Counting the Infinite

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'Ladder to Heaven'
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I am a number theorist, and as such I love to **count**.
The counting numbers are $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \ldots$

Most of what I count is **finite**, but not everything we want to count is.
Some collections are **infinite** but **countable**.

What is countable?

A collection of infinitely many objects is called **countable** if we can count them, meaning that we can establish a one-to-one correspondence between these objects and the counting numbers .

Weird things can happen with infinite sets (i.e. infinite collections).

For example, here is a one-to-one correspondence:

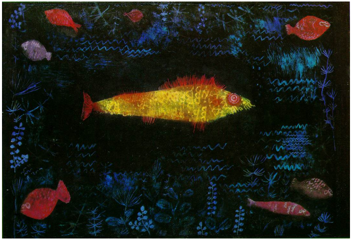
$$\text{counting numbers} \leftrightarrow \textbf{even} \text{ counting numbers}$$
$$1 \leftrightarrow 2$$
$$2 \leftrightarrow 4$$
$$3 \leftrightarrow 6$$
$$\vdots$$

So even though the set of **even** counting numbers is a subset (i.e. a subcollection) of the set of **all** counting numbers, these two sets have the same magnitude ("cardinality").

Claim: every infinite set has a countable subset.

How do we see this?

Imagine we've got a magic bag that's holding infinitely many fish. How do we select countably many of them?



Reach in, grab one, and call this $fish_1$.

Grab another, and call this $fish_2$.

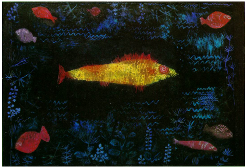
Keep going... after selecting 10,000,000 fish, there are still infinitely many left in the bag.

So we can keep doing this forever....

Or can we?

Does this method really work?
Well, almost...

But how, in our lifetimes, can we choose one fish at a time, countably many times?



We need to assume the Axiom of Countable Choice, which says that given a countable collection of nonempty sets, we can choose one element from each set.

Set notation:

What is the difference between

$$fish_1, \ fish_2$$

and

$$\{fish_1, \ fsh_2\}?$$

$fish_1, \ fish_2$ is a list of fish, while $\{fish_1, \ fsh_2\}$ denotes the set, or collection, containing $fish_1$ and $fish_2$.

So $\{fish_1, \ fsh_2\}$ is like a bag containing $fish_1$ and $fish_2$.

Proving every infinite set has a countable subset:

Let X be the **set** (i.e. collection) of the infinitely many fish in our magic bag.

Let A_0 be the collection of all subsets of X with exactly $1 = 2^0$ fish; so for any *fish* in X, $\{fish\}$ is an element of A_0.

Let A_1 be the collection of all subsets of X with exactly $2 = 2^1$ fish; so for any two distinct *fish*, *fish'* in X, $\{fish, fish'\}$ is an element of A_1.

For each counting number n, let A_n be the collection of all subsets of X with 2^n fish.

Thus $A_0, A_1, A_2, \ldots, A_n, \ldots$ is a list of countably many nonempty sets.

Now we use the **Axiom of Countable Choice**:

For each counting number n, we choose an element B_n from A_n.

So B_0 is of the form $B_0 = \{fish_1\}$ where $fish_1$ is a fish from X.

Also, B_1 is a set with $2 = 2^1$ fish from X, but as far as we know, we could have $B_1 = \{fish_2, fish_3\}$ **or** $B_1 = \{fish_1, fish_2\}$.

In general, B_n is a set with 2^n fish, but many of these could be from $B_0, B_1, B_2, \ldots, B_{n-1}$.

So we set $C_0 = B_0$,
and we set

$$C_1 = \{\text{all fish in } B_1 \text{ that are } \textbf{not} \text{ in } B_0\ \},$$

$$C_2 = \{\text{all fish in } B_2 \text{ that are } \textbf{not} \text{ in } B_0 \text{ or } B_1\ \},$$

and in general,

$$C_n = \{\text{all fish in } B_n \text{ that are } \textbf{not} \text{ in } B_0 \text{ or } B_1 \text{ or } B_2 \text{ or } \dots \text{ or } B_{n-1}\ \}.$$

Claim: For each counting number n, the set C_n has at least one fish.
To see this, we recall that B_n has 2^n fish, and **together**, the sets $B_0, B_1, B_2, \dots, B_{n-1}$ have **at most**

$$2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} = \frac{2^n - 1}{2 - 1} = 2^n - 1$$

fish.
So C_n has **at least** $2^n - (2^n - 1) = 1$ fish.

Now we use the **Axiom of Countable Choice** once more:

For each counting number n, we choose one fish α_n from C_n.

