Analysis Workshop
How fast is fast? Comparative growth of functions

Story. We know many sequences $(a_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} a_n = \infty$. However, some of these sequences approach ∞ faster than others. Understanding their rates of growth is crucial in some of our uses of sequences, such as the use of sequences to define "series", which are essentially infinite-degree polynomials that describe functions.

One method to explore comparative rates of growth of sequences is what is called l'Hopital's Rule (which we will develop rigourously later in the Analysis 1 course). In 1696, l'Hopital published the first textbook on the infinitesimal calculus. Based heavily on the ground-breaking work of Leibniz as well as work of the Bernoulli brothers Jacob and Johann, this book addresses differential calculus and is still praised for its lucidity. However, the origin of l'Hopital's Rule became a topic of bitter dispute. Some years before this publication, Johann Bernoulli had entered an agreement with l'Hopital, who paid Bernoulli 300 Francs per year to be the first (and only) person to be informed of Bernoulli's mathematical discoveries. After l'Hopital's death in 1704, Johann Bernoulli publicised this agreement and claimed credit for what we still call l'Hopital's Rule.

In this workshop you compare the rates of growth of certain functions, without using l'Hopital's Rule.

For further exploration: Assume that $(a_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers.

(a) Suppose that
$$\lambda' = \sup\left\{\frac{a_{n+1}}{a_n} : \; n \in \mathbb{N}\right\} < 1.$$
Does it follow that $\lim_{n\to\infty} a_n = 0$?

(b) Suppose that $\frac{a_{n+1}}{a_n} < 1$ for all $n \in \mathbb{N}$. Does it follow that $\lim_{n\to\infty} a_n = 0$?