RESTRICTING HECKE-SIEGEL OPERATORS TO JACOBI MODULAR FORMS

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Abstract. We compute the action of Hecke operators $T_j^J(p^2)$ on Jacobi forms of "Siegel degree" n and $m \times m$ index M, provided $1 \le j \le n-m$. We find they are restrictions of Hecke operators on Siegel modular forms, and we compute their action on Fourier coefficients. Then we restrict the Hecke-Siegel operators $T(p)$, $T_j(p^2)$ $(n-m < j \le n)$ to Jacobi forms of Siegel degree n, compute their action on Fourier coefficients and on indices, and produce lifts from Jacobi forms of index M to Jacobi forms of index M' where $\det M | \det M'$. Finally, we present an explicit choice of matrices for the action of the Hecke operators on Siegel modular forms, and for their restrictions to Jacobi modular forms.

§1. Introduction and statement of results

In the 1930's [17], [18], [19] Siegel introduced generalized theta series to study representations of quadratic forms by a given (positive definite) quadratic form Q. The variable of such a theta series is a symmetric $n \times n$ complex matrix $\underline{\tau}$ with $\Im \underline{\tau} > 0$, meaning that as a quadratic form, $\Im \underline{\tau}$ is positive definite (this is to ensure the theta series is analytic). Like the classical theta series, the Siegel theta series transforms under a congruence subgroup of the symplectic group $Sp_n(\mathbb{Z})$ (note that $Sp_1(\mathbb{Z}) = SL_2(\mathbb{Z})$). From this, we have that the Siegel theta series attached to Q has a Fourier series supported on symmetric matrices T, and the Fourier coefficient $c(T)$ tells us how many times Q represents T.

This began the study of functions now called Siegel modular forms, which are analytic function in the $n \times n$ variable $\underline{\tau}$ (as above) that behave like the Siegel theta series under the action on $\underline{\tau}$ by a congruence subgroup of the symplectic group. Such a modular form F has a Fourier series; as well, by decomposing the variable $\underline{\tau}$ into a 2×2 matrix of blocks, we can write F as a "Fourier-Jacobi" series where the coefficients are functions of two (matrix) variables. The Fourier-Jacobi coefficients inherit from F certain transformation properties under a subgroup of the congruence subgroup associated to F.

Jacobi modular forms are functions defined to behave like Fourier-Jacobi coefficients of Siegel modular forms under the "Jacobi subgroup" of the symplectic

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group. In the seminal work [5], the authors introduce and study Jacobi modular forms that map $\mathbb{H}\times\mathbb{C}$ into $\mathbb{C}$; similar to elliptic modular forms, these Jacobi forms have a weight, as well as another integer parameter called an index. One of the striking applications of these Jacobi forms is the proof of the Saito-Kurokawa conjecture due to Maass, Andrianov, and Zagier, explicitly identifying certain Siegel modular forms of degree 2 (meaning $\underline{\tau}$ is 2×2) as lifts of integral weight elliptic modular forms (see [13], [14], [15], [2], [20], or chapter II of [5]). (In an interesting intermediate step, Jacobi forms are identified with a subspace of the space of half-integral weight modular forms, which was studied further by Kohnen [10], [11].) In chapter I of [5], two types of "Hecke operators" are introduced; one of these changes the index, and it is this that is used to construct the lift of a Jacobi form to a degree 2 Siegel form.

In [4], Duke and Imamoḡlu gave a new proof of the Saito-Kurokawa correspondence using L-series and a converse theorem due to Imai [9]; they conjectured that this correspondence extends (under mild conditions), relating elliptic modular forms and Siegel modular forms of degree $2n$. Then in [8], Ikeda used representation theory and L-series to prove the Duke-Imamoḡlu conjecture (see also [12], in which Kohnen reformulates Ikeda's formula). Also, Ikeda shows the lift takes Hecke eigenforms to Hecke eigenforms.

In this paper we return to the study of Jacobi modular forms. In particular, we consider Hecke operators on Jacobi forms with variables τ, Z where $\tau\in\mathbb{C}^{r,r}$, $Z\in\mathbb{C}^{m,r}$ with τ symmetric, $\Im\tau>0$; such a Jacobi form has an $m\times m$ index M. We define Hecke maps $T^J(p)$ and $T^J_j(p^2)$ on Jacobi forms, and analyze their action on Fourier coefficients. Here we allow $1\le j\le n$ where $n=r+m$; these Hecke maps are the restrictions of Hecke operators on Siegel modular forms, in the sense that the action of one of these maps is given by a subset of the matrices giving the action of the corresponding Hecke-Siegel operator (Propositions 3.1, 4.1, and 4.2). We find that when $j\le r$ then $T^J_j(p^2)$ is truly an operator on the space of Jacobi modular forms with index M, but for $j>r$, $T^J_j(p^2)$ and $T^J(p)$ map Jacobi forms of index M to Jacobi forms of index M' where $M'\ne M$. This allows us to build lifts of Jacobi forms, raising a question this author is currently unable to answer: Can this lift be used to build a (partial) Fourier series of a Siegel modular form from the Fourier series of a Jacobi modular form (as done in [5] when $r=m=1$)?

We now summarize our results.

As with Siegel modular forms, we can identify the Fourier coefficients of a Jacobi form f with lattices equipped with quadratic forms: Fixing the rank m lattice Δ equipped with the quadratic form given by M, the index of f, we can realize f as a Fourier series supported on lattices $\Lambda\oplus\Delta$; we use $c_\Delta(\Lambda)$ to denote the Fourier coefficient of f corresponding to $\Lambda\oplus\Delta$. Then (Theorem 3.2) we find that for $j\le r=n-m$, the Λth coefficient of $f|T^J_j(p^2)$ is

$$\sum_{\Omega}\chi(p^{e_{j,\Delta}(\Lambda,\Omega)})\;p^{E'_{j,\Delta}(\Lambda,\Omega)}\;\alpha'_{j,\Delta}(\Lambda,\Omega)\;c_\Delta(\Omega);$$

here Ω varies subject to $p\Lambda\oplus\Delta\subseteq\Omega\oplus\Delta\subseteq\frac{1}{p}(\Lambda\oplus\Delta)$, $e_{j,\Delta},E'_{j,\Delta}$ are given in terms of n,j, and the multiplicities of the invariant factors of $\Omega\oplus\Delta$ in $\Lambda\oplus\Delta$, and

$\alpha'_{j,\Delta}(\Lambda,\Omega)$ is a sum of incomplete character sums that partially test for divisibility by p the codimension $n-j$ subspaces of $(\Lambda\oplus\Delta)\cap(\Omega\oplus\Delta)/p(\Lambda+\Omega+\Delta)$ that are independent of $\overline{\Delta}$ (so Ω only appears in our sum if the invariant factors of $\Omega\oplus\Delta$ in $\Lambda\oplus\Delta$ include at least $n-j$ factors 1). As in [7], we complete these character sums by replacing $T_j^J(p^2)$ by $\widetilde{T}_j^J(p^2)$, a simple linear combination of the $T_\ell^J(p^2)$, $0\le\ell\le j$; then (Corollary 3.3) the Λth coefficient of $f|\widetilde{T}_j^J(p^2)$ is

$$\sum_{\Omega}\chi(p^{e_{j,\Delta}(\Lambda,\Omega)})\ p^{E_{j,\Delta}(\Lambda,\Omega)}\ \alpha_{j,\Delta}(\Lambda,\Omega)\ c_\Delta(\Omega)$$

where $\alpha_{j,\Delta}(\Lambda,\Omega)$ is the number of totally isotropic codimension $n-j$ subspaces of $(\Lambda\oplus\Delta)\cap(\Omega\oplus\Delta)/p(\Lambda+\Omega+\Delta)$ that are independent of $\overline{\Delta}$.

When $j>n-m$, $T_j^J(p^2)$ annihilates f unless

$$\begin{pmatrix}\frac{1}{p}I_\ell & \\ & I_{m-\ell}\end{pmatrix} M \begin{pmatrix}\frac{1}{p}I_\ell & \\ & I_{m-\ell}\end{pmatrix}$$

is even integral, where $\ell=j-n+m$; similarly, $T^J(p)$ annihilates f unless $\frac{1}{p}M$ is even integral (Theorem 4.3). Then, quite similar to [5], we can define lifts of f by first conjugating M by $\begin{pmatrix}pI_t & \\ & I_{m-t}\end{pmatrix}$ $(t>\ell)$ and then applying $T_j^J(p^2)$ $(j>n-m)$, or by multiplying M by p and then applying $T(p)$. In Theorem 4.3 we describe the Fourier coefficients of $f|T_j^J(p^2)$ $(n-m<j\le n)$ and of $f|T^J(p)$; in Corollary 4.4 we discuss lifts.

In [7] we analysed the action of Hecke operators on Siegel modular forms by first explicitly describing a set of coset representatives giving the action of the Hecke operators; these matrices are completely determined except for a choice of matrix $G\in GL_n(\mathbb{Z})$. The situation here is the same; thus in Propositions 5.1 and 5.2, we describe how to choose G to get a complete, non-redundant set of matrices for each Hecke operator on Siegel modular forms, and in Propositions 5.3 and 5.4 we describe how to choose G to give us matrices for the corresponding maps on Jacobi modular forms.

§2. Definitions and notation

Let us begin with a brief discussion of Siegel modular forms and the Jacobi subgroup. Then we define Jacobi forms, Hecke-Siegel operators, and their analogues in the Jacobi case.

The degree n symplectic group is

$$Sp_n(\mathbb{Z}) = \left\{\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in GL_{2n}(\mathbb{Z}):\ A\,{}^tB,\ C\,{}^tD \text{ symmetric},\ A\,{}^tD-B\,{}^tC=I\right\}.$$

This group acts on the Siegel upper half-space

$$\mathbb{H}_{(n)}=\{\underline{\tau}=X+iY:\ X,Y\in\mathbb{R}^{n,n}\text{ symmetric},\ Y>0\ \}$$

by fractional linear transformation:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \circ \underline{\tau} = (A\underline{\tau}+B)(C\underline{\tau}+D)^{-1}.$$

(Proposition 1.2.1 of [1] shows $C\underline{\tau}+D$ is invertible; also, $Y>0$ means that as a quadratic form, Y is positive definite.)

Siegel modular forms (defined below) are analytic functions on $\mathbb{H}_{(n)}$ that transform under $Sp_n(\mathbb{Z})$, or under some subgroup of $Sp_n(\mathbb{Z})$ that contains a principle congruence subgroup

$$\Gamma(N)=\Gamma^{(n)}(N)=\{\gamma\in Sp_n(\mathbb{Z}):\ \gamma\equiv I_{2n}\ (\mathrm{mod}\ N)\ \}$$

($N\in\mathbb{Z}$). In this paper we focus on the subgroups

$$\Gamma_0(N)=\Gamma_0^{(n)}(N)=\left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in Sp_n(\mathbb{Z}):\ C\equiv 0\ (\mathrm{mod}\ N)\ \right\}.$$

Definition. A scalar-valued Siegel modular form of (integral) weight k, degree $n(>1)$, level N, and character χ is an analytic function $F:\mathbb{H}_{(n)}\to\mathbb{C}$ so that for all $\gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\Gamma_0^{(n)}(N)$,

$$F(\gamma\circ\underline{\tau})=\chi(\det D)\ \det(C\underline{\tau}+D)^k\ F(\underline{\tau}).$$

(Here χ is a Dirichlet character modulo N.) We let $\mathcal{M}_k(\Gamma_0^{(n)}(N),\chi)$ denote the complex vector space of all weight k, degree n Siegel modular forms with level N and character χ. We sometimes abuse the notation and write $\chi(\gamma)$ for $\chi(\det D)$ where $\gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in Sp_n(\mathbb{Z})$.

Note. When $n=1$, we need the additional condition that

$$\lim_{\tau\to i\infty}(c\tau+d)^{-k}F\left(\frac{a\tau+b}{c\tau+d}\right)<\infty$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\mathbb{Z})$; when $n>1$ this condition is automatic by the Koecher Principle (see Theorems 2.3.1, 2.3.4 of [1]).

To avoid the distraction of the automorphy factor $\det(C\underline{\tau}+D)^k$, we introduce an action of $Sp_n(\mathbb{Z})$ on $F:\mathbb{H}_{(n)}\to\mathbb{C}$, and more generally, an action of

$$GSp_n^+(\mathbb{Q})=\left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in GL_{2n}(\mathbb{Q}): A\,{}^tB,\ C\,{}^tD \text{ symmetric}, \right.$$
$$\left. A\,{}^tD-B\,{}^tC=rI_n,\ r\in\mathbb{Q}_+\ \right\}$$

by setting

$$F|\gamma(\underline{\tau}) = F|_k\gamma(\underline{\tau}) = (\det\gamma)^{k/2}\det(C\underline{\tau}+D)^{-k}F(\gamma\circ\underline{\tau})$$

for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_n^+(\mathbb{Q})$ (note that $GSp_n^+(\mathbb{Q})$ acts on $\mathbb{H}_{(n)}$ by fractional linear transformation).

