V. 50 MATHEMATICS AND ABSOLUTE TRUTH

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INTRODUCTION --Author

Mathematics differs from other sciences in its single-minded focus on abstraction. Because of the abstract nature of mathematical thought, its product is often portrayed as absolute and eternal truth. Faith in the absolute certainty of mathematical truth has affected both the culture of mathematicians and the way mathematics is viewed from the outside.

The mathematical culture formed in part by this vision of absolute truth has not been friendly to women. A recent study in *Science* magazine of the participation of women in science found "the persistence of rampant sexism [to be] almost unique to mathematics among scientific disciplines," with almost all of the dozens of female mathematicians interviewed reporting "a climate of hostility" for women in mathematics. [1] The latest National Science Foundation study [2] showed that among the sciences, mathematics had one of the lowest percentages of PhD's awarded to women in 1988 (16%), and by far the largest difference between women's representation at the bachelor's degree level and their representation at the PhD level (see Table 1). A report by the National Research Council on the status of women in science also comments on this unusual attrition rate among women in mathematics between the Bachelor's degree and the PhD. [3] The NRC report notes that mathematics stands out statistically in two other respects: the representation of women is not increasing in mathematics as much as in other sciences (see Table 1 and [3]), and only in mathematics is the likelihood of a female PhD to have a degree from a prestigious university substantially less than that of a male PhD. [3]

Table 1. REPRESENTATION OF WOMEN AMONG SCIENCE DEGREE RECIPIENTS IN U.S., AS REPORTED BY THE NATIONAL SCIENCE FOUNDATION

	% of women of all 1982 BA recipients	% of women of all 1988 PhD recipients	% attrition in women's share from '82 BAs to '88 PhDs	% increase in % of PhDs to women from 1978 to'88
Total Sciences	45%	32%	29%	45%
Math	43	16	63	14
Chemistry	32	21	34	62
Physics	13	10	23	100
Biology	45	37	18	48
Earth Science	26	20	23	100
Psychology	67	55	18	49
Social Science	44	33	25	38

These statistics point to the need for change in the mathematics community. However, mathematics' aura of absolute truth has also distanced it from feminist critiques of science. Many of these critiques are specifically aimed at the empirical nature of science as embodied in the scientific method. Pure mathematics floats safely above the fray. In *The Science Question in Feminism*, Harding directly challenges the timelessness of mathematical truth [4], but from the perspective of an outsider whom mathematicians are unwilling to take seriously. [5]

In this insider's critique, I argue that mathematics has a legitimate, but very limited, claim to absolute truth. I examine the nature of that claim and the ways in which its limits have been overlooked in the development of mathematical culture. I discuss ways in which the theories developed by feminist critics of science can be used to help us recognize these limits and reform mathematics without destroying it.

ABSOLUTE TRUTH

All sciences seek patterns; all use abstraction as a tool for finding connections between dissimilar things. Mathematics stands out not in its use of abstraction, but in the extent to which it takes its abstractions as its world. Mathematical objects are ideals which are only imperfectly realizable in the empirical world. For example, a circle, to a mathematician, is the set of points equidistant from a special point called the center. Such an object is not found in nature, nor can we draw one. Any concrete object we label as a circle has irregularities, however minute.

Mathematicians test their statements about objects such as the circle using criteria of internal consistency. The mathematical assertion that pi, the ratio between the circumference and diameter of any circle, is a constant (whose value has been calculated to more than 500,000 decimal places) could not possibly be tested empirically. Only individual circles could be tested, not all possible circles, and no individual circle could be tested to that degree of accuracy with even the most precise measuring equipment imaginable (see e.g. [6]). Instead, mathematicians test the truth of statements such as this by constructing a logical argument using other more basic mathematical beliefs. The fact that this formula is only approximately true of the objects we call circles in the empirical world points out the imperfections of the world rather than any limitation on the truth of the mathematical statement. (For a philosopher of mathematics' view of these aspects of mathematical truth, see e.g. [7].) This essentially nonempirical nature makes mathematical truth absolute.

