Analysis Workshop
Geometry and Irrationality

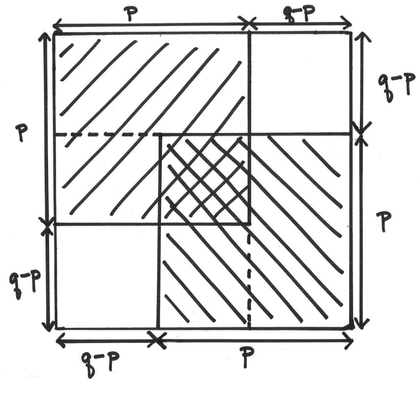
Story. As stories have it, the ancient Greek Pythagorean Brotherhood knew that $\sqrt{2}$ is irrational. At that time, it was initially posited that irrational numbers do not occur in nature. Of course, this is contradicted by the Pythagorean Theorem, as the right triangle whose legs both have length 1 has hypotenuse of length $\sqrt{2}$. Very possibly an apocryphal story, legend has it that one of the brethren revealed that an irrational number does indeed occur in nature, and for this he was thrown overboard and drowned at sea. (Note that there were women in ancient and early Greek history studying mathematics. According to "Rhetoric Retold: Regendering the Tradition from Antiquity Through the Renaissance" by Cheryl Glenn, women were allowed to study as Pythagoreans, but it is not clear from this resource whether the Pythagoreans allowed women to help develop mathematics. However, it is believed that later in early Greek history, women did participate in developing mathematics. In particular, Hypatia, born between 350 and 370 and then murdered in 415, was very famous in her time as a mathematician and astronomer, and people came from miles around to hear her lecture mathematics. There continue to be scholarly works written about her.)

In this workshop you used a geometric argument to show that $\sqrt{2}$ is irrational (these arguments are based on a clever argument of John H. Conway). You are allowed to use that for a, b real numbers, if $0 < a < b$ then $0 < a^2 < ab < b^2$ and $0 < \sqrt{a} < \sqrt{b}$.

For further exploration: In this workshop, for the sake of contradiction you supposed that $\sqrt{2}$ is rational. Thus there are integers p and q so that $\frac{q}{p} = \sqrt{2}$. As $\sqrt{2}$ is positive, we assumed that p and q are positive, with p the **smallest** positive integer so that there is a positive integer q with $\frac{q}{p} = \sqrt{2}$. Then using the picture below, you computed the area of the large square in two different ways, and from that showed that

$$\sqrt{2} = \frac{2p - q}{q - p}.$$

Since you had previously shown that $0 < q-p < p$, this led to a contradiction as to the choice of p (where $\sqrt{2} = q/p$, with the conditions given above).



(a) Is it possible to mimic the above argument to conclude that $\sqrt{3}$ cannot be rational?

(b) As we know that $\sqrt{4}$ is rational, it cannot be possible to mimic the above argument to conclude that $\sqrt{4}$ is irrational. Where does the previous line of argument fail when considering $\sqrt{4}$ rather than $\sqrt{2}$?

(c) Abu Kamil (circa 850-930) was an Egyptian Muslim mathematician, credited for being the first to systematically use irrational numbers in equations and as solutions to equations. He wrote the book whose title is translated as "The Book of Algebra", and his work is believed to have influenced the work of Al-Karaji and of Fibonacci.