Since a Siegel form F is analytic and $F(\underline{\tau}+B) = F(\underline{\tau})$ for any symmetric $B \in \mathbb{Z}^{n,n}$, F has a Fourier series:

$$F(\underline{\tau}) = \sum_T c(T)\mathrm{e}\{T\underline{\tau}\}$$

where $\mathrm{e}\{*\} = \exp(\pi i Tr(*))$ and T runs over all symmeric, $n \times n$ even integral matrices with $T \geq 0$. (T even integral means T has integer entries with even diagonal entries, and $T \geq 0$ means that the quadratic form defined by T is positive semi-definite. Note that some authors include a factor of 2 in the definition of $\mathrm{e}\{*\}$, and then the matrices T are half-integral with integral diagonal entries.) Decomposing $\underline{\tau}$ as $\begin{pmatrix} \tau & {}^tZ \\ Z & \tau' \end{pmatrix}$, $\tau \in \mathbb{H}_{(n-m)}$, $\tau' \in \mathbb{H}_{(m)}$, $Z \in \mathbb{C}^{m,n-m}$, we can also write

$$F(\underline{\tau}) = \sum_M f_M(\tau, Z)\mathrm{e}\{M\tau'\}$$

where M runs over $m \times m$, symmetric, even integral matrices, and

$$f_M(\tau, Z) = \sum_{N,R} c_M(N,R)\mathrm{e}\{N\tau + 2\,{}^tRZ\},\ c_M(N,R) = c\begin{pmatrix} N & {}^tR \\ R & M \end{pmatrix}.$$

The functions f_M are called Fourier-Jacobi coefficients; since F has level N and character χ, the f_M inherit from F the following transformations: for

$$\gamma_1 = \begin{pmatrix} A & & B & \\ & I_m & & 0_m \\ C & & D & \\ & 0_m & & I_m \end{pmatrix} \in Sp_n(\mathbb{Z}) \text{ where } C \equiv 0\ (N)$$

(i.e. $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n-m)}(N)$), we have

$$\begin{aligned} &f_M((A\tau+B)(C\tau+D)^{-1}, Z(C\tau+D)^{-1}) \\ &\quad = \chi(\det D)\ \det(C\tau+D)^k\ \mathrm{e}\{MZ(C\tau+D)^{-1}C\ {}^tZ\}\ f_M(\tau, Z), \end{aligned}$$

and for

$$\gamma_2 = \begin{pmatrix} I_{n-m} & & & {}^tV \\ U & I_m & V & W \\ & & I_{n-m} & -{}^tU \\ & & & I_m \end{pmatrix} \in Sp_n(\mathbb{Z})$$

we have

$$f_M(\tau, U\tau + Z + V) = \mathrm{e}\{-M(U\tau\ {}^tU + 2U\ {}^tZ)\}\ f(\tau, Z).$$

Let $Sp^J_{n,m}(\mathbb{Z})$ be the subgroup of $Sp_n(\mathbb{Z})$ generated by all such matrices γ_1, γ_2 with $N = 1$. So

$$Sp^J_{n,m}(\mathbb{Z}) = \left\{ \begin{pmatrix} A & & B & V' \\ U & I_m & V & W \\ C & & D & U' \\ & & & I_m \end{pmatrix} \in Sp_n(\mathbb{Z}) \right\},$$

and for f_M a Fourier-Jacobi coefficient of a level $N(\geq 1)$ Siegel form F, f_M transforms under $\Gamma^J_0(N) = \Gamma_0(N) \cap Sp^J_{n,m}(\mathbb{Z})$. We call $Sp^J_{n,m}(\mathbb{Z})$ the Jacobi subgroup of $Sp_n(\mathbb{Z})$ for indices of size $m \times m$. (Note that many authors instead use a semi-direct product of matrices of $Sp_{n-m}(\mathbb{Z})$ and triples (U, V, W) from $\mathbb{Z}^{m,n-m} \times \mathbb{Z}^{m,n-m} \times \mathbb{Z}^{m,m}_{\mathrm{sym}}$, or replacing W by $\det W$, triples from $\mathbb{Z}^{m,n-m} \times \mathbb{Z}^{m,n-m} \times \mathbb{Z}$.)

As a formal mechanism to ease our notation, we identify a function $f : \mathbb{H}_{(n-m)} \times \mathbb{C}^{m,n-m} \to \mathbb{C}$ and a symmetric $m \times m$ matrix M with the function

$$F_M(\underline{\tau}) = f(\tau, Z)\mathrm{e}\{M\tau'\}$$

where $\underline{\tau} = \begin{pmatrix} \tau & {}^tZ \\ Z & \tau' \end{pmatrix}$; given $\tau \in \mathbb{H}_{n-m}$, $Z \in \mathbb{C}^{m,n-m}$, we can choose τ' so that $\underline{\tau} \in \mathbb{H}_{(n)}$, or simply treat τ' as a formal variable. Then we use our definition of the slash operator $|$ on Siegel forms to induce on f an action of matrices $\gamma \in Sp^J_{n,m}(\mathbb{Z})$, or more generally, of matrices

$$\gamma = \begin{pmatrix} A & & B & V' \\ U & G & V & W \\ C & & D & U' \\ & & & {}^tG^{-1} \end{pmatrix} \in GSp^+_n(\mathbb{Q})$$

$(A, B, C, D\ (n-m) \times (n-m))$ by setting

$$f|_{k,M}\gamma(\tau, Z)\mathrm{e}\{M'\tau\} = F_M|_k\gamma(\underline{\tau})$$

where $M' = {}^tGMG$.

Definition. Suppose M is a symmetric, even integral $m \times m$ matrix, and $f : \mathbb{H}_{(n-m)} \times \mathbb{C}^{m,n-m} \to \mathbb{C}$ is analytic. Then f is a Jacobi modular form of Siegel degree n, weight k, level N, character χ, and index M if

$$f|\gamma = f|_{k,M}\gamma = \chi(\det D)f$$

for all $\gamma = \begin{pmatrix} A & & B & V' \\ U & I & V & W \\ C & & D & U' \\ & & & I \end{pmatrix} \in \Gamma^J_0(N) = \Gamma_0(N) \cap Sp^J_{n,m}(\mathbb{Z})$. Again, we abuse notation and write $\chi(\gamma)$ for $\chi(\det D)$ for γ as above. If $n - m = 1$, we need the additional condition that for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $\lim_{\tau\to i\infty} f|\widetilde{\gamma}(\tau, Z) < \infty$ where $\widetilde{\gamma} = \begin{pmatrix} a & & b & \\ & I_{n-1} & & 0_{n-1} \\ c & & d & \\ & 0_{n-1} & & I_{n-1} \end{pmatrix}$. (When $n-m > 1$, the Koecher Principle tells us this condition is automatically satisfied; see [21]). We let $\mathcal{M}_{k,M}(\Gamma_0^J(N), \chi)$ denote the complex vector space of Jacobi modular forms with Siegel degree n, weight k, level N, character χ, and index M. (By [21], this space is known to be finite dimensional.)

As discussed above, a degree n Siegel modular form F has a Fourier series expansion supported on symmetric, even integral $n \times n$ matrices T with $T \geq 0$. We consider each T to be a quadratic form on a rank n $\mathbb{Z}$-lattice Λ relative to some basis for Λ. As T varies, the pair (Λ, T) varies over all isometry classes of rank n lattices with even integral positive semi-definite quadratic forms. Also, the isometry class of (Λ, T) is that of (Λ, T') if and only if $T' = T[G]$ for some $G \in GL_n(\mathbb{Z})$ (here $T[G] = {}^tGTG$). Since $F(\underline{\tau}[G]) = \chi(\det G)\,(\det G)^k\,F(\underline{\tau})$ for all $G \in GL_n(\mathbb{Z})$, it follows that $c(T[G]) = \chi(\det G)\,(\det G)^k\,c(T)$ for all $G \in GL_n(\mathbb{Z})$, where $c(T)$ denotes the Tth Fourier coefficient of F. Hence, equipping the lattices Λ with an orientation when $\chi(-1)\,(-1)^k \neq 1$, we write $c(\Lambda) = c(T)$ where T is any matrix representing the quadratic form on Λ (denoted $\Lambda \simeq T$), and then we have

$$F(\underline{\tau}) = \sum_{\mathrm{cls}\Lambda} c(\Lambda)\mathrm{e}^*\{\Lambda\underline{\tau}\} \text{ where } \mathrm{e}^*\{\Lambda\underline{\tau}\} = \sum_G \mathrm{e}\{T[G]\underline{\tau}\};$$

here clsΛ varies over all isometry classes of positive semi-definite $\mathbb{Z}$-lattices of rank n, G varies over $O(T)\backslash GL_n(\mathbb{Z})$ if Λ is not oriented, and over $O^+(T)\backslash SL_n(\mathbb{Z})$ if Λ is oriented, $O(T)$ the orthogonal group of T, $O^+(T) = O(T) \cap SL_n(\mathbb{Z})$. We do something similar for Jacobi forms: First let

$$GL^J_{n,m}(\mathbb{Z}) = \left\{ \begin{pmatrix} G_1 & 0 \\ U & I_m \end{pmatrix} \in GL_n(\mathbb{Z}) \right\},$$

$SL^J_{n,m}(\mathbb{Z}) = GL^J_{n,m}(\mathbb{Z}) \cap SL_n(\mathbb{Z})$. Then with Δ a rank m $\mathbb{Z}$-lattice such that $\Delta \simeq M$,

$$F_M(\underline{\tau}) = f(\tau, Z)\mathrm{e}\{M\tau'\} = \sum_\Lambda c_\Delta(\Lambda)\mathrm{e}^*_M\{(\Lambda \oplus \Delta)\underline{\tau}\}$$

where Λ varies so that $\Lambda \oplus \Delta$ varies over all rank n, even integral, positive semi-definite isometry classes, oriented when $\chi(-1)\,(-1)^k \neq 1$,

$$\mathrm{e}^*_M\{(\Lambda \oplus \Delta)\underline{\tau}\} = \sum_G \mathrm{e}\{T[G]\underline{\tau}\}$$

with $\Lambda \oplus \Delta \simeq T = \begin{pmatrix} * & * \\ * & M \end{pmatrix}$, G varying over $(O(T) \cap GL^J_{n,m}(\mathbb{Z}))\backslash GL^J_{n,m}(\mathbb{Z})$ when k is even, and over $(O(T) \cap SL^J_{n,m}(\mathbb{Z}))\backslash SL^J_{n,m}(\mathbb{Z})$ when k is odd.

Now we define Hecke operators $T(p), T_j(p^2)$ on Siegel forms and their analogues $T^J(p), T_j^J(p^2)$ on Jacobi forms $(1 \le j \le n)$. As we will see, the $T_j^J(p^2)$ are indeed operators when $j \le n-m$ (meaning they are linear maps taking $\mathcal{M}_{k,M}(\Gamma_0^J(N),\chi)$ into itself), but $T^J(p)$ and $T_j^J(p^2)$, $j > m-n$, change the index. First, set $\Gamma = \Gamma_0(N) = \Gamma_0^{(n)}(N)$. Take $1 \le j \le n$, and set $\underline{\delta} = \begin{pmatrix} pI_n & \\ & I_n \end{pmatrix}$, $\delta_j = \begin{pmatrix} pI_j & \\ & I_{n-j} \end{pmatrix}$, $\underline{\delta}_j = \begin{pmatrix} \delta_j & \\ & \delta_j^{-1} \end{pmatrix}$; then set $\Gamma' = \underline{\delta}\Gamma\underline{\delta}^{-1}$, $\Gamma'_j = \underline{\delta}_j\Gamma\underline{\delta}_j^{-1}$. For $F \in \mathcal{M}_k(\Gamma,\chi)$, we set

$$F|T(p) = p^{n(k-n-1)/2} \sum_{\gamma} \overline{\chi}(\gamma) F|\underline{\delta}^{-1}\gamma$$

where γ varies over representatives for $(\Gamma' \cap \Gamma)\backslash\Gamma$; we set

$$F|T_j(p^2) = \sum_{\gamma} \overline{\chi}(\gamma) F|\underline{\delta}_j^{-1}\gamma$$

where γ varies over representatives for $(\Gamma'_j \cap \Gamma)\backslash\Gamma$. For $f \in \mathcal{M}_{k,M}(\Gamma^J,\chi)$, $\Gamma^J = \Gamma_0^J(N) = \Gamma_0(N) \cap Sp_{n,m}^J(\mathbb{Z})$, we define

$$f|T(p) = p^{n(k-n-1)/2} \sum_{\gamma} \overline{\chi}(\gamma) f|\underline{\delta}^{-1}\gamma$$

where γ varies over representatives for $(\Gamma' \cap \Gamma^J)\backslash\Gamma^J$, and

$$f|T_j^J(p^2) = \sum_{\gamma} \overline{\chi}(\gamma) f|\underline{\delta}_j^{-1}\gamma$$

where γ varies over representatives for $(\Gamma'_j \cap \Gamma^J)\backslash\Gamma^J$.