> A chair or a star is not in the least what it seems to be; the more we think of it, the fuzzier its out-lines become in the haze of sensation which surrounds it; but "20" or "317" has nothing to do with sensation, and its properties stand out more clearly the more closely we scrutinize it [8] .

Mathematical truth depends only on a path from agreed upon definitions, axioms, and rules of inference to their intricate consequences. We cannot conceive of a world in which the same definitions, the same axioms, and the same rules of inference result in different conclusions.

Many mathematicians and outsiders are intoxicated by the possibility mathematics affords of reaching absolute truth. Twentieth century mathematician G.H. Hardy writes in *A Mathematician's Apology,* "Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not." [8] However, insofar as mathematical statements are absolute, they are also absolutely meaningless. Mathematical truths themselves say nothing about their own applicability to the empirical world or their relative interest or importance to mathematicians. When we use mathematics to solve problems in the world or when we use models in the world to discover or understand mathematics, we step outside the realm of absolute truth. Our choices of mathematical definitions, axioms, and questions to pursue are also outside this realm.

Mathematicians have had many reminders of the limits of their claims to absolute truth. The invention of non-Euclidean geometries at the beginning of the 19th century dissolved hopes of finding a complete self-evident set of axioms that yield the one true picture of space. Non-Euclidean geometries describe the world we see as well as Euclidean geometry does. However, many of the geometrical "facts" we take for granted are not true in the non-Euclidean geometries. For example, the circumference of a circle is not pi times its diameter; the ratio of the circumference to the diameter is not even constant (see, e.g. [9]). In the mid 20th century, Kurt Gödel showed that this problem of having more than one model fit a set of axioms is unavoidable. He proved that in fact no mathematical system rich enough to be of interest to mathematicians can be both consistent and complete. [10] Our choices of definitions and axioms thus necessarily fall short of any claim to absolute truth.

MATHEMATICAL CULTURE

Most mathematicians have at least a passing acquaintance with these developments; most would agree that mathematics' claim to absolute truth is in some way limited. However, in spite of this acknowledgement, mathematical culture reflects a much larger belief in absolute truth. Mathematicians exhibit faith that mathematical standards are exact and timeless, that mathematical success results from innate talent and is accurately predicted by exams, and that both the practice and teaching of mathematics are safely insulated from the culture at large.

The most important mathematical standard is that of truth; mathematicians behave as if this standard can be applied absolutely, even in light of Gödel's result. Many philosophers, and even mathematicians, have commented on the irony of this behavior. As Michael Polyani writes,

> We can now turn to the paradox of a mathematics based on a system of axioms which are not regarded as self-evident and indeed cannot be known to be mutually consistent. To apply the utmost ingenuity and the most rigorous care to prove the theorems of logic or mathematics, while the premises of these inferences are cheerfully accepted, without any grounds being given for doing so . . .might seem altogether absurd. It reminds one of the clown who solemnly sets up in the middle of the arena two gateposts with a securely locked gate between them, pulls out a large

> bunch of keys, and laboriously selects one which opens the lock, then passes through the gate and carefully locks it after himself-- all the while the whole arena lies open on either side of the gateposts where he could go round unhindered [9].

Many mathematicians who would laugh good naturedly at this caricature, would still insist that at least within a given axiom system, a proof is either eternally true or eternally false (see e.g.[10]).

