Analysis Workshop
Evaluating $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$

Story. Previously, you may have seen how to evaluate

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$$

using "l'Hopital's Rule". Another way to evaluate this limit is by using an "infinite series" expansion for $\frac{\sin\theta}{\theta}$. However, there is a rather simple geometric method to evaluate this limit, as you explore below.

It seems unclear who first found this geometric argument. Ancient Egyptians and Babylonians knew theorems regarding ratios of the sides of similar triangles, however it seems they did not articulate the concept of measuring angles. This concept was articulated in the Hellinistic period (which began in the 4th century BC with Alexander the Great's conquest of the eastern Mediterranean, Egypt, Mesopotamia, the Iranian plateau, Central Asia, and parts of India). Scholars report that during this period, Greek mathematics merged with Egyptian and Babylonian mathematics, so they had all the tools and knowledge to have created the geometric argument you recreate below – but perhaps they did not ask what this limit should be?

For further exploration. In chapter VIII of his book "Introductio", Euler put forward the definitions of the functions sin and cos that we use today, making use of radian measure and a circle of radius 1. He also referred to an inductive argument to prove "DeMoivre's identity"

$$(\cos(x)+i\sin(x))^n=\cos(nx)+i\sin(nx)$$

where $x\in\mathbb{R}$, $i=\sqrt{-1}$ (so $i^2=-1$) and $n\in\mathbb{N}$. Further, using the Binomial Theorem he presented an argument resulting in what we now call Taylor series (or Maclaurin series) for $\sin(x)$ and $\cos(x)$, although his argument uses ε as an *infinitely small* number and N as an *infinitely large* integer. (See the section "Euler's exponential and logarithmic functions" in chapter 10 of "The Historical Development of the Calculus" by C.H. Edwards, Jr. for more discussion and exercises regarding these topics.)