Proposition 2.1. *Say $f \in \mathcal{M}_{k,M}(\Gamma_0^J(N),\chi)$. Then for $1 \le j \le n-m$ and p prime, $f|T_j^J(p^2) \in \mathcal{M}_{k,M}(\Gamma_0^J(N),\chi)$. For $n-m < j \le n$, set $\ell = j-n+m$ and*

$$M_j = \begin{pmatrix} \frac{1}{p}I_\ell & \\ & I_{m-\ell} \end{pmatrix} M \begin{pmatrix} \frac{1}{p}I_\ell & \\ & I_{m-\ell} \end{pmatrix};$$

if M_j is even integral then $f|T_j^J(p^2) \in \mathcal{M}_{k,M_j}(\Gamma_0^J(N),\chi)$. If $\frac{1}{p}M$ is even integral then $f|T^J(p) \in \mathcal{M}_{k.M/p}(\Gamma_0^J(N),\chi)$.

Remark. In Theorem 4.3 we will prove that if $j > n-m$ and M_j is not even integral, then$f|T_j^J(p^2) = 0$, and that if $\frac{1}{p}M$ is not even integral then $f|T^J(p) = 0$.

Proof. Since $\det(C\underline{\tau}+D)^{-k}$ and $\gamma \circ \underline{\tau}$ are rational functions in the entries of $\underline{\tau}$, and f is analytic, $f|T_j^J(p^2)$ and $f|T^J(p)$ are analytic. Just as with elliptic modular forms, these maps preserve the analyticity at the cusps when $n-m=1$.

We now show $f|T_j^J(p^2)$ and $f|T^J(p)$ transform as claimed.

Set $F(\underline{\tau}) = f(\tau, Z)\mathrm{e}\{M\tau'\}$; suppose first that $j \le n-m$. For $\gamma' \in \Gamma'_j \cap \Gamma^J$ and $\gamma = \underline{\delta}_j \gamma' \underline{\delta}_j^{-1}$, we have

$$F|\underline{\delta}_j^{-1}|\gamma' = F|\gamma|\underline{\delta}_j^{-1} = \chi(\gamma)F|\underline{\delta}_j^{-1} = \chi(\gamma')F|\underline{\delta}_j^{-1};$$

so $f|T_j^J(p^2)$ is well-defined. Also, for $\{\gamma\}$ a set of representatives for $(\Gamma'_j \cap \Gamma^J)\backslash\Gamma^J$ and $\gamma' \in \Gamma^J$, we know $\{\gamma\gamma'\}$ is also a set of coset representatives for $(\Gamma'_j \cap \Gamma^J)\backslash\Gamma^J$, and so

$$\begin{aligned} f|T_j^J(p^2)|\gamma'(\tau, Z)\mathrm{e}\{M\tau'\} &= F|T_j^J(p^2)|\gamma'(\underline{\tau}) \\ &= \sum_\gamma \overline{\chi}(\gamma)\ F|\underline{\delta}_j^{-1}\gamma|\gamma'(\underline{\tau}) \\ &= \chi(\gamma') \sum_\gamma \overline{\chi}(\gamma\gamma')\ F|\underline{\delta}_j^{-1}\gamma\gamma'(\underline{\tau}) \\ &= \chi(\gamma')\ F|T_j^J(p^2)(\underline{\tau}) \\ &= \chi(\gamma')\ f|T_j^J(p^2)(\tau, Z)\mathrm{e}\{M\tau'\}. \end{aligned}$$

Consequently, when $j \le n-m$, $f|T_j^J(p^2)$ transforms under Γ^J with character χ, weight k and index M.

Now say $j > n-m$; as with $j \le n-m$, $f|T_j^J(p^2)$ is well-defined. However,

$$F|T_j^J(p^2)(\underline{\tau}) = f|T_j^J(p^2)(\tau, Z)\mathrm{e}\{M'\tau'\}.$$

Consequently, when $j > n-m$, $f|T_j^J(p^2)$ transforms under Γ^J with weight k, character χ, and index M'.

Similarly, $f|T^J(p)$ is well-defined, and

$$F|T^J(p)(\underline{\tau}) = f|T^J(p)(\tau, Z)\mathrm{e}\left\{\frac{1}{p}M\tau'\right\};$$

so $f|T^J(p)$ transforms under Γ^J with weight k, character χ, and index $\frac{1}{p}M$. □

§3. Hecke-Jacobi operators

We first find a set of matrices giving the action of $T_j^J(p^2)$ on $f \in \mathcal{M}_{k,M}(\Gamma_0^J(N), \chi)$ when M is $m \times m$, and $j \le n-m$ (Proposition 3.1); these matrices are uniquely determined up to certain choices of $G \in G_{n,m}^J(\mathbb{Z})$ (defined in the preceding section). Then we analyse the action of $T_j^J(p^2)$ on the Fourier coefficients of f (Theorem 3.2). The formulas in Theorem 3.2 involve incomplete character sums, so as in [7], we complete these by replacing $T_j^J(p^2)$ by a weighted average of $T_\ell^J(p^2)$, $\ell \le j$; in Corollary 3.3 we describe the action on Fourier coefficients of the modified operators.

Proposition 3.1. *Let $f \in \mathcal{M}_{k,M}(\Gamma_0^J(N),\chi)$ where M is $m\times m$, $\Gamma_0^J(N)\subseteq Sp_n(\mathbb{Z})$, and let p be a prime; fix j so that $1\le j\le n-m$. Let Λ,Δ be fixed reference lattices of ranks $n-m, m$ respectively.*

(a) If $p\nmid N$ then

$$f|T_j^J(p^2)=\sum_{(\Omega,\Lambda_1,Y)}\chi(detD/detG)\ f|\underline{\delta}_j^{-1}\begin{pmatrix}D & Y\\ & D^{-1}\end{pmatrix}\begin{pmatrix}G^{-1} & \\ & {}^tG\end{pmatrix}$$

where Ω,Λ_1 vary subject to

$$p\Lambda\oplus\Delta\subseteq\Omega\oplus\Delta\subseteq\frac{1}{p}(\Lambda\oplus\Delta),$$

and $\overline{\Lambda}_1$ is a codimension $n-j$ subspace of $(\Lambda\oplus\Delta)\cap(\Omega\oplus\Delta)/p(\Lambda+\Omega+\Delta)$ that is independent of $\overline{\Delta}$. Here

$$D=D(\Omega)=\begin{pmatrix}I_{r_0} &&&\\ & pI_{r_1} &&\\ && p^2I_{r_2} &\\ &&& I_{n-j}\end{pmatrix}$$

is an $n\times n$ matrix, and $G=G(\Omega,\Lambda_1)\in GL_{n,m}^J(\mathbb{Z})$ so that

$$\Omega\oplus\Delta=(\Lambda\oplus\Delta)GD^{-1}\delta_j,\ \Lambda_1=(\Lambda\oplus\Delta)G\begin{pmatrix}0_{r_0}&&\\ & I_{r_1} & \\ && 0\end{pmatrix};$$

also,

$$Y=\begin{pmatrix}Y_0 & {}^tY_2 & 0 & {}^tY_3\\ pY_2 & Y_1 && \\ 0 &&&\\ Y_3 &&&\end{pmatrix}$$

is integral with Y_0 symmetric, $r_0\times r_0$, varying modulo p^2, Y_1 symmetric, $r_1\times r_1$ varying modulo p with $p\nmid detY_1$, Y_2 $r_1\times r_0$ varying modulo p, Y_3 $(n-j)\times r_0$ varying modulo p.

(b) If $p|N$ then

$$f|T_j^J(p^2)=\sum_{(\Omega,Y)}\overline{\chi}(detG)\ f|\underline{\delta}_j^{-1}\begin{pmatrix}I & Y\\ & I\end{pmatrix}\begin{pmatrix}G^{-1} & \\ & {}^tG\end{pmatrix}$$

where Ω varies so that for some $G=G(\Omega)\in GL_{n,m}^J(\mathbb{Z})$, $\Omega\oplus\Delta=(\Lambda\oplus\Delta)G\delta_j$, and $Y=\begin{pmatrix}Y_0 & {}^tY_3\\ Y_3 & \end{pmatrix}$ is integral with Y_0 symmetric, $j\times j$, varying modulo p^2, Y_3 $(n-j)\times j$, varying modulo p.

Proof. We will write $\mathrm{rank}_p A$ to denote the rank of the matrix A over $\mathbb{Z}/p\mathbb{Z}$; similarly, we will write $\mathrm{span}_p A$ to denote the $\mathbb{Z}/p\mathbb{Z}$-span of the columns of the matrix A. The proof parallels that of Proposition 2.1 of [7].

(a) First suppose $p \nmid N$. Choose $M \in \Gamma^J$ and let M_j denote the top j rows of M. So

$$M_j = (A, A'|B, B')$$

where A, B are $j \times (n-m)$, A' is a $j \times m$ matrix of zeros, B' is $j \times m$. Now we choose $G_0' \in GL_{n-m}(\mathbb{Z})$ so that $AG_0' = (A_0, pA_1)$ where A_0 is $j \times r_0$ with $\mathrm{rank}_p A_0 = r_0$, and A_1 is integral. Set $G_0 = \begin{pmatrix} G_0' & \\ & I_m \end{pmatrix}$; then

$$(A, A'|B, B') \begin{pmatrix} G_0 & \\ & {}^tG_0^{-1} \end{pmatrix} = (A_0, pA_1, A'|B_0, B_1, B')$$

where B_0 is $j \times r_0$. Since (A, B) are the top j rows of a matrix in $Sp_{n-m}(\mathbb{Z})$, so are $(A_0, pA_1|B_0, B_1)$ and thus by Lemma 7.2 of [7], $B_0 \subseteq \mathrm{span}_p A_0$ and $\mathrm{rank}_p(A_0, B_1) = j$. Hence $B' \subseteq \mathrm{span}_p(A_0, B_1)$, so we choose an $m \times (n - m - r_0)$ (integral) matrix W_0 so that $B'' = B_1 W_0 + B' \subseteq \mathrm{span}_p A_0$. Also, choose $G_1' \in GL_{n-m-r_0}(\mathbb{Z})$ so that $B_1 G_1' = (B_1', B_3)$ with B_1' of size $j \times (j - r_0)$, $\mathrm{rank}_p(A_0, B_1') = j$, and $B_3 \subseteq \mathrm{span}_p A_0$. Set

$${}^tG_1^{-1} = \begin{pmatrix} I_{r_0} & & \\ & G_1' & W_0 \\ & & I_m \end{pmatrix} \begin{pmatrix} I_{r_0} & & \\ & G_1' & \\ & & I_m \end{pmatrix};$$

then

$$(A_0, pA_1, A'|B_0, B_1, B') \begin{pmatrix} G_1 & \\ & {}^tG_1^{-1} \end{pmatrix} = (A_0, pA_1', pA_3, A'|B_0, B_1', B_3, B'')$$

where A_1', B_1' are $j \times (j - r_0)$, A_3, B_3 are $j \times (n - j - m)$, and $B_3, B'' \subseteq \mathrm{span}_p A_0$. Now choose a permutation matrix E so that

$$(A_0, A_1') \begin{pmatrix} I_{r_0} & \\ & E \end{pmatrix} = (A_0, A_1'', A_2)$$

with A_1'' of size $j \times r_1$, A_2 of size $j \times r_2$, and $\mathrm{rank}_p(A_0, A_1') = \mathrm{rank}_p(A_0, A_1'') = r_0 + r_1$. Now choose X_0, X_1 so that

$$A_0 X_0 + A_1'' X_1 \equiv A_2 \pmod{p},$$

and set

$$G_2 = \begin{pmatrix} I_{r_0} & & \\ & E & \\ & & I_{n-j} \end{pmatrix}, \quad G_3 = \begin{pmatrix} I_{r_0} & & X_0 & \\ & I_{r_1} & X_1 & \\ & & I_{r_2} & \\ & & & I_{n-j} \end{pmatrix}.$$

Then with $G = G_0G_1G_2G_3$, we have $\begin{pmatrix} G & \\ & {}^tG^{-1} \end{pmatrix} \in \Gamma^J$ and so $M \begin{pmatrix} G & \\ & {}^tG^{-1} \end{pmatrix} \in \Gamma^J$ with top j rows

$$(A_0, pA_1'', p^2A_2, pA_3, A'|B_0, B_1'', B_2, B_3, B'')$$

where B_1'' is $j \times r_1$, B_2 is $j \times (j - r_0 - r_1)$.