In spite of the prevalence of this attitude in the mathematical community, our understanding of truth has in fact changed over time. As Judith Grabiner writes, "Perhaps mathematical truth is eternal, but our knowledge of it is not." [11] Grabiner documents the changes in standards of rigor in analysis in the two centuries since the birth of calculus. Standards of mathematical proof can change because of shifts in the axioms that form the starting point; documenting a complete path from the axioms to the statement at hand is usually too large a task to be possible. For example, one logician has estimated that a formal demonstration of one of the mathematician Ramanujan's conjectures would take about two thousand pages. [12]

The proofs of the approximately 200,000 theorems per year [13] that appear in mathematical journals rely instead on intermediate results and techniques. Some of these have been proven using other intermediate results; some are folklore. Some are later shown to be false. Many of the proofs in contemporary mathematics are so long and complicated, even making use of intermediate results, that checking even the written details becomes almost impossible. The Classification Theorem for finite simple groups is an extreme example, about which it has been said that "the probability of an error in its proof is virtually 1," even though no error has yet been found and most mathematicians accept the result. [14]

These difficulties in applying the standard of truth to mathematical results are even more pronounced in applying standards of beauty or relative worth. The history of mathematics contains numerous examples of results whose value and beauty have changed radically over time because of shifting mathematical fashions. [15] Yet G. H. Hardy writes that "no other subject has such clear-cut or unanimously accepted standards of beauty, seriousness, significance, generality and depth." Hardy spends many pages struggling to define these terms. While he ultimately admits defeat, he maintains that mathematicians have no difficulty in recognizing these qualities. [7] Other contemporary mathematicians who write about mathematical culture echo this belief. [13]

The misplaced faith that mathematical theorems can be judged objectively leads to an equally misplaced faith in our judging of mathematicians themselves. No other field believes so strongly that success results from innate and measurable talent, more than from hard work or experience. [16] Ability to grasp eternal and absolute truth must be God-given, and directly related to pure intelligence. We claim to judge ability with the same precision that we judge other mathematical matters.

Mathematicians show this faith in many ways. One way is in our admiration for the quick. Since everything that is true is in some sense tautological, good mathematicians can grasp truth instantly. Mistakes and even questions reveal stupidity. This attitude sets the mood in many mathematics classrooms, "Those who do not instantly understand--including many thoughtful, reflective, creative students--are made to feel deeply dumb, like outsiders who do not get the point of an in-joke." [17] A similar atmosphere is found in research seminars, where mathematicians are unnecessarily intolerant and necessarily careful of one another. We write papers and give talks in a condensed form, including many uses of "it is obvious that," to show how much quicker we are than the reader.

Because the good mathematician is the quick mathematician, and because experience is not as important as brains, we believe mathematicians do their best work when they are young. "No mathematician should ever allow himself to forget that mathematics, more than any other art or science, is a young man's game." [17] The Fields medal, the most prestigious prize in mathematics, is by tradition (though not by written rule) only awarded to mathematicians under the age of forty. However, in spite of the universal acceptance of the belief that mathematicians are best when young, a study of productivity and citation counts shows it to be a myth. [18]

The mathematics community's belief that mathematics ability is a matter of innate intelligence leads to a belief in the value of competitive examinations. In other sciences and the humanities, the culminating experience of undergraduate education is often a research experience or the writing of an integrative paper. In mathematics, such opportunities are rare. Instead, undergraduates in the United States are offered the chance to achieve fellowships and fame through the annual Putnam Competition. This day-long, twelve problem exam is so difficult that often at least ten percent of the scores are zero, and the median score is usually less than two out of twelve. [19] The exam is supposed to measure only problem solving ability, and not require specialized mathematical knowledge. However, critics have charged that the exam is aimed only at a "very sophisticated type of student." [20] Students from the same prestigious universities, and from schools which emphasize training sessions, win year after year. For example, Harvard has been among the top five teams in all but fourteen of the contest's fifty-three years.

