Analysis Workshop
Irrational Numbers and The Jumping Flea

Story. We know that the Pythagoreans realised that $\sqrt{2}$ is irrational. As we can show (using, for instance "geometric series") that any decimal expansion that is eventually repeating represents a rational number, this means that any irrational number (such as $\sqrt{2}$) has a non-repeating decimal expansion.

Long before the Pythagoreans, mathematical tablets from the Old Babylonian age (c. 1800-1600 BC) reveal that the Babylonians had a surprisingly good estimate for $\sqrt{2}$. They used a positional "sexagesimal" (i.e. base 60) numerical system, and calculated $\sqrt{2}$ to be

$$1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} = 1.414213.$$

(However, they used 3 as the value of π. While many scholars believe that overall Babylonian mathematics was more advanced than that of the Egyptians during the "Middle Kingdom" (c. 2000-1800 BC), the Egyptians used 3.16 as the value of π.)

In this workshop you explore the following sort of question: is there a multiple of $\sqrt{2}$ whose decimal expansion after the dot begins with, say, 57?

For further exploration.

(a) Suppose that x is a positive irrational number. For any $k \in \mathbb{N}$ and any string of digits $c_1, c_2, \ldots, c_k$, show that there is some $n \in \mathbb{N}$ so that in the decimal expansion of nx, the first k digits following the dot are $c_1 c_2 \cdots c_k$.

(b) For more information on the mathematics of the Babylonians, Egyptians, and early Greeks, see (for instance) the first chapter of "The History of The Calculus" by C.H. Edwards, Jr. (which includes some interesting exercises), and "A Concise History of Mathematics" by Dirk J. Struik.