Next, choose $r_0 \times r_0$ (integral) Y_0' so that $A_0Y_0' - B_0 \equiv 0 \pmod p$, and $(n-j) \times j$ (integral) Y_3 so that $A_0\,{}^tY_3 - (B_3, B'') \equiv 0 \pmod p$ (note that Y_0', Y_3 are uniquely determined modulo p). Since $M \begin{pmatrix} G & \\ & {}^tG^{-1} \end{pmatrix} \in \Gamma^J$, we know $A_0\,{}^tB_0$ is symmetric, so we can choose Y_0' to be symmetric. Then

$$M \begin{pmatrix} G & \\ & {}^tG^{-1} \end{pmatrix} \begin{pmatrix} I_n & -W \\ & I_n \end{pmatrix} \in \Gamma^J \text{ where } W = \begin{pmatrix} Y_0 & 0 & {}^tY_3 \\ 0 & & \\ Y_3 & & \end{pmatrix};$$

with $pB_0' = B_0 - A_0Y_0'$ and $pB''' = B'' - A_0\,{}^tY_3$, we have $(B_0', B_1'') \subseteq \mathrm{span}_p(A_0, A_1'')$ by Lemma 7.2 of [7]. Thus we can choose a symmetric (integral) $(r_0+r_1) \times (r_0+r_1)$ matrix $W' = \begin{pmatrix} Y_0'' & {}^tY_2 \\ Y_2 & Y_1 \end{pmatrix}$ so that

$$(A_0, A_1'')W' - (B_0', B_1'') \equiv 0 \pmod p;$$

note that the matrices Y_0'', Y_1, Y_2 are uniquely determined modulo p. Set $Y_0 = Y_0' + pY_0''$; then with

$$Y = \begin{pmatrix} Y_0 & {}^tY_2 & 0 & {}^tY_3 \\ pY_2 & Y_1 & & \\ 0 & & & \\ Y_3 & & & \end{pmatrix}, \ D = \begin{pmatrix} I_{r_0} & & & \\ & pI_{r_1} & & \\ & & p^2I_{r_2} & \\ & & & I_{n-j} \end{pmatrix},$$

we have

$$\begin{aligned}(A_0, pA_1'', p^2A_2, pA_3, A'|B_0, B_1'', B_2, B_3, B'') \begin{pmatrix} D^{-1} & -{}^tY \\ & D \end{pmatrix} \\ = (A_0, A_1'', A_2, pA_3, A'|p^2B_0'', p^2B_1''', p^2B_2, pB_3', pB''').\end{aligned}$$

Although we will see that with $X = \begin{pmatrix} D & Y \\ & D^{-1} \end{pmatrix}$, $X \begin{pmatrix} G^{-1} & \\ & {}^tG \end{pmatrix}$ gives the action of the coset of M, it is clear that $X \notin \Gamma^J$ unless $D = I$. However, (D', NY') is a symmetric coprime pair, where

$$D' = \begin{pmatrix} I_{r_0} & & \\ & pI_{r_1} & \\ & & p^2I_{r_2} \end{pmatrix}, \ Y' = \begin{pmatrix} Y_0 & {}^tY_2 & 0 \\ pY_2 & Y_1 & 0 \\ 0 & 0 & I_{r_2} \end{pmatrix}.$$

(A pair (R,S) of $j\times j$ matrices is a symmetric coprime pair if $R\,{}^tS$ is symmetric, and $G(R,S)$ integral implies G is integral; by Lemma 2.1.17 of [1], a pair (R,S) of $j\times j$ matrices is symmetric and coprime exactly when (U,V) is the bottom row of blocks of an element of $Sp_j(\mathbb{Z})$, or equivalently, (S,R) is the top row of blocks of an element of $Sp_n(\mathbb{Z})$.) Thus there exist matrices U',V' so that $\begin{pmatrix} D' & Y' \\ NU' & V' \end{pmatrix}\in Sp_j(\mathbb{Z})$, and with

$$\widetilde{Y}=\begin{pmatrix} Y' & {}^tY_3 \\ Y_3 & \end{pmatrix},\ U=\begin{pmatrix} U' & \\ & 0_{n-j} \end{pmatrix},\ V=\begin{pmatrix} V' & U'\,{}^tY_3 \\ & I_{n-j} \end{pmatrix},$$

we have $X'=\begin{pmatrix} D & \widetilde{Y} \\ NU & V \end{pmatrix}\in\Gamma^J$. Also, by Lemma 7.1 of [7], $M(X')^{-1}\in\Gamma_j'\cap\Gamma^J$, and hence X' represents the coset of M. One easily verifies that $\underline{\delta}_j^{-1}X'=X''\underline{\delta}_j^{-1}X$ where $X''\in\Gamma^J$ and $\chi(X'')=1$. Thus

$$\overline{\chi}(X')\ f|\underline{\delta}_j^{-1}X'=\overline{\chi}(\mathrm{det}V)\ f|\underline{\delta}_j^{-1}X=\chi(\mathrm{det}D)\ f|\underline{\delta}_j^{-1}X.$$

This shows that for $j\le n-m$, $p\nmid N$,

$$f|T_j^J(p^2)=\sum_{(G,D,Y)}\chi(\mathrm{det}D)f|\underline{\delta}_j^{-1}\begin{pmatrix} D & Y \\ & D^{-1} \end{pmatrix}\begin{pmatrix} G^{-1} & \\ & {}^tG \end{pmatrix}$$

where (G,D,Y) varies over the triples constructed above. These triples constitute a subset of the triples constructed in [7] for determining the action of $T_j(p^2)$ on Siegel modular forms; in that case we showed that the triples (G,D,Y) are in one-to-one correspondence with triples (Ω,Λ_1,Y) where Ω varies over all lattices such that $p\Lambda\subseteq\Omega\subseteq\frac{1}{p}\Lambda$, Λ is a fixed rank n reference lattice, and $\overline{\Lambda}_1$ varies over all codimension $n-j$ subspaces of $\Lambda\cap\Omega/p(\Lambda+\Omega)$. While the choice of G is not uniquely determined by a coset, the pair $(\Omega,\overline{\Lambda}_1)$ is, and $G=G(\Omega,\overline{\Lambda}_1)$ can be any element of $GL_n(\mathbb{Z})$ so that

$$\Omega=\Lambda GD^{-1}\delta_j,\ \Lambda_1=\Lambda G\begin{pmatrix} 0_{r_0} & & \\ & I_{r_1} & \\ & & 0 \end{pmatrix}.$$

In the Jacobi case, we replace Λ by $\Lambda\oplus\Delta$ where Λ has rank $n-m$, Δ rank m; then the role of Λ_3 is played by $\Lambda_3\oplus\Delta$, and the triples (G,D,Y) constructed above correspond to triples (Ω,Λ_1,Y) where Ω varies so that $p\Lambda\oplus\Delta\subseteq\Omega\oplus\Delta\subseteq\frac{1}{p}(\Lambda\oplus\Delta)$, and $\overline{\Lambda}_1$ varies over all codimension $n-j$ subspaces of

$$(\Lambda\oplus\Delta)\cap(\Omega\oplus\Delta)/p(\Lambda+\Omega+\Delta)$$

that are independent of $\overline{\Delta}$. Here $G\in GL_{n,m}^J(\mathbb{Z})$ so that

$$\Omega\oplus\Delta=(\Lambda\oplus\Delta)GD^{-1}\delta_j,\ \Lambda_1=(\Lambda\oplus\Delta)G\begin{pmatrix} 0_{r_0} & & \\ & I_{r_1} & \\ & & 0 \end{pmatrix}.$$

The proposition now follows for $p \nmid N$.

(b) Now consider the case that $p|N$; again take $M \in \Gamma^J$ and let M_j denote its top j rows. As before, $M_j = (A, A'|B, B')$ where A, B are $j \times n-m$, and A' is a $j \times m$ matrix of zeros. Since $M \in \Gamma^J = \Gamma_0^J(N)$, the upper left $(n-m)\times(n-m)$ block of M must have $\mathbb{Z}/p\mathbb{Z}$-rank $n-m$, and so $\text{rank}_p A$ is maximal, meaning $\text{rank}_p A = j$. Thus we can find $G_0 \in GL_{n-m}(\mathbb{Z})$ so that $AG_0 = (A_0, pA_3)$ where A_0 is $j \times j$ (and A_3 is integral). Then with $G = \begin{pmatrix} G_0 & \\ & I_m \end{pmatrix}$,

$$(A, A'|B, B')\begin{pmatrix} G & \\ & {}^tG^{-1} \end{pmatrix} = (A_0, pA_3, A'|B_0, B_3, B')$$

where B_0 is $j \times j$, A_3, B_3 are $j \times (n-m-j)$, and since $\text{rank}_p A_0 = j$, $B_0, B_3, B' \subseteq \text{span}_p A_0$. Choose symmetric $j \times j$ Y_0' and $(n-j)\times j$ Y_3 so that $A_0Y_0' - B_0 \equiv 0 \pmod{p}$, $A_0\ {}^tY_3 - (B_3, B') \equiv 0 \pmod{p}$. (So Y_0', Y_3 are uniquely determined modulo p.) Then with $pB_0' = -A_0Y_0' + B_0$, choose Y_0'' so that $A_0Y_0'' - B_0' \equiv 0 \pmod{p}$ (so Y_0'' is uniquely determined modulo p); set $Y_0 = Y_0' + pY_0''$, $Y = \begin{pmatrix} Y_0 & {}^tY_3 \\ Y_3 & \end{pmatrix}$. Then

$$(A_0, pA_3, A'|B_0, B_3, B')\begin{pmatrix} I & -Y \\ & I \end{pmatrix} = (A_0, pA_3, A'|p^2B_0'', pB_3'pB'')$$

where B_0'' is $j \times j$. Thus $\begin{pmatrix} I & Y \\ & I \end{pmatrix}\begin{pmatrix} G^{-1} & \\ & {}^tG \end{pmatrix}$ represents the coset of M; the proposition now follows for the case $p|N$. $\square$

We now apply these coset representatives to a Jacobi form to determine the action of the Hecke-Jacobi operators on Fourier coefficients.

Theorem 3.2. *Let $f \in \mathcal{M}_{k,M}(\Gamma_0^J(N), \chi)$ where M is $m \times m$, $\Gamma_0^J(N) \subseteq Sp_n(\mathbb{Z})$, and let p be prime. Let $c_\Delta(\Lambda)$ denote the Λth coefficient of f. Choose $1 \le j \le n-m$, and p prime. The Λth coefficient of $f|T_j^J(p^2)$ is*

$$\sum_{\Omega} \chi(p^{j+r_2-r_0})\ p^{E'_{j,\Delta}(\Lambda,\Omega)}\ \alpha'_{j,\Delta}(\Lambda,\Omega)\ c_\Delta(\Omega);$$

here Ω varies so that for some Λ' with $\Lambda' \oplus \Delta = \Lambda \oplus \Delta$, we have $p\Lambda' \subseteq \Omega \subseteq \frac{1}{p}\Lambda'$. Also, r_0 is the multiplicity of p in $\{\Lambda \oplus \Delta : \Omega \oplus \Delta\}$, r_2 the multiplicity of $\frac{1}{p}$, and $E'_{j,\Delta}(\Lambda,\Omega) = k(r_2 - r_0) + r_0(n - r_2 + 1)$. Also,

$$\alpha'_{j,\Delta}(\Lambda,\Omega) = \sum_{\Lambda_1} \alpha'(\Lambda_1)$$

where $\overline{\Lambda}_1$ varies over all codimension $n-j$ subspaces of $(\Lambda\oplus\Delta)\cap(\Omega\oplus\Delta)/p(\Lambda+\Omega+\Delta)$ that are independent of $\overline{\Delta}$, and with T_1 an $r_1 \times r_1$ symmetric matrix giving the *quadratic form on* Λ_1, $\alpha'(\Lambda_1) = \sum_{Y_1} e\{T_1 Y_1/p\}$ *where* Y_1 *varies over all (integral)* $r_1 \times r_1$ *symmetric matrices modulo* p *with* $p \nmid \det Y_1$.

Remark. When $p|N$ this simplifies to give us that the Λth coefficient of $f|T_j^J(p^2)$ is

$$p^{j(n+1-k)} \sum_{\Omega} c_\Delta(\Omega)$$

where Ω varies subject to $p\Lambda \oplus \Delta \subseteq \Omega \oplus \Delta \subseteq \Lambda \oplus \Delta$ with $[\Lambda \oplus \Delta : \Omega \oplus \Delta] = p^j$.

Proof. Set $F(\underline{\tau}) = f(\tau, Z)\mathrm{e}\{M\tau'\}$ where $\underline{\tau} = \begin{pmatrix} \tau & {}^tZ \\ Z & \tau' \end{pmatrix}$. We will see that the $\Lambda \oplus \Delta$th coefficient of $F|T_j^J(p^2)$ is built out of $\Omega \oplus \Delta$th coefficients of F where each $\Omega \oplus \Delta$ is even integral and

$$\Omega \oplus \Delta = (\Lambda \oplus \Delta) G D^{-1} \delta_j \text{ with } D = \begin{pmatrix} I_{r_0} & & & \\ & pI_{r_1} & & \\ & & p^2 I_{r_2} & \\ & & & I_{n-j} \end{pmatrix}$$

and $G \in GL_{n,m}^J(\mathbb{Z})$ (or $G \in SL_{n,m}^J(\mathbb{Z})$ if k is odd); the sum on Y for this choice of G and D gives us a character sum to test whether $\Lambda \oplus \Delta$ is even integral. (When $p|N$, $r_1 = r_2 = 0$.)