Most mathematicians would admit that the Putnam exam gives only limited information about mathematical ability. Many great mathematicians did not take the exam or did poorly; many successful Putnam contestants did not go on to fulfill their mathematical promise. Indeed, a rough count shows that only about half of the winners have gone on to a career in mathematics (where this is measured by membership in some major mathematical organization). However the publicity accorded the exam and the effect it has on the confidence of its participants, gives it an importance beyond what its supporters might claim. For example, Julian Stanley said in *Science* that in light of women's poor performance on the Putnam and other competitions, it is not surprising that so few women receive tenure in top mathematics departments. [21]

The mathematical community has many other competitive examinations. At the high school level, the International Mathematics Olympiad is a contest in which fifty countries participate. In the United States, students qualify for the International Math Olympiad through a tiered system of examinations followed by a rigorous four week training session. Other exams figure prominently in the history of mathematics. The Cambridge Tripos examination in Great Britain was legendary in the 19th century for its effect on both mathematicians and mathematics. The top scorer was "not invariably a great mathematician, but it was virtually certain that he could if he wished be an influential one." The exam "demanded accuracy and speed in the manipulation of mathematical formulas, a shallow cleverness perhaps, but not real insight." [22] Because it determined what mathematics was studied, the exam "effectively ruined serious mathematics in England for a hundred years." [7]

Examinations figure prominently in arguments that purport to show gender or racial difference in mathematical ability. For example, the Stanley and Benbow study claimed to have determined that males have more innate mathematical talent by administering the Scholastic Aptitude Test to precocious seventh and eighth grade children. [23] If the Putnam is ineffective in determining mathematical talent, how much faith can be placed on the multiple-choice SAT? Indeed, SATs have been shown to under predict women's success in college, whether that success is measured in freshman GPA, total GPA, or probability of graduating. [24] Furthermore, scores on the SAT, like scores on the Putnam, are affected by the totality of the test takers' mathematical experiences, not just innate ability. Nevertheless, the Stanley and Benbow study is still quoted. [25]

All of the competitive exams mentioned above are taken by individuals working alone. On the Putnam exam, even though the exam's endowment particularly emphasized the importance of college students competing on academic teams, no teamwork is allowed to take place during the exam itself. This points to another aspect of mathematical culture: the widespread belief that mathematics is a solitary activity. While many mathematicians speak of the joy of collaboration, a study of published articles in pure mathematics from 1939-1957 revealed that 92 percent had single authors. [26]

Mathematical education stresses competition over collaboration. This is most clearly expressed in the Moore method, a popular (though controversial) technique of forcing students to discover mathematics on their own. Mathematician Paul Halmos expresses its philosophy of competition: " Do not read, do not collaborate--think, work by yourself, beat the other guy." [27] Mathematics attracts students who enjoy this competition. G. H. Hardy describes this as the driving force of his own initial interest in mathematics. "I thought of mathematics in terms of examinations and scholarships: I wanted to beat other boys, and this seemed to be the way in which I could do so most decisively." [7]

If the mathematician is somewhat isolated from mathematical colleagues, he or she is even more isolated from society at large. "Science works for evil as well as for good; and both Gauss and lesser mathematicians may be justified in rejoicing that there is one science at any rate, and that their own, whose very remoteness from ordinary human activities should keep it gentle and clean." [7] Mathematicians are not responsible for the use that is made of their work. Neither the uses of mathematics nor its history should be important to a mathematician. A result of this is the mathematician's austere definition of mathematical research, one which excludes the professional activity of half the nation's mathematics faculty. [17]

Another result is the self-contained style in which mathematics is taught. At most schools, a mathematics major is one of the few that has no outside requirements. Most mathematics departments do not require even a course in the history or philoso phy of mathematics and science, and most mathematics courses themselves spend little time on applications of the ideas they teach. In its worst form, the belief that mathematical ideas speak for themselves leads to the the paring away of even mathematical context and justification. One woman writes:

> On the eighth day, God created mathematics. He took stainless steel, and he rolled it out thin, and he made it into a fence, forty cubits high, and infinite cubits long. And on this fence, in fair capitals, he did print rules, theorems, axioms and pointed reminders. "Invert and multiply." "The square on the hypotenuse is three decibels louder than one hand clapping." "Always do what's in the parentheses first." And when he was finished, he said, "On the one side of this fence will reside those who are good at math. And on the other will remain those who are bad at math, and woe unto them, for they shall weep and gnash their teeth ." [28]

