{(\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2)}: \mathcal{N}_0 \mathcal{N}_1 \mathcal{N}_2 = \mathcal{N}, \ \chi^2_{\mathcal{N}_1} = 1\}
\]
for the space \(\mathcal{E}^{(2)}_k(\mathcal{N}, \chi)\) of degree 2 Siegel Eisenstein series of weight \(k\), level \(\mathcal{N}\), character \(\chi\) so that for any prime \(p\), \(\overline{\mathcal{E}}_{(\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2)}|T(p) = \lambda(p)\overline{\mathcal{E}}_{(\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2)}\) and \(\overline{\mathcal{E}}_{(\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2)}|T_1(p^2) = \lambda_1(p^2)\overline{\mathcal{E}}_{(\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2)}\) where
\[
\lambda(p) = (\chi_{\mathcal{N}_0}(p)p^{k-1} + \chi_{\mathcal{N}_1}(p)\chi_{\mathcal{N}_2}(p))(\chi_{\mathcal{N}_0}(p)p^{k-2} + \chi_{\mathcal{N}_2}(p))
\]
and
\[
\lambda_1(p^2) = (p + \chi_{\mathcal{N}_1}(p^2))(\chi_{\mathcal{N}_0}(p^2)p^{2k-3} + \chi(p)p^{k-3}(p - 1) + \chi_{\mathcal{N}_2}(p^2)).
\]
Proof. Fix a partition $\rho = (N_0, N_1, N_2)$ of $\mathcal{N}$. For $q$ an odd prime dividing $\mathcal{N}$, set $\epsilon = (\frac{-1}{q})$, and set

$$a_\rho(q) = \begin{cases} \frac{-\chi_{N_1}N_2(q)q^{(q-1)\epsilon(q-1)}}{\chi_{N_0}(q)q^{(q-1)\epsilon}-\chi_{N_1}N_2(q)} & \text{if } \chi_q = 1, \\ 0 & \text{otherwise}; \end{cases}$$

$$b_\rho(q) = \begin{cases} \frac{-\chi_{N_2}(q^2)q^{(q-1)(\chi_{N_0}(q)q^{(q-3)})\chi_{N_1}N_2(q)}}{(\chi_{N_0}(q)q^{(q-1)\epsilon}-\chi_{N_2}(q))^2} & \text{if } \chi_q = 1, \\ \frac{-\chi_{N_1}N_2(q)(q^2-\chi_{N_2}(q))}{\chi_{N_0}(q)q^{(q-1)\epsilon}-\chi_{N_2}(q^2)} & \text{if } \chi_q \neq 1, \chi_q^2 = 1, \\ 0 & \text{otherwise}; \end{cases}$$

$$c_\rho(q) = \begin{cases} \frac{-\chi_{N_2}(q^2)q^{(q-2)(q-1)}}{\chi_{N_0}(q)q^{(q-1)\epsilon}-\chi_{N_2}(q)} & \text{if } \chi_q = 1, \\ 0 & \text{otherwise}. \end{cases}$$

Extend these functions multiplicatively, and set

$$\tilde{E}_\rho = \sum_{Q_0Q_0'|N_0 \atop Q_1|N_1} a_\rho(Q_0)b_\rho(Q_0')c_\rho(Q_1)E_{(N_0/(Q_0Q_0'),Q_0N_1/Q_1,Q_0Q_1N_2)}.$$  

Since $a_\rho(Q_0) = 0$ unless $\chi_q = 1$, $b_\rho(Q_0') = 0$ unless $\chi_q^2 = 1$, and $c_\rho(Q_1) = 0$ unless $\chi_q = 1$, Propositions 3.3, 3.4, 3.5, and 3.8 show that $\tilde{E}_\rho$ is an eigenform for all $T(p)$, $T_1(p^2)$ where $p \nmid N_0N_1$, with eigenvalues as claimed in the theorem. For a prime $q|N_1$, write

$$\tilde{E}_\rho = \sum_{Q_0Q_0'|N_0 \atop Q_1|N_1/q} a_\rho(Q_0)b_\rho(Q_0')c_\rho(Q_1)$$

$$\cdot \left( \tilde{E}_{(N_0/(Q_0q^\epsilon),Q_0N_1/Q_1,q_0Q_1N_2)} + c_\rho(q)E_{(N_0/(Q_0q^\epsilon),Q_0N_1/qQ_1,q_0Q_1N_2)} \right).$$