Write $F(\underline{\tau}) = \sum_T c(T)\mathrm{e}\{T\underline{\tau}\}$ where $T = \begin{pmatrix} N & {}^tR \\ R & M \end{pmatrix} \in \mathbb{Z}^{n,n}$ (symmetric). With $c(N, R) = c(T)$, we can write

$$f(\tau, Z) = \sum_{N,R} c(N, R)\mathrm{e}\{N\tau + 2\,{}^tRZ\}.$$

First consider $p \nmid N$. Then

$$F(\underline{\tau})|T_j^J(p^2) = \sum_{\substack{T \\ (\Omega, \Lambda_1', Y)}} \chi(\det D)\det(\delta_j^{-1}D)^k c(T)\mathrm{e}\{T[\delta_j^{-1}DG^{-1}]\underline{\tau}\}\mathrm{e}\{T\delta_j^{-1}YD\delta_j^{-1}\}$$

where Ω, Λ_1, Y vary as in Proposition 3.1, $D = \begin{pmatrix} I_{r_0} & & & \\ & pI_{r_1} & & \\ & & p^2 I_{r_2} & \\ & & & I_{n-j} \end{pmatrix}$ $G = G(\Omega, \Lambda_1') \in GL_{n,m}^J(\mathbb{Z})$ so that

$$\Omega \oplus \Delta = (\Lambda \oplus \Delta) G D^{-1} \delta_j,$$

and

$$\Lambda_1 = (\Lambda \oplus \Delta) G \begin{pmatrix} 0_{r_0} & & \\ & I_{r_1} & \\ & & 0 \end{pmatrix}.$$

Identify T as the matrix for an even integral quadratic form on $\Omega\oplus\Delta$, relative to some ordered bases $(y_1,\dots,y_{n-m})$, $(y_{n-m+1},\dots,y_n)$ for Ω,Δ (resp.). Using this basis for Ω, write $\Omega=\Omega_0\oplus\Omega_1\oplus\Omega_2\oplus\Omega_3$ where $\mathrm{rank}\Omega_i=r_i$ and $r_3=n-m-j$ (so $\Omega_0=\mathbb{Z}y_1\oplus\cdots\oplus\mathbb{Z}y_{r_0}$, etc.). Then $T[\delta_j^{-1}DG^{-1}]$ is the matrix for the quadratic form on $\Lambda\oplus\Delta=(\Omega\oplus\Delta)\delta_j^{-1}DG^{-1}=(\frac{1}{p}\Omega_0\oplus\Omega_1\oplus p\Omega_2\oplus\Omega_3\oplus\Delta)G^{-1}$. Note that since $G\in GL_n(\mathbb{Z})$, $T[\delta_j^{-1}DG^{-1}]$ is even integral if and only if $T[\delta_j^{-1}D]$ is. Also, $\mathrm{e}\{T\delta_j^{-1}YD\delta_j^{-1}\}=\mathrm{e}\{T[\delta_j^{-1}D]D^{-1}Y\}$. Recall that Y is built from the matrices Y_0,Y_1,Y_2,Y_3, so we can split the sum on Y into a product of sums on the Y_i. The sum on Y_0 is a complete character sum modulo p^2, and this sum tests whether the quadratic form on $\frac{1}{p}\Omega_0$ is integral, yielding a contribution of $p^{r_0(r_0+1)}$ or 0. The sums on Y_2,Y_3 test whether the bilinear form between $\frac{1}{p}\Omega_0$ and $\Omega_1\oplus\Omega_3\oplus\Delta$ is integral, yielding a contribution of $p^{r_0(r_1+n-j)}=p^{r_0(n-r_0-r_2)}$ or 0. The sum on Y_1 is an incomplete character sum since $p\nmid\det Y_1$; this sum yields

$$\alpha'(\Omega_1)=\sum_{\substack{Y_1\\ p\nmid\det Y_1}}\mathrm{e}\left\{\frac{1}{p}T_1Y_1\right\}$$

where $\Omega_1\simeq T_1$. Thus we can restrict our attention to those T,D,G where $T'=T[\delta_j^{-1}DG^{-1}]$ is integral.

Making a change of variables, we now identify T as the matrix for a quadratic form on $\Lambda\oplus\Delta$, and $T[GD^{-1}\delta_j]$ as the matrix for a quadratic form on $\Omega\oplus\Delta$. Thus

$$F(\underline{\tau})|T_j^J(p^2)=\sum_{T,G,D}\chi(\det D)p^{E'_{j,\Delta}(\Lambda,\Omega)}\alpha'(\Lambda_1)c_F(T[GD^{-1}\delta_j])\mathrm{e}\{T\underline{\tau}\}$$

where $\Lambda\oplus\Delta\simeq T$, $\Omega\oplus\Delta=(\Lambda\oplus\Delta)GD^{-1}\delta_j=p\Lambda_0\oplus\Lambda_1\oplus\frac{1}{p}\Lambda_2\oplus\Lambda_3\oplus\Delta$, and $E'_{j,\Delta}(\Lambda,\Omega)=k(-j+r_1+2r_2)+r_0(r_0+r_1+n-j+1)=k(r_2-r_0)+r_0(n-r_2+1)$ (since $j=r_0+r_1+r_2$). As discussed in the proof of Proposition 3.1, for each choice of Ω we have various G, one for each choice of Λ_1 (which is the same as Ω_1 in the previous paragraph); Λ_1 varies so that in the quotient $(\Lambda\oplus\Delta)\cap(\Omega\oplus\Delta)/p(\Lambda+\Omega+\Delta)$, $\overline{\Lambda}_1$ varies over all codimension $n-j$ subspaces independent of $\overline{\Delta}$. Thus with

$$\alpha'_{j,\Delta}(\Lambda,\Omega)=\sum_{\Lambda_1}\alpha'(\Lambda_1),$$

we have

$$F(\underline{\tau})|T_j^J(p^2)=\sum_{T}c'_j(T)\mathrm{e}\{T\underline{\tau}\}=\sum_{N,R}c'_j\left(\begin{pmatrix}N & {}^tR\\ R & M\end{pmatrix}\right)\mathrm{e}\{N\tau+2\,{}^tRZ\}\mathrm{e}\{M\tau'\}$$

where

$$c'_j(T)=\sum_{\Omega}p^{E'_{j,\Delta}(\Lambda,\Omega)}\alpha'_{j,\Delta}(\Lambda,\Omega)c_F(T[GD^{-1}\delta_j]),$$

Ω varying as described in Proposition 3.1. Since we have identified $\Lambda \oplus \Delta$ with T and $\Omega \oplus \Delta$ with $T[GD^{-1}\delta_j]$, this yields the result when $p \nmid N$.

Now consider $p|N$. Then

$$F(\underline{\tau})|T_j^J(p^2) = \sum_{\substack{T \\ (\Omega,Y)}} \det\delta_j^{-k}\ c_F(T)\ \mathrm{e}\{T[\delta_j^{-1}G^{-1}]\underline{\tau}\}\ \mathrm{e}\{T\delta_j^{-1}Y\delta_j^{-1}\}$$

where $Y, G = G(\Omega)$ vary as in Proposition 3.1. So the analysis is similar to the case $p \nmid N$, but simpler; adapting the argument for $p \nmid N$ easily yields the result. $\square$

In the above analysis we encounter incomplete character sums when $p \nmid N$. We can complete these by replacing the operator $T_j^J(p^2)$ by

$$\widetilde{T}_j^J(p^2) = p^{j(k-n-1)} \sum_{0\le\ell\le j} \chi(p^{j-\ell})p^{m(j-\ell)}\beta(n-m-\ell, j-\ell)T_\ell^J(p^2)$$

where $\beta(s,r) = \prod_{i=0}^{r-1} \frac{p^{s-i}-1}{p^{r-i}-1}$, which is the number of r-dimensional subspaces of an s-dimensional space over $\mathbb{Z}/p\mathbb{Z}$. Thus, with V, V' vector spaces over $\mathbb{Z}/p\mathbb{Z}$, $\dim V = n-m-r$, $\dim V' = m$, and U a dimension $\ell - r$ subspace of $V \oplus V'$ that is independent of V', the number of ways to extend U to a dimension $j-r$ subspace of $V \oplus V'$ that is independent of V' is $p^{m(j-\ell)}\beta(n-m-\ell, j-\ell)$. Then almost exactly as shown in Theorem 4.1 of [7] (see also Proposition 5.1 [3]), we get

Corollary 3.3. *Let $f \in \mathcal{M}_{k,M}(\Gamma_0^J(N),\chi)$, and let $c_\Delta(\Lambda)$ denote the Λth coefficient of f. Let p be a prime so that $p \nmid N$. Then for $1 \le j \le n-m$, the Λth coefficient of $f|\widetilde{T}_j^J(p^2)$ is*

$$\sum_\Omega \chi(p^{j+r_2-r_0})\ p^{E_{j,\Delta}(\Lambda,\Omega)}\ \alpha_{j,\Delta}(\Lambda,\Omega)\ c_\Delta(\Omega);$$

here Ω varies subject to $p\Lambda \oplus \Delta \subseteq \Omega \oplus \Delta \subseteq \frac{1}{p}(\Lambda \oplus \Delta)$, r_0 is the multiplicity of p in $\{\Lambda \oplus \Delta : \Omega \oplus \Delta\}$, r_2 the multiplicity of $\frac{1}{p}$, $r_1 = j - r_0 - r_2$, $E_{j,\Delta}(\Lambda,\Omega) = k(r_2 - r_0 + j) + r_0(n - r_2 + 1) + r_1(r_1+1)/2 - j(n+1)$, and $\alpha_j(\Lambda,\Omega)$ is the number of totally isotropic, codimension $n-j$ subspaces $\overline{\Lambda}_1$ of $(\Lambda\oplus\Delta)\cap(\Omega\oplus\Delta)/p(\Lambda+\Omega+\Delta)$ that are independent of $\overline{\Delta}$.

§4. Index-changing Hecke maps

In this section we consider $T_j^J(p^2)$ where $n-m < j \le n$, as well as $T^J(p)$. Proposition 2.1 shows that the operators considered in this section change the index of the Jacobi form, and so our algorithm for finding coset representatives giving the action of these operators is similar, but not identical to the algorithm of Proposition 3.1.

Proposition 4.1. *Let $f \in \mathcal{M}_{k,M}(\Gamma_0^J(N),\chi)$ where M is $m\times m$, $\Gamma_0^J(N) \subseteq Sp_n(\mathbb{Z})$, and let p be prime; fix j so that $n-m<j\le n$. Let Λ,Δ be fixed reference lattices of ranks $n-m,m$ respectively. Fix a basis for Δ, and set $\Delta'=\Delta\begin{pmatrix}\frac{1}{p}I_\ell & \\ & I_{n-j}\end{pmatrix}$ where $\ell=j-n+m$.*

(a) If $p\nmid N$ then

$$f|T_j^J(p^2)=\sum_{(\Omega,\Lambda_1,Y)}\chi(detD/detG)\ f|\underline{\delta}_j^{-1}\begin{pmatrix}D & Y\\ & D^{-1}\end{pmatrix}\begin{pmatrix}G^{-1} & \\ & {}^tG\end{pmatrix}$$

where Ω varies subject to

$$p\Lambda\oplus\Delta\subseteq\Omega\oplus\Delta\subseteq\frac{1}{p}(\Lambda\oplus\Delta'),$$

so that for some Λ' with $\Lambda'\oplus\Delta'=\Lambda\oplus\Delta'$, we have $p\Lambda'\subseteq\Omega\subseteq\frac{1}{p}\Lambda'$, and $\overline{\Lambda}_1$ varies over all codimension $n-j$ subspaces of $(\Lambda\oplus\Delta')\cap(\Omega\oplus\Delta)/p(\Lambda+\Omega+\Delta')$ that are independent of $\overline{\Delta}'$. Here

$$D=D(\Omega)=\begin{pmatrix}I_{r_0} &&&\\ & pI_{r_1} &&\\ && p^2I_{r_2} &\\ &&& I_m\end{pmatrix}$$

and $G=G(\Omega,\Lambda_1)\in GL_{n,m}^J(\mathbb{Z})$ so that

$$\Omega\oplus\Delta=(\Lambda\oplus\Delta')GD^{-1}\delta_j,\ \ \Lambda_1=(\Lambda\oplus\Delta')G\begin{pmatrix}0_{r_0} &&\\ & I_{r_1} &\\ && 0\end{pmatrix};$$

also,

$$Y=\begin{pmatrix}Y_0 & Y_2 & & W_1 & Y_3\\ p\ {}^tY_2 & Y_1 & & p\ {}^tW_2 & \\ & & 0 & & \\ {}^tW_1 & W_2 & & W_0 & W_3\\ {}^tY_3 & & & {}^tW_3 & \end{pmatrix}$$

is integral with Y_0 symmetric, $r_0\times r_0$, varying modulo p^2, W_0 symmetric, $\ell\times\ell$, varying modulo p^2, W_1 $r_0\times\ell$, varying modulo p^2, Y_1 symmetric, $r_1\times r_1$, varying modulo p so that $p\nmid detY_1$, Y_2 $r_0\times r_1$, varying modulo p, Y_3 $r_0\times(n-j)$, varying modulo p, W_3 $\ell\times(n-j)$, varying modulo p.