THE FEMINIST CRITIQUE

The distortions of mathematical culture described above, work to exclude those who are not attracted by a competitive and insular world. Leone Burton argues at the Conference Femmes et Mathematiques, Quebec 1986:

> The genderization of mathematics rests not only on the social climate, nor on the personal and social experiences of girls within the classroom, although it is clear that these influences are far from being trivial. I believe that the discipline itself and the style through which it is encountered is rendered masculine by the misguided stress which is laid on those very attributes of mathematics which are no longer acceptable to mathematicians, that is, completeness, certainty and absolutism. . . .It would appear that there is a private and a public world of mathematics. The private world is where struggle, failure, incomprehension, intuition and creativity dominate. . . . The public world is where the results of the private struggle make their appearance in a formal, conventional abstract formulation from which all evidence of false trials, inadequate reasoning or misunderstandings have been eliminated. [29]

Current movements for reform, many led by feminists, address some of the distortions of the mathematical culture described above. A joint task force of the Mathematical Association of America and the Association of American Colleges call for an end to the cult of "geniusism" in mathematics, for increasing student opportunities for research, and for requiring all mathematics students "to engage in serious study of the historical context and contemporary impact of mathematics." [17]

Marilyn Frankenstein has successfully applied a more radical version of these ideas to reach adult students who have rejected or been rejected by traditional math education. She agrees with Paulo Freire that "knowledge does not exist apart from how and why it is used, in whose interest," [30] so that mathematical word problems must come from "ideas and experiences which give meaning to students' lives" [30]. Frankenstein has students read newspaper articles on topical social, political, or economic issues and examine the mathematics they suggest. Also students are encouraged to respond critically to techniques of mathematics education.

Proposed reforms show more willingness to change how we teach than to change how we think about mathematics. However, any genuine reform will require the latter as well. How are we to think about the practice of mathematics if our knowledge and judgment cannot be absolute? One answer lies in Helen Longino's vision of socially constructed knowledge. [31] Although mathematical truth endures in a way other scientific truth does not, our knowledge of it is built and tested by a social process. The process in mathematics is described by three computer scientists:

> No mathematician grasps a proof, sits back, and sighs happily at the knowledge that he can now be certain of the truth of his theorem. He runs out into the hall and looks for someone to listen to it. He bursts into a colleague's office and commandeers the blackboard. He throws aside his scheduled topic and regales a seminar with his new idea. He drags his graduate students away from their dissertations to listen. He gets on the phone and tells colleagues in Texas and Toronto...If they find it tolerably interesting and believable, he writes it up. After it has circulated in draft far a while, if it still seems plausible, he does a polished version and submits it for publication. The mathematician who reads and believes the proof will attempt to paraphrase it, to put it in his own terms, to fit it into his own personal view of mathematical knowledge. No two mathematicians are likely to internalize a mathematical concept in exactly the same way, so this process leads usually to multiple versions of the same theorem, each reinforcing belief, each adding to the feeling of the mathematical community that the original statement is likely to be true. [12]

This characterization of the process by which a mathematical argument gains credibility echoes Helen Longino's belief that scientific objectivity results from a social process of criticism and revision. While we are almost never completely certain that a given proof is true, the process increases our certainty and eventually refines our notion of truth. The complexity of the mathematical argument determines the amount of social testing we require. The absolute nature of mathematical truth means that in the case of mathematics, this complexity involves only ideas and not the empirical world. Agreement can be easier to reach in mathematics, but the path is no less essential.

Mathematics holds a unique position in the sciences because of its widespread use as a tool. Mathematics requirements keep many women out of science;

arguments of the impregnability of mathematics are used to dismiss feminist critiques of science in general. [4] Dale Spender writes that "Mathematics is part of the 99% of the world resources that is owned by men, and they guard it well." [32] The insights into mathematics that feminists offer benefit not just mathematics, but all science.

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