Mathematical Investigations Workshop
Newton's law of cooling

Story. In 1701 Isaac Newton formulated a law governing how the temperature of a body changes in time if it is surrounded by an environment at a different temperature. His article appeared anonymously in Latin in the journal 'Philosophical Transactions'. Formulated in present terms this law states that an object approaches the temperature of the environment with a rate of change that is proportional to the temperature difference between the body and the environment. This law is used today for example in forensics to determine that time at which a murder occurred. A correction to this law for large temperature differences was formulated in 1817 by Dulong and Petit.

Newton's law of cooling leads to the following differential equation for the temperature $T(t)$ of an object (e.g. a cup of coffee) in a surrounding environment (e.g. a kitchen) at temperature $S(t)$:

$$\frac{dT}{dt} = -\gamma(T - S).$$

Here γ is a constant real number. We assume that the temperature $T(0)$ at time 0 is known. We also assume that the temperature of the surroundings $S(t)$ is a known function of t. We want to find $T(t)$ for later times $t > 0$.

This problem was considered in ODEs, Curves and Dynamics for the case that $S(t)$ is a constant. Here we will extend this calculation to situations where $S(t)$ varies in time.

In this workshop, you found the integrating factor for the differential equation

$$\frac{dT}{dt} + \gamma T = \gamma S.$$

Then taking $S(t)$ to be a general function, you used the integrating factor show that we obtain the following result for $T(t)$, for a given initial temperature $T(0)$,

$$T(t) = e^{-\gamma t}T(0) + \gamma e^{-\gamma t}\int_0^t e^{\gamma\tau}S(\tau)d\tau.$$

Also, you determined $T(t)$ for the case

$$S(t) = \bar{S} + \sin t,$$

where $\bar{S}$ is a constant.

For further exploration:

(1) We are now interested in what happens for large times, in the case of question of the workshop as recalled above. What would be a good approximation of $T(t)$ for large times?

(2) Try to simplify the r.h.s. of the result obtained in 2(a) by finding A and ϕ such that

$$\gamma \sin t - \cos t = A\sin(t - \phi).$$

(3) Now consider the case

$$S(t) = \bar{S} + \Delta \sin t.$$

where Δ is a real number. This situation is more realistic than the one in 1(e), because it allows to consider oscillations of the temperature of the environment that have an arbitrary amplitude. Is there a way to get the answer for this case just using the final result of 1(e), without repeating the whole calculation?

One could use similar ideas to generalise to $S(t) = \bar{S} + \Delta \sin \omega t$ where $\omega \in \mathbb{R}$ but this case will be omitted here. Then one could use this $S(t)$ as a model for the change of the outside temperature during day and night.