(b) If $p|N$ then

$$f|T_j^J(p^2)=\sum_{(\Omega,Y)}\overline{\chi}(detG)\ f|\underline{\delta}_j^{-1}\begin{pmatrix}I & Y\\ & I\end{pmatrix}\begin{pmatrix}G^{-1} & \\ & {}^tG\end{pmatrix}$$

where Ω *varies so that for some* $G = G(\Omega) \in GL^J_{n,m}(\mathbb{Z})$, $\Omega \oplus \Delta = (\Lambda \oplus \Delta')G\delta_j$, *and* $Y = \begin{pmatrix} Y_0 & {}^tY_3 \\ Y_3 & \end{pmatrix}$ *is integral with* Y_0 *symmetric,* $j \times j$*, varying modulo* p^2*,* Y_3 $(n-j) \times j$*, varying modulo* p*.*

Proof. Take $M \in \Gamma^J = \Gamma^J_0(N)$ and let M_j denote the top j rows of M. So

$$M_j = (A, A_0', A' | B, B_0', B')$$

where A, B are $j \times (n-m)$, A_0', B_0' are $j \times \ell$, $\ell = j - n + m$, A', B' are $j \times (n-j)$, $A_0' = \begin{pmatrix} 0 \\ I_\ell \end{pmatrix}$, and A' is a matrix of zeros. Choose $G_0' \in GL^J_{j,\ell}(\mathbb{Z})$ so that $(A, A_0')G_0' = (A_0, pA_1, A_0')$ where A_0 is $j \times (r_0 - \ell)$ and $\text{rank}_p(A_0, A_0') = r_0$. So (A_0, A_0') plays the role played by A_0 in Proposition 3.1. Then essentially following the proof of Proposition 3.1 (with the added inconvenience that A_0, A_0' are typically not adjacent) yields the result. □

Proposition 4.2. *Let* $f \in \mathcal{M}_{k,M}(\Gamma^J_0(N), \chi)$ *where* M *is* $m \times m$, $\Gamma^J_0(N) \subseteq Sp_n(\mathbb{Z})$*, and let* p *be prime. Let* Λ, Δ *be fixed reference lattices of ranks* $n-m, m$ *respectively.*

(a) If $p \nmid N$ *then*

$$f|T^J(p) = p^{n(k-n-1)/2} \sum_{(\Omega,Y)} \chi(detD/detG) f|\underline{\delta} \begin{pmatrix} D & Y \\ & D^{-1} \end{pmatrix} \begin{pmatrix} G^{-1} & \\ & {}^tG \end{pmatrix}$$

where Ω *varies subject to*

$$\Lambda \oplus \Delta \subseteq \Omega \oplus \Delta \subseteq \frac{1}{p}(\Lambda \oplus \Delta).$$

Here

$$D = D(\Omega) = \begin{pmatrix} I_r & & \\ & pI_{n-r-m} & \\ & & I_m \end{pmatrix}$$

and $G = G(\Omega) \in GL^J_{n,m}(\mathbb{Z})$ *so that* $\Omega \oplus \Delta = (\Lambda \oplus \Delta)GD^{-1}$*; also,*

$$Y = \begin{pmatrix} W_0 & 0 & {}^tW_2 \\ 0 & 0 & 0 \\ W_2 & 0 & W_1 \end{pmatrix}$$

is $n \times n$ *and integral with* W_0 *symmetric,* $r \times r$*, varying modulo* p*,* W_1 *symmetric,* $m \times m$*, varying modulo* p*, and* W_2 $m \times r$*, varying modulo* p*.*

(b) If $p|N$ *then*

$$f|T^J(p) = p^{n(k-n-1)/2} \sum_Y \overline{\chi}(detG) f|\underline{\delta}^{-1} \begin{pmatrix} I & Y \\ & I \end{pmatrix} \begin{pmatrix} G^{-1} & \\ & {}^tG \end{pmatrix}$$

where Y is symmetric, $n\times n$, varying modulo p.

Proof. Take $\gamma\in\Gamma^J=\Gamma_0^J(N)$. Then the top n rows of γ are $(A,A'|B,B')$ where A,B are $n\times(n-m)$, A',B' are $n\times m$, and $A'=\begin{pmatrix}0\\ I_m\end{pmatrix}$. Note that if $p|N$ then we necessarily have $\operatorname{rank}_p(A,A')=n$.

Choose $G\in GL_{n,m}^J(\mathbb{Z})$ so that $(A,A')G=(A_0,pA_1,A')$ with A_0 of size $n\times r$, $\operatorname{rank}_p(A_0,A')=r+m$. Write

$$(A,A'|B,B')\begin{pmatrix}G&\\&{}^tG^{-1}\end{pmatrix}=(A_0,pA_1,A'|B_0,B_1,B')$$

where B_0 is $n\times r$. By Lemma 7.2 of [7], $(B_0,B')\subseteq\operatorname{span}_p(A_0,A')$, so we can choose symmetric $(r+m)\times(r+m)$ matrix W so that

$$(A_0,A')W\equiv-(B_0,B')\ (\mathrm{mod}\ p).$$

Decompose W as $W=\begin{pmatrix}W_0&{}^tW_2\\W_2&W_1\end{pmatrix}$ where W_0 is $r\times r$, W_1 is $m\times m$; define $n\times n$ matrices

$$Y=\begin{pmatrix}W_0&&{}^tW_2\\&0&\\W_2&&W_1\end{pmatrix},\ D=\begin{pmatrix}I_r&&\\&pI_{n-r-m}&\\&&I_m\end{pmatrix}.$$

Then we have

$$(A,A'|B,B')\begin{pmatrix}G&\\&{}^tG^{-1}\end{pmatrix}\begin{pmatrix}D^{-1}&Y\\&D\end{pmatrix}=(A_0,A_1,A'|pB_0',pB_1,pB'').$$

While the choice of G is by no means uniquely determined by γ, the corresponding lattice $\Omega\oplus\Delta$ is, as $p(\Omega\oplus\Delta)=p(\Lambda\oplus\Delta)GD^{-1}$ is the kernel of the homomorphism that takes the ith basis vector of $\Lambda\oplus\Delta$ to the ith column of (A,A') modulo p.

While $\begin{pmatrix}D&-{}^tY\\&D^{-1}\end{pmatrix}\notin\Gamma_0^J(N)$ when $D\neq I$, just as explained in the proof of Proposition 3.1,

$$\begin{pmatrix}D&-{}^tY\\&D^{-1}\end{pmatrix}\begin{pmatrix}G^{-1}&\\&{}^tG\end{pmatrix}$$

gives the action of γ.

Note that when $p|N$, $\operatorname{rank}_p(A,A')=n$ and hence $r=n-m$ and $D=I$. □

Now we evaluate the action of the Hecke maps on Fourier coefficients.

Theorem 4.3. *Let p be prime, and take $f\in\mathcal{M}_{k,M}(\Gamma_0^J(N),\chi)$, and let $c_\Delta(\Omega)$ denote the Ωth coefficient of f where Ω denotes a lattice of rank $n-m$, Δ a (fixed) lattice of rank m.*

(a) Fix j so that $n-m<j\le n$; let $\ell=n-j-m$ and set $\Delta'=\Delta\begin{pmatrix}\frac{1}{p}I_\ell & \\ & I_{n-j}\end{pmatrix}$. *Then if* $M'=M\left[\begin{pmatrix}\frac{1}{p}I_\ell & \\ & I_{n-j}\end{pmatrix}\right]$ *is not even integral,* $f|T_j^J(p^2)=0$*; otherwise, the Λth coefficient of* $f|T_j^J(p^2)$ *is*

$$\sum_{\Omega}\chi(p^{j+r_2-r_0})p^{E'_{j,\Delta}(\Lambda,\Omega)}\alpha'_{j,\Delta}(\Lambda,\Omega)c_\Delta(\Omega)$$

where Ω varies subject to $p\Lambda\oplus\Delta\subseteq\Omega\oplus\Delta\subseteq\frac{1}{p}(\Lambda\oplus\Delta')$ *so that for some Λ' with* $\Lambda'\oplus\Delta'=\Lambda\oplus\Delta'$*, we have* $p\Lambda'\subseteq\Omega\subseteq\frac{1}{p}\Lambda'$*. Also,*

$$\alpha'_{j,\Delta}(\Lambda,\Omega)=\sum_{\Lambda_1}\alpha'(\Lambda_1)$$

with $\overline{\Lambda}_1'$ varying over all codimension $n-j$ subspaces of $(\Lambda\oplus\Delta')\cap(\Omega\oplus\Delta)/p(\Lambda+\Omega+\Delta')$ *that are independent of $\overline{\Delta}'$; here $\alpha'(\Lambda_1)$ is as in Theorem 3.2. Also, with $r_0+\ell$ the multiplicity of p in* $\{\Lambda\oplus\Delta':\Omega\oplus\Delta\}$*, r_2 the multiplicity of $\frac{1}{p}$, we have*

$$E'_{j,\Delta}(\Lambda,\Omega)=k(r_2-r_0-\ell)+(r_0+\ell)(n-r_2+1).$$

(b) The Λth coefficient of $f|T^J(p)$ is 0 if $\frac{1}{p}M$ is not even integral; otherwise, it is

$$\sum_{\Omega}\chi(p^{n-m-r})p^{E_\Delta(\Lambda,\Omega)}c_\Delta(\Omega)$$

where Ω varies subject to $\Lambda\oplus\Delta\subseteq\Omega\oplus\Delta\subseteq\frac{1}{p}(\Lambda\oplus\Delta)$, $\left[\frac{1}{p}(\Lambda\oplus\Delta):\Omega\oplus\Delta\right]=p^{r+m}$*, and* $E_\Delta(\Lambda,\Omega)=k(n-m-r)+(r+m)(r+m+1)/2-n(n+1)/2$*.*

Proof. Set $F(\underline{\tau})=f(\tau,Z)\mathrm{e}\{M'\tau'\}$ where $\underline{\tau}=\begin{pmatrix}\tau & {}^tZ\\ Z & \tau'\end{pmatrix}$.

(a) The $\Lambda\oplus\Delta'$th coefficient of $F|T_j^J(p^2)$ is built out of $\Omega\oplus\Delta$th coefficients of F where each $\Omega\oplus\Delta$ is even integral and

$$\Omega\oplus\Delta=(\Lambda\oplus\Delta')GD^{-1}\delta_j \text{ with } D=\begin{pmatrix}I_{r_0} &&&\\ & pI_{r_1} &&\\ && p^2I_{r_2} &\\ &&& I_m\end{pmatrix}$$

and $G\in GL^J_{n,m}(\mathbb{Z})$. The argument now follows that of the proof for Proposition 3.2, and so is left to the reader.

(b) With G,D,Y varying as in Proposition 4.2,

$$F|T^J(p)(\underline{\tau}) = p^{-n(n+1)/2}\sum_{T,G,D}\chi(\det D)\,(\det D)^k\,c(T)\,\mathrm{e}\left\{\frac{1}{p}T[DG^{-1}]\underline{\tau}\right\}\sum_Y\left\{\frac{1}{p}TY\,{}^tD\right\}.$$

Identify T as the matrix for an even integral lattice on $\Omega\oplus\Delta$, and let $\Lambda\oplus\Delta=(\Omega\oplus\Delta)DG^{-1}$. Thus $\Lambda\oplus\Delta\simeq T[DG^{-1}]$, and $(\Lambda\oplus\Delta)^{1/p}\simeq\frac{1}{p}T[DG^{-1}]$. So the sum on Y tests whether $\frac{1}{p}T[D]$ is even integral, or equivalently, whether $\frac{1}{p}T[DG^{-1}]$ is even integral. Hence the coefficient of $F|T^J(p)$ attached to $(\Lambda\oplus\Delta)^{1/p}$ is 0 if $(\Lambda\oplus\Delta)^{1/p}$ is not even integral, and otherwise it is as claimed in the theorem. □

As in [5], we can first shift Z, thereby changing the index, and then apply a Hecke map that changes the index, lifting a Jacobi modular form from index M to index M'' where $\det M|\det M''$.

Corollary 4.4. *Let $f\in\mathcal{M}_{k,M}(\Gamma_0^J(N),\chi)$.*

(a) Take j so that $n-m<j\le n$, and take $s\ge\ell=j-n+m$. Define g by $g(\tau,Z)=f(\tau,\begin{pmatrix}pI_s&\\&I_{m-s}\end{pmatrix}Z)$. Then $g|T_j^J(p^2)\in\mathcal{M}_{k,M''}(\Gamma_0^J(N),\chi)$ where

$$M''=M\left[\begin{pmatrix}I_\ell&&\\&pI_{s-\ell}&\\&&I_{m-s}\end{pmatrix}\right].$$

If $s=\ell$ then $g|T_j^J(p^2)$ is a multiple of $f|T_{n-m}^J(p^2)$.

