Counting and Sizes of Infinity

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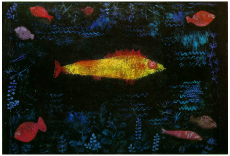
Why do mathematicians get addicted to maths?



Finding order and structure allows us to see **beauty** instead of **chaos**, and besides developing our *analytical minds*, it can bring us pleasure and satisfaction.

I am a **number theorist**, which means that in my research I study questions primarily about the integers, which are

$$0,\ 1,\ -1,\ 2,\ -2,\ 3,\ -3,\ 4,\ -4,\ 5,\ -5,\ 6,\ -6, \ldots$$

Number theorists love to **count**. For instance, a classical question in number theory is: *Given a positive integer n, how many ways can we write n as a sum of k squares of integers?*

For instance, with $n = 12$ and $k = 8$, we have:

$$\begin{aligned} 12 &= 3^2 + 1^2 + 1^2 + 1^2 + 0^2 + 0^2 + 0^2 + 0^2 \\ &= 2^2 + 2^2 + 2^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 \\ &= 2^2 + 2^2 + 1^2 + 1^2 + 1^2 + 1^2 + 0^2 + 0^2. \end{aligned}$$

The Integers: $0,\ 1,\ -1,\ 2,\ -2,\ 3,\ -3,\ 4,\ -4,\ 5,\ -5,\ 6,\ -6, \dots$

The Integers: 0, 1, −1, 2, −2, 3, −3, 4, −4, 5, −5, 6, −6, . . .

$$12 = 3^2 + 1^2 + 1^2 + 1^2 + 0^2 + 0^2 + 0^2 + 0^2$$

When we consider changes of signs and rearrangements of the order of the squares, the above solution gives us other solutions, such as:

$$\begin{aligned} 12 &= (-3)^2 + 1^2 + 1^2 + 1^2 + 0^2 + 0^2 + 0^2 + 0^2 \\ &= 1^2 + 3^2 + 1^2 + 1^2 + 0^2 + 0^2 + 0^2 + 0^2. \end{aligned}$$

Altogether, we find that there are **31, 808** (ordered) **8**-tuples of integers so that the **sum of their squares** gives us **12**.

More generally, we have a **formula** for $r_8(n)$, which denotes how many ways we can write a positive integer n as a sum of 8 squares:

$$\boxed{r_8(n) = 16 \sum_{d|n} (-1)^{n-d}\, d^3}$$

where the notation

$$\sum_{d|n}$$

means the sum over all positive integers d that divide n.

$$r_8(n) = 16 \sum_{d|n} (-1)^{n-d}\, d^3$$

With $n = 12$, the positive divisors d of 12 are $1,\ 2,\ 3,\ 4,\ 6,\ 12,$ so

$$\sum_{d|12} (-1)^{12-d}\, d^3 = -1 + 8 - 27 + 64 + 216 + 1728 = 1988$$

thus $$\mathbf{r_8(12)} = 16 \cdot 1988 = \mathbf{31,808}.$$

Why study such a question? Why integers?

We use integers to model problems in the digital world.
The generalised Pythagorean Theorem tells us the the square of the length of the vector from the origin to a point $(x_1, x_2, x_3, \dots, x_k)$ is

$$x_1^2 + x_2^2 + x_3^2 + \cdots + x_k^2.$$

How do we derive such a formula?

We build an infinite degree polynomial with coefficients $r_8(n)$:

$$f(X) = r_8(0)X^0 + r_8(1)X^1 + r_8(2)X^2 + r_8(3)X^3 + \cdots$$

We show that this function has some nice symmetries to deduce that it lives in a space we understand fairly well, and then we use this to better understand $f(X)$.

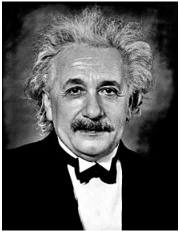
In my research, I look at generalisations of this: Given a k-dimensional space with a certain geometry, I count how many smaller dimensional spaces lying inside this space have a given geometry.

Moving the problem to a more *abstract* setting, and studying the *structure* of this abstract setting allows us to more easily derive *specific* symmetries, patterns, and formulas.

In this problem my answer is always **finite**, but...

In the 1st year course *Foundations and Proof*, we explore **SIZES OF INFINITY**.

Let's ask first: **Does infinity exist?**



Albert Einstein, 1879 – 1955

Two things are infinite: the universe and human stupidity; and I'm not sure about the universe.

Everything that can be counted does not necessarily count; everything that counts cannot necessarily be counted.

We use the numbers $1, 2, 3, 4, 5, \ldots$ to count, so we sometimes call these the *counting numbers*, or more formally, the *positive integers*.
We use $\mathbb{Z}_+$ to denote the set (that is, the collection) of all the positive integers, of which there are **infinitely** many.

We say that a set X is *countable* if there is a one-to-one correspondence between the elements of X and the elements of $\mathbb{Z}_+$, and when there is we write

$$|X| = |\mathbb{Z}_+|.$$

(So we think of $|\mathbb{Z}_+|$ as capturing the "size" of the set of positive integers.)

