Analysis Workshop
Life without Limits: The Finite Calculus

Story. "Limits" are fundamental in our study of analysis. We see the beginnings of our modern notion of a limit in the works of Archimedes (circa 287–212 BC). Often hailed as the greatest mathematician of antiquity, Archimedes used the "method of exhaustion" to compute things like the area of a circle and the area between a parabola and a line. To compute the area of a circle, Archimedes inscribes the circle inside a regular hexagon, and inscribes a regular hexagon inside the circle, thereby being able to compute an upper and a lower estimate for the area of the circle. Then Archimedes proceeds to repeatedly double the number of sides of each regular polygon, and thus squeezes the area of the circle between the areas of these polygons, with greater precision of the estimate of the area of the circle as the number of sides of the outer and inner polygons grow. So Archimedes essentially uses a limiting process to compute the area of the circle. Archimedes uses a similar method to compute the area between a parabola and a line, yielding what we now call a "geometric series", which he evaluates.

However, it is surprising what we can accomplish without the use of the limit. Here we look at functions $f : \mathbb{Z} \to \mathbb{Q}$. Instead of the derivative, we have the difference operator Δ, defined by

$$\Delta f(n) = f(n+1) - f(n).$$

To develop an analogue of differentiating polynomials, we introduced the "factorial power" $n^{\underline{k}}$ as follows. For $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, we set

$$n^{\underline{k}} = n(n-1)(n-2)\cdots(n-k+1)$$

(so this is the product of the k integers m where $n \geq m > n-k$). We also set $n^{\underline{0}} = 1$. (Note that if $k > n \geq 0$ then $n^{\underline{k}} = 0$, as $n \geq 0 > n-k$. For instance, $2^{\underline{4}} = 2 \cdot 1 \cdot 0 \cdot (-1) = 0$.) A significant result is that $\Delta n^{\underline{k}} = k \cdot n^{\underline{k-1}}$.

Instead of the integral, for $a, b \in \mathbb{Z}$ with $a < b$, we have the summation operator $\displaystyle\sum_{n:a\to b}$ defined by

$$\sum_{n:a\to b} f(n) = f(a) + f(a+1) + f(a+2) + \cdots + f(b-2) + f(b-1)$$

(so this is a sum of $b-a$ terms). Then we get an analogue of the Fundamental Theorem of Calculus:

$$\sum_{n:a\to b} \Delta f(n) = f(b) - f(a).$$

In this workshop, you demonstrated that for $f : \mathbb{Z} \to \mathbb{Q}$, we have

$$\Delta(f+g)(n) = \Delta f(n) + \Delta g(n),$$

and with $c \in \mathbb{Q}$, we have

$$\Delta(cf)(n) = c(\Delta f(n)).$$

Also, you verified that

$$\sum_{n:1\to m+1} \Delta f(n) = f(m+1) - f(1).$$

Note: for your own amusement, you can also verify analogues of the "product rule" and the "quotient rule" for derivatives; more precisely, you can verify that

$$\Delta(fg)(n) = f(n+1) \cdot \Delta g(n) + (\Delta f(n)) \cdot g(n),$$

and when $g(n), g(n+1) \neq 0$,

$$\Delta(f/g)(n) = \frac{(\Delta f(n)) \cdot g(n) - f(n) \cdot \Delta g(n)}{g(n)g(n+1)}.$$

You can also verify that with $a, b \in \mathbb{Z}$ and $a < b$, we have

$$\sum_{n:a\to b} \Delta f(n) = f(b) - f(a),$$

an analogue of the Fundamental Theorem of Calculus.

In this workshop, you found values for a, b, c so that $\Delta(an^{\underline{3}}+bn^{\underline{2}}+cn^{\underline{1}}) = n^2$, and then you evaluated $\sum_{n:1\to m+1} \Delta f(n)$ in two ways to obtain a concise formula for $1^2 + 2^2 + 3^2 + \cdots + m^2$.

For further exploration:

(i) Using this method, with $m \in \mathbb{N}$, find formulas for

$$1^3 + 2^3 + 3^3 + \cdots + m^3$$

and for

$$1^4 + 2^4 + 3^4 + \cdots + m^4.$$

(ii) The Prime Number Theorem has to do with the distribution of prime numbers, or put another way, the probability that a large integer is a prime number. Many eminent mathematicians are known to have made conjectures toward the distribution of primes, and the theorem was finally proved independently but simultaneously by de la Vallée Poussin and Hadamard in 1896, using "continuous" methods of complex analysis (and Riemann's now famous "zeta function"). But it is impressive how close people came to proving this theorem using "discrete" methods, such as you have used in this workshop. For a brief but nice discussion of this history and development of the mathematics, see Chapter 7 of "Multiplicative Number Theory" by H. Davenport; for a very nice and accessible development of the progress made with discrete methods, see "A Primer of Analytic Number Theory: From Pythagoras to Riemann" by J. Stopple.