(b) Define h by $h(\tau,Z)=f(\tau,pZ)$. Then $h|T^J(p)$ is a Jacobi form with index pM.

Proof. Let $F(\underline{\tau})=f(\tau,Z)\mathrm{e}\{M\tau'\}$ where $\underline{\tau}=\begin{pmatrix}\tau&{}^tZ\\Z&\tau'\end{pmatrix}$.

(a) Set

$$\gamma_s=\begin{pmatrix}I_{n-m}&&\\&pI_s&\\&&I_{m-s}\end{pmatrix};$$

then

$$\begin{aligned}G(\underline{\tau})&=p^{-ks}F|\begin{pmatrix}\gamma_s&\\&\gamma_s^{-1}\end{pmatrix}(\underline{\tau})\\&=f(\tau,\begin{pmatrix}pI_s&\\&I_{m-s}\end{pmatrix}Z)\mathrm{e}\{M'\tau'\}\\&=g(\tau,Z)\mathrm{e}\{M',\tau\}\end{aligned}$$

where $M'=M\left[\begin{pmatrix}pI_t&\\&I_{m-s}\end{pmatrix}\right]$. So G transforms with weight k and character χ under

$$\begin{pmatrix}\gamma_s^{-1}&\\&\gamma_s\end{pmatrix}\Gamma^J\begin{pmatrix}\gamma_s&\\&\gamma_s^{-1}\end{pmatrix}\supseteq\Gamma^J=\Gamma_0^J(N),$$

so $g\in\mathcal{M}_{k,M'}(\Gamma^J,\chi)$. Hence by Proposition 2.1, $g|T_j^J(p^2)\in\mathcal{M}_{k,M''}(\Gamma^J,\chi)$.

Now suppose $s=\ell$ (so $M''=M$). Then with γ running through a set of representatives for $(\Gamma'_j\cap\Gamma^J)\backslash\Gamma^J$, we have

$$p^{-k\ell}G|T_j^J(p^2)=\sum_\gamma F|\begin{pmatrix}\gamma_s & \\ & \gamma_s^{-1}\end{pmatrix}|\underline{\delta}_j^{-1}\gamma=\sum_\gamma F|\underline{\delta}_j^{-1}\gamma.$$

Also, $(\Gamma'_j\cap\Gamma^J)\subseteq(\Gamma'_{n-m}\cap\Gamma^J)$; consequently,

$$F|\begin{pmatrix}\gamma_s & \\ & \gamma_s^{-1}\end{pmatrix}|T_j^J(p^2)=\kappa F|T_{n-m}^J(p^2)$$

where $\kappa=[\Gamma'_{n-m}\cap\Gamma^J:\Gamma'_j\cap\Gamma^J]$.

(b) We have

$$\begin{aligned}H(\underline{\tau})&=p^{-km}\,F|\begin{pmatrix}\gamma_m & \\ & \gamma_m^{-1}\end{pmatrix}(\underline{\tau})\\&=f(\tau,pZ)\mathrm{e}\{p^2M\tau'\}\\&=h(\tau,Z)\mathrm{e}\{p^2M,\tau\}.\end{aligned}$$

So H transforms with weight k and character χ under

$$\begin{pmatrix}\gamma_m^{-1} & \\ & \gamma_m\end{pmatrix}\Gamma^J\begin{pmatrix}\gamma_m & \\ & \gamma_m^{-1}\end{pmatrix}\supseteq\Gamma^J=\Gamma_0^J(N),$$

so $h\in\mathcal{M}_{k,p^2M}(\Gamma^J,\chi)$. Hence by Proposition 2.1, $h|T^J(p)\in\mathcal{M}_{k,pM}(\Gamma^J,\chi)$. □

Remark. For $C_0\in GL_m(\mathbb{Z})$, $C=\begin{pmatrix}I_{n-m} & \\ & C_0\end{pmatrix}$, and $F(\underline{\tau})=f(\tau,Z)\mathrm{e}\{M\tau'\}$ as above,

$$F|\begin{pmatrix}C & \\ & {}^tC^{-1}\end{pmatrix}(\underline{\tau})=f(\tau,{}^tC_0Z)\mathrm{e}\{M[C_0]\tau'\},$$

so $f|\begin{pmatrix}C & \\ & {}^tC^{-1}\end{pmatrix}$ is a Jacobi form with index $M[C_0]$. Hence the above corollary can be used to construct from f Jacobi forms of various indices M'', $\det M|\det M''$.

§5. Explicit choices of matrices for the Hecke-Siegel operators and their restrictions

As discussed earlier, in Propositions 2.1 and 3.1 of [7] we described a set of matrices giving the action of the Hecke operators on Siegel modular forms; these matrices are explicitly given except for a particular choice of the change of basis matrices $G(\Omega,\Lambda_1)$, $G(\Omega)$. Here we construct explicit choices for these G, and then give a description of the matrices G corresponding to the restrictions of the Hecke operators to Jacobi modular forms.

We first introduce some notation. Fix j, n, $1 \le j \le n$. For (nonnegative) integers r_0, r_2 with $r_0 + r_2 \le j$ and $r_1 = j - r_0 - r_2$, we call $\mathcal{P}$ a partition of type (r_0, r_2) for (n, j) if $\mathcal{P}$ is an ordered partition

$$(\{d_1, \dots, d_{r_0}\}, \{b_1, \dots, b_{r_1}\}, \{a_1, \dots, a_{r_2}\}, \{c_1, \cdots, c_{n-j}\})$$

of $\{1, 2, \dots, n\}$. (Note that if some $r_i = 0$ or $n - j = 0$, a set in the partition could be empty.) Given a partition $\mathcal{P}$ of type (r_0, r_2) for (nj), we let $\mathcal{G}_{\mathcal{P}}(j) \subseteq GL_n(\mathbb{Z})$ consist of all matrices $G = (G_0, G_1, G_2, G_3)$ constructed as follows. G_0 is the $n \times r_0$ matrix with s, t-entry 1 if $s = d_t$, and 0 otherwise. G_1 is an $n \times r_1$ matrix with s, t-entry β_{st} where $\beta_{st} = 1$ if $s = b_t$, $\beta_{st} = 0$ if $s < b_t$ or $s = a_i$ (some i) or $s = b_i$ (some $i \ne t$), and otherwise $\beta_{st} \in \{0, 1, \dots, p-1\}$. G_2' is an $n \times r_2$ matrix with s, t-entry α_{st} where $\alpha_{st} = 1$ if $s = a_t$, $\alpha_{st} = 0$ if $s < a_t$ or $s = a_i$ (some $i \ne t$), and otherwise $\alpha_{st} \in \{0, 1, \dots, p-1\}$. G_2'' is an $n \times r_2$ matrix with s, t-entry ρ_{st} where $\rho_{st} \in \{0, 1, \dots, p-1\}$ if $s > a_t$ and $s = d_i$ (some i), and otherwise $\rho_{st} = 0$. $G_2 = G_2' + pG_2''$. G_3 is an $n \times (n - j)$ matrix with s, t-entry γ_{st} where $\gamma_{st} = 1$ if $s = c_t$, $\gamma_{st} \in \{0, 1, \dots, p-1\}$ if $s > c_t$ and $s = d_i$ (some i), and otherwise $\gamma_{st} = 0$.

Note that (G_0, G_1, G_2', G_3) is a (column) permutation of an integral lower triangular matrix with 1's on the diagonal, and thus is an element of $GL_n(\mathbb{Z})$. Also, it is easy to see that there is an elementary matrix E so that

$$(G_0, G_1, G_2', G_3)E = (G_0, G_1, G_2' + pG_2'', G_3) = G,$$

and so $G \in GL_n(\mathbb{Z})$.

We let $\mathcal{G}_{r_0,r_2}(j) = \cup_{\mathcal{P}} \mathcal{G}_{\mathcal{P}}(j)$ where $\mathcal{P}$ varies over all partitions of type (r_0, r_2) for (n, j), and we set

$$D_{r_0,r_2}(j) = \begin{pmatrix} I_{r_0} & & & \\ & pI_{r_1} & & \\ & & p^2 I_{r_2} & \\ & & & I_{n-j} \end{pmatrix}$$

where $r_1 = j - r_0 - r_2$.

Proposition 5.1. *Let p be prime, and j an integer so that $1 \le j \le n$. Let Λ be a fixed reference lattice of rank n.*

(a) If $p \nmid N$, the pairs (Ω, Λ_1) in Proposition 2.1 of [7] are in one-to-one correspondence with the pairs (D, G) where, for some non-negative r_0, r_2 so that $r_0 + r_2 \le j$, $D = D_{r_0,r_2}(j)$ and $G \in \mathcal{G}_{r_0,r_2}(j)$, via the correspondence

$$\Omega = \Lambda G D^{-1} \delta_j, \ \Lambda_1 = \Lambda \begin{pmatrix} 0_{r_0} & & \\ & I_{r_1} & \\ & & 0 \end{pmatrix}$$

where $r_1 = j - r_0 - r_2$.

(b) If $p|N$ then we only need those (Ω, Λ_1) corresponding to $r_0 = j$, $r_2 = 0$.

Proof. (a) In Proposition 2.1 of [7], Ω varies subject to $p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda$, and $\overline{\Lambda}_1$ is a codimension $n-j$ subspace of $\Lambda \cap \Omega/p(\Lambda+\Omega)$. Here we construct all pairs $(\Omega, \overline{\Lambda}_1)$, simultaneously constructing G. It is then evident that the pairs $(\Omega, \overline{\Lambda}_1)$ are in one-to-one correspondence with the elements of $\cup_{r_0+r_2\le j}\mathcal{G}_{r_0,r_2}(j)$.

Notice that when $p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda$, the Invariant Factor Theorem (81:11 of [16]) tells us we have compatible decompositions:

$$\Lambda = \Lambda_0 \oplus \Lambda_1' \oplus \Lambda_2,$$
$$\Omega = p\Lambda_0 \oplus \Lambda_1' \oplus \frac{1}{p}\Lambda_2.$$

On the other hand, given Λ, such an Ω is determined by $\Omega' = \Lambda_2 + p\Lambda$ and $(p\Lambda_1' \oplus \Lambda_2) + p\Omega'$. Also, in $\Lambda \cap \Omega/p(\Lambda+\Omega)$, $\overline{\Lambda}_2 = 0$, so Λ_1 can be chosen so that in $\Omega'/p\Omega'$, $\overline{\Lambda}_1 \subseteq \overline{p\Lambda_1'} \subseteq \overline{p\Lambda}$.

So to begin our construction of Ω, Λ_1 and $G = G(\Omega, \Lambda_1)$, in $\Lambda/p\Lambda$ we choose a dimension r_2 subspace $\overline{C}'$; let $(\overline{v}_1', \dots, \overline{v}_{r_2}')$ be a basis for $\overline{C}'$. Each $\overline{v}_t'$ is a linear combination over $\mathbb{Z}/p\mathbb{Z}$ of the $\overline{x}_i$; by adjusting the $\overline{v}_t'$ we can assume

$$\overline{v}_t' = \overline{x}_{a_t} + \sum_{s>a_t} \overline{\alpha}_{st}\overline{x}_s$$

where $a_1, \dots, a_{r_2}$ and distinct and $\overline{\alpha}_{st} = 0$ if $s = a_i$ (some $i \neq t$). Let $\alpha_{st} \in \{0, 1, \dots, p-1\}$ be a preimage of $\overline{\alpha}_{st}$.

Now let Ω' be the preimage in Λ of $\overline{C}'$. In $\Omega'/p\Omega'$ we will construct a dimension $n-r_0$ subspace $\overline{C}$ so that $\dim(\overline{C} \cap \overline{p\Lambda}) = n - r_0 - r_2$, distiguishing a dimension r_1 subspace $\overline{p\Lambda_1}$ of $\overline{C} \cap \overline{p\Lambda}$. We begin by choosing $\overline{p\Lambda_1}$ to be a dimension r_1 subspace of $\overline{p\Lambda}$; let $\overline{pu_1}, \dots, \overline{pu_{r_1}}$ be a basis for $\overline{p\Lambda_1}$. Since $\overline{px_{a_i}} = 0$ in $\Omega'/p\Omega'$, we can adjust the $\overline{pu_t}$ so that

$$\overline{pu_t} = \overline{px_{b_t}} + \sum_{s>b_t} \overline{\beta}_{st}\overline{px_s}$$

where $b_1, \dots, b_{r_1}$ are distinct, $b_t \neq a_i$ (any i), and $\overline{\beta}_{st} = 0$ if $s = a_i$ (some i) or $s = b_i$ (some $i \neq t$). Let $\beta_{st} \in \{0, 1, \dots, p-1\}$ be a preimage of $\overline{\beta}_{st}$.