Is every infinite set countable?

First, we note that every **infinite** set X contains a countable subset.

Intuitively: Suppose X is an infinite set. Choose a_1 from X.
Next, choose a_2 from X so that $a_2 \neq a_1$ (possible since X is infinite).
Then, choose a_3 from X so that $a_3 \neq a_1$ and $a_3 \neq a_2$.
Continue... We get a subset $A = \{a_1, a_2, a_3, \ldots\}$ of X with $|A| = |\mathbb{Z}_+|$.

WARNING! The argument above is *inductive*, meaning that it describes an *algorithm*, which in our lifetime we can only apply finitely many times. So to make this argument **rigourous** one needs to use the technique of *Transfinite Induction*, or *The Axiom of Countable Choice*, or ...

Conclusion: $|\mathbb{Z}_+|$ is the "smallest magnitude of infinity".

Thus there must be as many **even** positive integers as there are positive integers (since the set of **even** positive integers is **infinite** and is **contained in** the set of positive integers).

Here is a one-to-one correspondence between the positive integers and the **even** positive integers:

positive integers $\leftrightarrow$ even positive integers

$$1 \leftrightarrow 2$$
$$2 \leftrightarrow 4$$
$$3 \leftrightarrow 6$$
$$\vdots$$
$$n \leftrightarrow 2n$$
$$\vdots$$

Now consider $\mathbb{Z}_+ \times \mathbb{Z}_+$, the set of ordered pairs of positive integers.
Surely there are **more** of these than there are positive integers...

or are there?

Claim: $|\mathbb{Z}_+ \times \mathbb{Z}_+| = |\mathbb{Z}_+|$.

To prove this, we need to set up a one-to-one correspondence between the collection of ordered pairs of positive integers and the positive integers.

A grid of the ordered pairs of positive integers

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: ONE

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: TWO

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: THREE

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: FOUR

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: FIVE

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: SIX

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: SEVEN

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: EIGHT

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: NINE

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: TEN

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: ELEVEN

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: TWELVE

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & \color{red}{(2,4)} & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: THIRTEEN

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: FOURTEEN

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: FIFTEEN

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Formula? Note that we have been counting along successive cross-diagonals.

$$\begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

$(3,2)$ is the 3rd pair on the 4th cross-diagonal.
On the first 3 cross-diagonals, there are $1+2+3=6$ pairs.
So the pair $(3,2)$ is the $3+6$th pair that we count.

More generally, (m,n) is the $m+\frac{(m+n-2)(m+n-1)}{2}$th pair that we count.

We can use this to show that there are as many positive integers as there are positive rational numbers (which are all numbers of the form $\frac{m}{n}$ where m and n are positive integers).

Each positive rational number can be written **uniquely** as $\frac{m}{n}$ where m and n are positive integers with $\text{hcf}(m, n) = 1$.

So we can identify the set of positive rational numbers with a **subset** of $\mathbb{Z}_+ \times \mathbb{Z}_+$, the set of ordered pairs of positive integers: $\frac{m}{n} \leftrightarrow (m, n)$.

Hence with $\mathbb{Q}_+$ denoting the set of positive rational numbers, we have

$$|\mathbb{Q}_+| \leq |\mathbb{Z}_+ \times \mathbb{Z}_+| = |\mathbb{Z}_+|.$$

But we also know that each positive integer is a positive rational number, so

$$|\mathbb{Z}_+| \leq |\mathbb{Q}_+|.$$

Thus $|\mathbb{Q}_+| = |\mathbb{Z}_+|$.

Are all infinite sets countable, meaning the same "size" as $\mathbb{Z}_+$?
No!

We can show that there are more **real** numbers in the interval between 0 and 1 than there are positive integers.

To do this, we argue by **contradiction**: We assume there **are** as many positive integers as there are real numbers between 0 and 1, and we then derive a **contradiction**.

Technical fact: Each real number has a **unique** decimal expansion, **provided** we not use any decimal expansion that ends in all 9's.

Using geometric progressions, we can show that $0.99999\cdots = 1$, *so for instance,* $0.356799999\ldots = 0.35680000\ldots$

Suppose that the real numbers between 0 and 1 are countable, meaning that they are in one-to-one correspondence with the positive integers. Then we can enumerate these real numbers as

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \ldots$$

Each of these has a decimal expansion. For instance,

$$\alpha_1 = 0.a_{1,1}a_{1,2}a_{1,3}a_{1,4}a_{1,5}\cdots$$

where each $a_{1,j}$ is a digit in the decimal expansion of α_1.

(So $a_{1,j} = 0, 1, 2, 3, 4, 5, 6, 7, 8,$ or 9.)

