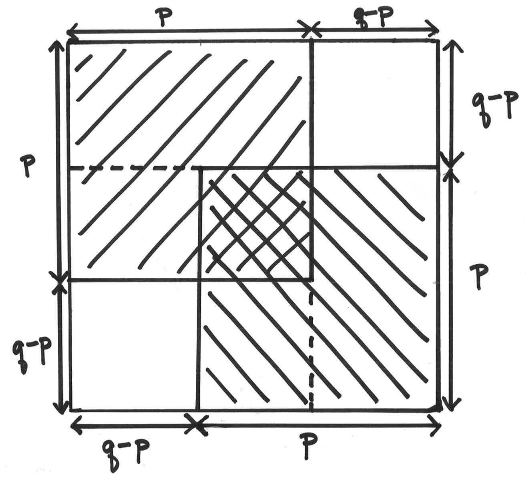
Analysis Workshop
Geometry and Irrationality

As stories have it, the ancient Greek Pythagorean Brotherhood knew that $\sqrt{2}$ is irrational. At that time, it was initially posited that irrational numbers do not occur in nature. Of course, this is contradicted by the Pythagorean Theorem, as the right triangle whose legs both have length 1 has hypotenuse of length $\sqrt{2}$. Very possibly an apocryphal story, legend has it that one of the brethren revealed that an irrational number does indeed occur in nature, and for this he was thrown overboard and drowned at sea. (Note that there were women in ancient and early Greek history studying mathematics. According to "Rhetoric Retold: Regendering the Tradition from Antiquity Through the Renaissance" by Cheryl Glenn, women were allowed to study as Pythagoreans. Later in early Greek history, Hypatia, born between 350 and 370 and then murdered in 415, was very famous in her time as a mathematician and astronomer, and people came from miles around to hear her lecture mathematics. There continue to be scholarly works written about her.)

Here you develop arguments regarding the irrationality of certain numbers (these arguments are based on a clever argument of John H. Conway).

Present all your answers in complete sentences.

1. For the sake of contradiction, suppose that $\sqrt{2}$ is rational. Thus there are integers p and q so that $\frac{q}{p} = \sqrt{2}$. As $\sqrt{2}$ is positive, we can assume that p and q are positive. We can also assume that we have chosen p to be the **smallest** positive integer so that there is a positive integer q with $\frac{q}{p} = \sqrt{2}$.
   (a) Using the assumption that $\frac{q}{p} = \sqrt{2}$, explain why $p < q$. (Suggestion: begin by explaining why $1 < \sqrt{2}$; if it were not the case that $1 < \sqrt{2}$, what would you be able to conclude about the relative sizes of 1 and 2?)
   (b) Explain why $q < 2p$. (Suggestion: begin by explaining why $\sqrt{2} < 2$.)
   (c) Now consider the picture below.



So the outer square has dimensions $q \times q$, and thus has area q^2. Inside this square, we have 2 squares of size$(q-p)\times(q-p)$, and 2 squares of size $p\times p$; these 2 squares of size $p\times p$ overlap to create the central inner square. Explain why this central inner square has area $(2p-q)^2$. (You may want to label the hyphenated line segments in the picture to help with your explanations; if you do this, make sure that you turn in this annotated picture with your workshop solutions! For convenience, an enlarged copy of this picture is on the back page.)

(d) Now use the initial assumptions, the picture in (c) and your answer to (c) to explain why

$$2p^2 = q^2 = 2p^2 + 2(q-p)^2 - (2p-q)^2.$$

(e) Using (d), explain why

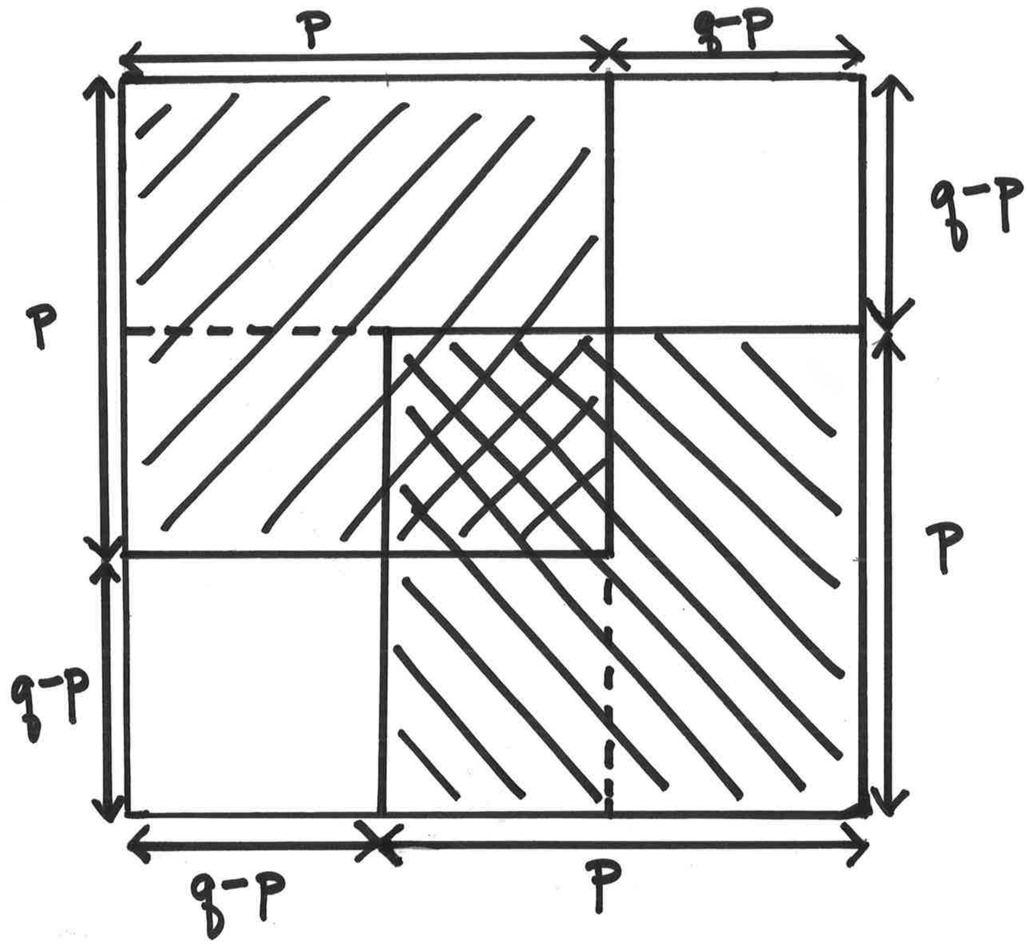
$$\sqrt{2} = \frac{2p-q}{q-p}.$$

(f) Using (b), explain why the conclusion of (e) contradicts our initial assumption that p is the smallest positive integer so that there is a positive integer q with $\frac{q}{p} = \sqrt{2}$. (Note that this means that by **assuming** that $\sqrt{2}$ is rational, we deduce a **contradiction**, and thus our initial assumption cannot be true. In Latin, this method of arguing is called *reductio ad absurdum.*)

2. **For further exploration:**

(a) Is it possible to mimic the above argument to conclude that $\sqrt{3}$ cannot be rational?

(b) As we know that $\sqrt{4}$ is rational, it cannot be possible to mimic the above argument to conclude that $\sqrt{4}$ is irrational. Where does the previous line of argument fail when considering $\sqrt{4}$ rather than $\sqrt{2}$?


p
q−p
q−p
p
p
q−p
q−p
p