Now extend $\overline{p\Lambda_1}$ to a dimension $n-r_0-r_2$ subspace $\overline{p\Lambda_1'}$ of $\overline{p\Lambda}$ in $\Omega'/p\Omega'$. Extend $(\overline{pu_1}, \dots, \overline{pu_{r_1}})$ to a basis

$$(\overline{pu_1}, \dots, \overline{pu_{r_1}}, \overline{pw_1}, \dots, \overline{pw_{n-j}})$$

for $\overline{p\Lambda_1'}$ so that

$$\overline{pw_t} = \overline{px_{c_t}} + \sum_{s>c_t} \overline{\gamma}_{st}\overline{px_s},$$

where $c_1, \dots, c_{n-j}$ are distinct, $c_t \neq a_i, b_i$ (any i), and $\overline{\gamma}_{st} = 0$ if $s = a_i$ (some i), or $s = b_i$ (some $i \neq t$). Let $\gamma_{st} \in \{0, 1, \dots, p-1\}$ be a preimage of $\overline{\gamma}_{st}$.

Now we extend $\overline{p\Lambda_1'}$ to a dimension $n-r_0$ space $\overline{C}$ so that the dimension of $\overline{C}\cap\overline{p\Lambda}$ is $n-r_0-r_2=r_1+n-j$, and we extend $(\overline{pu_1},\dots,\overline{pw_1},\dots)$ to a basis

$$(\overline{pu_1},\dots,\overline{pu_{r_1}},\overline{pw_1},\dots,\overline{pw_{n-j}},\overline{pv_1},\dots,\overline{pv_{r_2}})$$

for $\overline{C}$. Taking $d_1,\dots,d_{r_0}$ so that

$$(\{d_1,\dots,d_{r_0}\},\{b_1,\dots,b_{r_1}\},\{a_1,\dots,a_{r_2}\},\{c_1,\dots,c_{n-j}\})$$

is a partition of $\{1,\dots,n\}$, we can take

$$\overline{v}_t=\overline{v}_t'+\sum_{m=1}^{r_0}\overline{\delta}_{mt}\overline{px_{d_m}}$$

for some $\overline{\delta}_{mt}$; let $\delta_{mt}\in\{0,1,\dots,p-1\}$ be a preimage of $\overline{\delta}_{mt}$.

Now let $p\Omega$ be the preimage in Ω' of $\overline{C}$. So with

$$\begin{aligned}u_t&=x_{b_t}+\sum_{s>b_t}\beta_{st}x_s\ (1\le t\le r_1),\\ v_t&=x_{a_t}+\sum_{s>a_t}\alpha_{st}x_s+p\sum_m\delta_{mt}x_{d_m}\ (1\le t\le r_2),\\ w_t&=x_{c_t}+\sum_{s>c_t}\gamma_{st}x_s\ (1\le t\le n-j),\end{aligned}$$

the vectors

$$(px_{d_1},\dots,px_{d_{r_0}},u_1,\dots,u_{r_1},\frac{1}{p}v_1,\dots,\frac{1}{p}v_{r_2},w_1,\dots,w_{n-j})$$

form a basis for Ω, and $(\overline{u}_1,\dots,\overline{u}_{r_1})$ is a basis for $\overline{\Lambda}_1$ in $\Lambda\cap\Omega/p(\Lambda+\Omega)$.

(b) When $p|N$, we necessarily have $r_1=r_2=0$ and $r_0=j$, since the upper left block of a matrix in $\Gamma_0^{(n)}(N)$ necessarily has rank n over $\mathbb{Z}/p\mathbb{Z}$, and so its top j rows have rank j over $\mathbb{Z}/p\mathbb{Z}$. □

We follow a similar procedure to construct matrices giving the action of $T(p)$ on $\mathcal{M}_k(\Gamma_0^{(n)}(N),\chi)$: For $0\le r\le n$, we let $\mathcal{G}_r$ be the set of matrices G constructed as follows. Let $(\{d_1,\dots,d_r\},\{a_1,\dots,a_{n-r}\})$ be an ordered partition of $\{1,2,\dots,n\}$. G_0 is the $n\times r$ matrix whose s,t-entry is 1 if $s=d_t$, and 0 otherwise. G_1 is an $n\times(n-r)$ matrix whose s,t-entry is α_{st} where α_{st} is 1 if $s=a_t$, $\alpha_{st}=0$ if $s<a_t$ or $s=a_i$ (some $i\ne t$), and $\alpha_{st}\in\{0,1,\dots,p-1\}$ otherwise. $G=(G_0,G_1)\in GL_n(\mathbb{Z})$. Let $D_r=\begin{pmatrix}I_r&\\&pI_{n-r}\end{pmatrix}$.

Proposition 5.2. *Let p be prime, and Λ a fixed reference lattice of rank n.*

(a) If $p \nmid N$, the lattices Ω in Proposition 3.1 of [7] are in one-to-one correspondence with the pairs (D, G) where, for some non-negative $r \le n$, $D = D_r$ and $G \in \mathcal{G}_r$, via the correspondence

$$\Omega = p\Lambda G D^{-1}.$$

(b) If $p|N$ then we only need those Ω corresponding to $r = n$.

Proof. Note that in Proposition 4.2, we used the lattice $\Omega \oplus \Delta$, corresponding not to Ω, but rather to $\frac{1}{p}\Omega$ from Proposition 3.1 of [7].

(a) Using Proposition 3.1 of [7], we only need to show that as G varies over $\mathcal{G}_r$, $\Omega = p\Lambda G D_r^{-1}$ varies once over all lattices Ω where $p\Lambda \subseteq \Omega \subseteq \Lambda$, $[\Lambda : \Omega] = p^r$. So, similar to the proof of Proposition 5.1, we construct all the Ω as well as a specific basis for each Ω.

Let $\overline{C}$ be a dimension $n - r$ subspace of $\Lambda/p\Lambda$. Choose a basis $\overline{v}_1, \dots, \overline{v}_{n-r}$ so that

$$\overline{v}_t = \overline{x}_{a_t} + \sum_{s > a_t} \overline{\alpha}_{st} \overline{x}_s$$

where $a_1, \dots, a_{n-r}$ are distinct, $\overline{\alpha}_{st} = 0$ if $s = a_i$ (some $i \ne t$); for each $\overline{\alpha}_{st}$, take a preimage $\alpha_{st} \in \{0, 1, \dots, p-1\}$. Then with $(\{d_1, \dots, d_r\}, \{a_1, \dots, a + n - r\})$ an ordered partition of $\{1, 2, \dots, n\}$ and G constructed according to our recipe preceding this proposition, we have $\Omega = \Lambda G p D_r^{-1}$.

(b) When $p|N$, we necessarily have $r = n$ since the upper left block of any matrix in $\Gamma = \Gamma_0^{(n)}(N)$ has rank n over $\mathbb{Z}/p\mathbb{Z}$. $\square$

Now we consider the Jacobi case; thus the role played by Λ in the Siegel case is now played by $\Lambda \oplus \Delta$ where Λ has rank $n - m$ and Δ has rank m.

Suppose first $j \le n - m$. We let $\mathcal{G}^J_{r_0, r_2}(j, m)$ be the matrices of $\mathcal{G}_{r_0, r_2}(j)$ that satisfy the additional conditions (i) $c_t = j + t$ for $t > n - j - m$, and (ii) $\gamma_{st} = 0$ for $s \ne c_t$, $t > n - j - m$. Then $\mathcal{G}^J_{r_1, r_2}(j, m)$ is the subset of $\mathcal{G}_{r_0, r_2}(j)$ consisting of those G that fix the basis for Δ under right multiplication of $\Lambda \oplus \Delta$ by G.

Now suppose $j > n - m$. Let $\ell = n - j - m$, and relative to the fixed basis for Δ, write $\Delta = \Delta_0 \oplus \Delta_1$ where Δ_0 has rank ℓ. With $\Delta_0' = \frac{1}{p}\Delta_0$, $\Delta' = \Delta_0' \oplus \Delta_1$, we need G so that

$$\Omega \oplus \Delta = (\Lambda \oplus \Delta') G D_{r_0, r_2}^{-1} \delta_j = p\Lambda_0 \oplus \Lambda_1 \oplus p\Lambda_2 \oplus p\Delta_0' \oplus \Delta_1$$

($r_i = \text{rank}_{\mathbb{Z}} \Lambda_i$). Since Λ_0, Δ_0' are not adjacent, we need to modify how we construct G. (The G we construct will be permutations of a subset of those G of Proposition 5.1.) So with $\mathcal{P}' = \{(d_1, \dots, d_{r_0}), (b_1, \dots, b_{r_1}), (a_1, \dots, a_{r_2})\}$ an ordered partition of $\{1, 2, \dots, n - m\}$, we let $\mathcal{G}^J_{r_0, r_2}(j, m)$ consist of all matrices G constructed as follows. G_0 is the $n \times r_0$ matrix with s, t-entry 1 if $s = d_t$, and 0 otherwise. G_1 is an $n \times r_1$ matrix with s, t-entry β_{st} where $\beta_{st} = 1$ if $s = b_t$, $\beta_{st} = 0$ if $s < b_t$ or $s = a_i$ (some i) or $s = b_i$ (some $i \ne t$), and otherwise $\beta_{st} \in \{0, 1, \dots, p-1\}$. G_2' is an $n\times r_2$ matrix with s,t-entry α_{st} where $\alpha_{st}=1$ if $s=a_t$, $\alpha_{st}=0$ if $s<a_t$ or $s=a_i$ (some $i\neq t$), and otherwise $\alpha_{st}\in\{0,1,\dots,p-1\}$. G_2'' is an $n\times r_2$ matrix with s,t-entry ρ_{st} where $\rho_{st}\in\{0,1,\dots,p-1\}$ if $s>a_t$ and either $s=d_i$ (some i) or $n-m<s\le j$; otherwise $\rho_{st}=0$. $G_2=G_2'+pG_2''$, and G_3 is the $n\times m$ matrix $\begin{pmatrix}0\\ I_m\end{pmatrix}$. Set $G=(G_0,G_1,G_2,G_3)\in GL^J_{n,m}(\mathbb{Z})$. Also, let

$$D_{r_0,r_2}(j,m)=\begin{pmatrix}I_{r_0}&&&\\&pI_{r_1}&&\\&&p^2I_{r_2}&\\&&&I'_m\end{pmatrix}$$

where $m'=\max(n-j,m)$.

Then Propositions 3.1, 4.1, and 5.1 immediately gives us the following.

Proposition 5.3. *Let p be prime, and j an integer so that $1\le j\le n$. Let Λ,Δ be a fixed reference lattices of ranks $n-m,m$ respectively. If $j\le n-m$, set $\Delta'=\Delta$; if $j>n-m$, set $\Delta'=\Delta\begin{pmatrix}\frac{1}{p}I_\ell&\\&I_{m-\ell}\end{pmatrix}$ where $\ell=n-j-m$.*

(a) If $p\nmid N$, the pairs (Ω,Λ_1) in Propositions 3.1 and 4.1 are in one-to-one correspondence with the pairs (D,G) where, for some non-negative r_0,r_2 so that $r_0+r_2\le j'=\min(j,n-m)$, $D=D_{r_0,r_2}(j,m)$ and $G\in\mathcal{G}^J_{r_0,r_2}(j,m)$, via the correspondence

$$\Omega\oplus\Delta=(\Lambda\oplus\Delta')GD^{-1}\delta_j,\ \Lambda_1=(\Lambda\oplus\Delta')\begin{pmatrix}0_{r_0}&&\\&I_{r_1}&\\&&0\end{pmatrix}$$

where $r_1=j'-r_0-r_2$.

(b) If $p|N$ then we only need those (Ω,Λ_1) corresponding to $r_0=j'$, $r_2=0$.

Finally, for $0\le r\le n-m$, we let $\mathcal{G}^J_r(m)$ be the set of matrices G constructed as follows. Let $(\{d_1,\dots,d_r\},\{a_1,\dots,a_{n-r}\})$ be an ordered partition of $\{1,2,\dots,n-m\}$. G_0 is the $n\times r$ matrix whose s,t-entry is 1 if $s=d_t$, and 0 otherwise. G_1 is a matrix whose s,t-entry is α_{st} where α_{st} is 1 if $s=a_t$, $\alpha_{st}=0$ if $s<a_t$ or $s=a_i$ (some $i\neq t$), and $\alpha_{st}\in\{0,1,\dots,p-1\}$ otherwise. G_2 is the $n\times n$ matrix $\begin{pmatrix}0\\ I_m\end{pmatrix}$, and $G=(G_0,G_1,G_2)$. Also, let $D_r(m)=\begin{pmatrix}I_r&&\\&pI_{n-m-r}&\\&&I_m\end{pmatrix}$. Then by Propositions 4.1 and 4.2, we have the following.

Proposition 5.4. *Let p be prime, n a positive integer, and Λ,Δ a fixed reference lattices of rank $n-m,m$ respectively.*

(a) If $p\nmid N$, the lattices Ω in Proposition 4.2 are in one-to-one correspondence with the pairs (D,G) where, for some non-negative $r\le n-m$, $D=D_r(m)$ and $G\in\mathcal{G}^J_r(m)$, via the correspondence

$$\Omega\oplus\Delta=(\Lambda\oplus\Delta)GD^{-1}.$$

(b) If $p|N$ then we only need those Ω corresponding to $r=n$.

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