Propositions 3.5, 3.6, 3.8 and 3.9 show that $\tilde{E}_\rho$ is an eigenform for $T(q)$ and $T_1(q^2)$, with eigenvalues as claimed. For $q|N_0$, using Propositions 3.5 through 3.10 and a similar rearrangement of the sum defining $\tilde{E}_\rho$, we find $\tilde{E}_\rho$ is an eigenform for $T(q)$ and $T_1(q^2)$, with eigenvalues as claimed. \[ \square \]

Note that for $q$ a prime dividing $\mathcal{N}$ with $\chi_q^2 = 1$, Propositions 3.6, 3.7, 3.9 and 3.10 give us Hecke relations among Eisenstein series. In particular, when cond $\chi^2 < \mathcal{N}$ we can generate some of the Eisenstein series from $E_{(\mathcal{N},1,1)}$. To see this, let $q$ be a prime dividing $\mathcal{N}$ so that $\chi_q^2 = 1$. If $\chi_q = 1$, set $c(q) = \frac{q^2}{(q-1)(\chi_q^2/q)q^{(q-1)}}$,

$$S_1(q) = c(q)\left[ T_1(q^2) - q^{-1}(q+1)T(q) - q^{-1}(q^2-1) \right]$$

and

$$S_2(q) = c(q)\left[ (\chi_{\mathcal{N}/q}(q)q^{k-1}+1)T(q) - T_1(q^2) - q(\chi_{\mathcal{N}/q}(q)q^{k-2} - 1) \right];$$

if $\chi_q \neq 1$, set

$$S_2(q) = \frac{(-1)^2}{(q-1)} T_1(q) - T(q).$$

Extending these maps multiplicatively, we have the following.
Theorem 3.12. Suppose $N$ is square-free, $N_0N_1N_2 = N$, $\chi_{N_1} = 1$, and $\chi_{N_2}^2 = 1$. Then

$$E(N_0, N_1, N_2) = E(N, 1, 1)|S_1(N_1)S_2(N_2).$$

In particular, when $\chi = 1$,

$$\left\{E(N, 1, 1)|S_1(N_1)S_2(N_2) : N_1, N_2 \mid N\right\}$$

is a basis for $\mathcal{E}^{(2)}(N, 1)$.

Proof. When $\chi_q = 1$, we use Propositions 3.7, 3.10 to solve for $E(N_0/q, qN_1, N_2)$ and for $E(N_0/qN_1, qN_2)$ in terms of $E(N_0, N_1, N_2)$, presuming $q \mid N_0$. When $\chi_q \neq 1$ we use Proposition 3.7 to get $E(N_0/qN_1, qN_2)$ in terms of $E(N_0, N_1, N_2)$, again presuming $q \mid N_0$. Now a simple induction argument yields the result. □

Remarks. (1) When $f$ is a Siegel modular form with Fourier coefficients $a(T)$, we have $a(TGT) = a(T)$ for all $G \in \text{GL}_2(\mathbb{Z})$ if $k$ is even, and for all $G \in \text{SL}_2(\mathbb{Z})$ if $k$ is odd. Thus we can consider the Fourier series of $f$ to be supported on lattices $\Lambda$ equipped with a positive, semi-definite quadratic form given by $T$ (relative to some basis), with $\Lambda$ oriented if $k$ is odd; for such $\Lambda$ we set $a(\Lambda) = a(T)$. Then by Theorem 6.1 of [4], with $QP$ square-free, the $\Lambda$th coefficient of $f|T_1(Q^{-2})T(P)$ is

$$\sum_{Q, A \subset \Omega \subset \Lambda \atop \{A: \Omega\} = Q} a(\Omega^P),$$

where $\Omega^P$ denotes the lattice $\Omega$ whose quadratic form has been scaled by $P$. (Note that an orientation on $\Lambda$ induces an orientation on $\Omega \subset \Lambda$.)

(2) With $\mathcal{E}$ the degree 2 Eisenstein series of level 1, we have

$$\mathcal{E}(\tau) = \sum_{N_0N_1N_2 = N} E(N_0, N_1, N_2).$$

Thus formulas for the Fourier coefficients of $\mathcal{E}$ together with our Hecke relations can be used to generate Fourier coefficients of all degree 2, square-free level $N$ Eisenstein series with trivial character.

References