We create a grid of these numbers with their decimal expansions:

$$\begin{aligned}
\alpha_1 &= 0.a_{1,1}a_{1,2}a_{1,3}a_{1,4}a_{1,5}\cdots \\
\alpha_2 &= 0.a_{2,1}a_{2,2}a_{2,3}a_{2,4}a_{2,5}\cdots \\
\alpha_3 &= 0.a_{3,1}a_{3,2}a_{3,3}a_{3,4}a_{3,5}\cdots \\
\alpha_4 &= 0.a_{4,1}a_{4,2}a_{4,3}a_{4,4}a_{4,5}\cdots \\
\alpha_5 &= 0.a_{5,1}a_{5,2}a_{5,3}a_{5,4}a_{5,5}\cdots \\
&\vdots
\end{aligned}$$

Now we construct another real number β between 0 and 1 that is **not** in the list $\alpha_1, \alpha_2, \alpha_3, \cdots$:

We look at the "diagonal" of our grid of decimal expansions:

$$\alpha_1 = 0.a_{1,1}a_{1,2}a_{1,3}a_{1,4}a_{1,5}\cdots$$
$$\alpha_2 = 0.a_{2,1}a_{2,2}a_{2,3}a_{2,4}a_{2,5}\cdots$$
$$\alpha_3 = 0.a_{3,1}a_{3,2}a_{3,3}a_{3,4}a_{3,5}\cdots$$
$$\alpha_4 = 0.a_{4,1}a_{4,2}a_{4,3}a_{4,4}a_{4,5}\cdots$$
$$\alpha_5 = 0.a_{5,1}a_{5,2}a_{5,3}a_{5,4}a_{5,5}\cdots$$
$$\vdots$$

We set $\beta = 0.b_1 b_2 b_3 b_4 b_5 \cdots$ where

$$b_i = \begin{cases} 1 & \text{if } a_{i,i} \neq 1, \\ 2 & \text{if } a_{i,i} = 1. \end{cases}$$

We know that β is not equal to any α_i since the ith digit in the decimal expansion of β is not equal to the ith digit in the decimal expansion of α_i.

This is our **contradiction**:

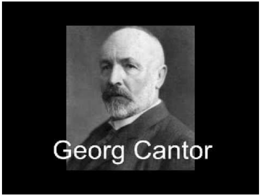
We *assumed* that the set of real numbers between 0 and 1 was countable, meaning that we could enumerate all of them as $\alpha_1, \alpha_2, \alpha_3, \cdots$, and then we *constructed* a real number β between 0 and 1 that is not in this list.

Hence it **cannot** be the case that the set of real numbers between 0 and 1 is countable.
Thus the magnitude of the set of real numbers between 0 and 1 is **larger** than that of the positive integers.

Mathematicians have proved that there is a one-to-one correspondence between **all** real numbers, and those between 0 and 1 (we will prove this). (So in some sense, the magnitude of the unit interval is the same as the magnitude of the real line.)

Mathematicians have also proved that there are magnitudes of infinity **larger** than that of the real line (we will prove this).

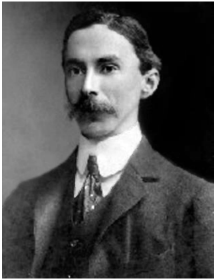
Unknown: Is there a magnitude of infinity **between** that of the positive integers and that of the real numbers?



Georg Cantor, 1845 – 1918

Cantor introduced and studied these notions of magnitudes of infinity, which he believed came directly from God.

Although Cantor's work is now fundamental in mathematics, initially there were mathematicians and philosophers who objected vehemently to Cantor's claims and methods of argument, calling him a *scientific charlatan* and a *corrupter of youth*, whose work was *laughable, wrong* and *contradicted the uniqueness of the absolute infinity in the nature of God*.



Bertrand Russell, 1872 – 1970

Another significant outcome of Cantor's work was the discovery of certain **paradoxes**, exemplifying that not everything one can think to write down gives an unambiguous defnition, and in particular, that one needs to be careful when defining a set.

For instance, in 1901 Russell presented the following:

Suppose there is a set R consisting of all sets that do not contain themselves as elements.

But then we have an unresolvable paradox:

If $R \notin R$ then $R \in R$, and if $R \in R$ then $R \notin R$.

A variation of this paradox is the following:

A PARADOX:

The barber is the man in town who shaves all the men who do not shave themselves.

Who shaves the barber?

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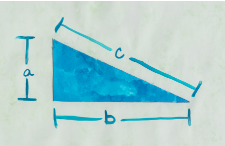
As mathematics instructors at Bristol, we are passionate about mathematics, and we want to infect others with the enthusiasm we feel for our subjects.

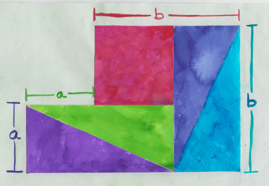
As a student here, we want you to engage with us in the mathematics, to work hard, and to feel that joyous sense of accomplishment when the hard work leads to those "Aha!" moments.

Learning mathematics is not a spectator sport! As a student here, we want you to engage with us in lectures, work seriously on the homework, ask us questions, ask your tutors questions, ask your friends questions, ask yourself questions.

We are here to help you learn, and we are eager that you do, but the most important ingredient in this process is ... **YOU!**

Mathematics captures truth and beauty





$$\boxed{a^2 + b^2 = c^2}$$