Correction to Proof of Lemma 1.1 in paper "On lifting Hecke eigenforms"

Here h' the class number (not the strict class number). For $\tilde{s}$ and adele, we set $\chi_\infty(\tilde{s}) = \text{sgn}(\tilde{s})^k$. (By $\text{sgn}(\tilde{s})$ we mean $\prod_{i=1}^n \text{sgn}(\tilde{s}_{\varphi_i})$ where the φ_i are the embeddings of $\mathbb{K}$ into $\mathbb{Q}$ and $\tilde{s}_{\varphi_i}$ is the corresponding component of $\tilde{s}$.)

Let ε be a character on the ideal class group. Using ε, we define a Hecke character on the adeles of $\mathbb{K}$ extending $\chi_{N\infty}$ as follows.

The ideal class group can be decomposed as a product of cyclic subgroups; using Dirichlet's theorem on primes, we let $\mathcal{P}_1, \dots, \mathcal{P}_r$ be prime ideals not dividing $\mathcal{N}$ so that the $\text{cls}\mathcal{P}_j$ generate the ideal class group. For each j, let $o(\mathcal{P}_j)$ denote the order of $\text{cls}\mathcal{P}_j$. Take $p_j \in \mathcal{O}$ so that $\mathcal{P}_j^{o(\mathcal{P}_j)} = p_j\mathcal{O}$, and choose an $o(\mathcal{P}_j)$th root ρ_j of $\overline{\chi}_{\mathcal{N}\infty}(p_j)$. For $\tilde{s}$ an adele away from $\mathcal{N}$, write $\tilde{s}\mathcal{O} = \alpha\mathcal{P}_i^{e_1}\cdots\mathcal{P}_r^{e_r}$ where $\alpha \in \mathbb{K}^\times$. Set

$$\chi(\tilde{s}) = \varepsilon(\mathcal{P}_1^{e_1}\cdots\mathcal{P}_r^{e_r})\rho_1^{e_1}\cdots\rho_r^{e_r}\chi_{\mathcal{N}\infty}(\alpha)\overline{\chi}_\infty(\tilde{s}).$$

To show $\chi(\tilde{s})$ is well-defined, say we also have $\tilde{s}\mathcal{O} = \beta\mathcal{P}_1^{f_1}\cdots\mathcal{P}_r^{f_r}$. Then $f_j = e_j + m_j o(\mathcal{P}_j)$ for $m_j \in \mathbb{Z}$ and $\beta/\alpha = up_1^{m_1}\cdots p_r^{m_r}$ where $u \in \mathcal{O}^\times$. Then

$$\begin{aligned}
&\varepsilon(\mathcal{P}_1^{f_1}\cdots\mathcal{P}_r^{f_r})\rho_1^{f_1}\cdots\rho_r^{f_r}\chi_{N\infty}(\beta)\\
&\quad= \varepsilon(\mathcal{P}_1^{e_1}\cdots\mathcal{P}_r^{e_r})\rho_1^{e_1}\cdots\rho_r^{e_r}\overline{\chi}_{N\infty}(p_1^{m_1}\cdots p_r^{m_r})\chi_{N\infty}(\beta)\\
&\quad= \varepsilon(\mathcal{P}_1^{e_1}\cdots\mathcal{P}_r^{e_r})\rho_1^{e_1}\cdots\rho_r^{e_r}\chi_{N\infty}(\alpha)
\end{aligned}$$

(recall that for a unit u, $\chi_{N\infty}(u) = 1$). One easily checke that $\chi(a\tilde{s}) = \chi(\tilde{s})$ where $a \in \mathbb{K}$, a a unit at all primes dividing N.

If $\tilde{s}$ is an adele not away from $\mathcal{N}$, then choose $a \in \mathbb{K}^\times$ so that $a\tilde{s} = \tilde{u}$ where $\tilde{u}$ is away from $\mathcal{N}$; define $\chi(\tilde{s}) = \chi(\tilde{u})$. As above, one easily verifies $\chi(\tilde{s})$ is well-defined.

Thus χ is a Hecke character.