We know that C_1 has at least one fish, and this fish is not in C_0, so α_1 is not the same as α_0.

We know that C_2 has at least one fish, and this fish is not in C_0 or C_1, so α_2 is not the same as α_0 or α_1.

In general, C_n has at least one fish, and this fish is not in C_0 or C_1 or ... or C_{n-1}, so α_n is not the same as α_0 or α_1 or ... or α_{n-1}.

So in this (rather lengthy) way, we construct a set $Y = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \ldots\}$ where all the fish in Y are from X, and the fish in are in **one-to-one correspondence with the counting numbers**.

Can we find an infinite set with magnitude bigger than countable?

Let's try: let's first consider the set X of all pairs of counting numbers.
So this is the set of all pairs (k, m) where k and m are counting numbers.

Certainly X has a "copy" of the counting numbers:

(1,1), (2,1), (3,1), (4,1), (5,1), (6,1), (7,1), ...

Actually, X has many copies of the counting numbers:

(1,2), (2,2), (3,2), (4,2), (5,2), (6,2), (7,2), ...
(1,3), (2,3), (3,3), (4,3), (5,3), (6,3), (7,3), ...

How many similar copies of the counting numbers does X have?
Countably many!

So X is comprised of countably many copies of the counting numbers...
Surely this means the magnitude of X is bigger than countable?

A grid of the ordered pairs of counting numbers

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: ONE

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: TWO

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: THREE

$$
\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Counting: FOUR

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: FIVE

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: SIX

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: SEVEN

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: EIGHT

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: NINE

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: TEN

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
{\color{red}(4,1)} & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: ELEVEN

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: TWELVE

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: THIRTEEN

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: FOURTEEN

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & \color{red}{(4,2)} & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: FIFTEEN

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Formula? Note that we have been counting along successive cross-diagonals.

$$\begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

$(3,2)$ is the 3rd pair on the 4th cross-diagonal.
On the first 3 cross-diagonals, there are $1+2+3=6$ pairs.
So the pair $(3,2)$ is the $3+6$th pair that we count.

More generally, (m,n) *is the* $m+\frac{(m+n-2)(m+n-1)}{2}$*th pair that we count.*

So this shows that there is a one-to-one correspondence between the counting numbers and the **pairs** of counting numbers, meaning...
the set of pairs of counting numbers is COUNTABLE!

Are all infinite sets countable?
No!

We can show that the magnitude of **real** numbers between 0 and 1 is larger than the magnitude of counting numbers.

Each real number between 0 and 1 has a decimal expansion of the form

$$0.b_1 b_2 b_3 \cdots$$

where each $b_1, b_2, b_3, \cdots$ denotes a digit between 0 and 9.

Technical fact: Each real number has a **unique** decimal expansion, **provided** we not use any decimal expansion that ends in all 9's.

Using "geometric progressions", we can show that $0.99999\cdots = 1$, *so for instance,* $0.356799999\ldots = 0.35680000\ldots$

So certainly we have **infinitely many** real numbers between 0 and 1.

To **prove** that the magnitude of real numbers between 0 and 1 is larger than that of counting numbers, we argue by **contradiction**:

Suppose that the real numbers between 0 and 1 are countable, meaning that they are in one-to-one correspondence with the counting numbers. Then we can enumerate these real numbers as

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \ldots$$

Each of these has a unique decimal expansion that does not end in all 9s. For instance, we can write

$$\alpha_1 = 0.a_{1,1}a_{1,2}a_{1,3}a_{1,4}a_{1,5}\cdots$$

where each $a_{1,j}$ is $0, 1, 2, 3, 4, 5, 6, 7, 8,$ or 9.

We create a grid of these numbers with their decimal expansions:

$$
\begin{aligned}
\alpha_1 &= 0.a_{1,1}a_{1,2}a_{1,3}a_{1,4}a_{1,5}\cdots \\
\alpha_2 &= 0.a_{2,1}a_{2,2}a_{2,3}a_{2,4}a_{2,5}\cdots \\
\alpha_3 &= 0.a_{3,1}a_{3,2}a_{3,3}a_{3,4}a_{3,5}\cdots \\
\alpha_4 &= 0.a_{4,1}a_{4,2}a_{4,3}a_{4,4}a_{4,5}\cdots \\
\alpha_5 &= 0.a_{5,1}a_{5,2}a_{5,3}a_{5,4}a_{5,5}\cdots \\
&\vdots
\end{aligned}
$$

Now we construct another real number β between 0 and 1 that is **not** in the list $\alpha_1, \alpha_2, \alpha_3, \cdots$:

We look at the "diagonal" of our grid of decimal expansions:

$$\begin{aligned}
\alpha_1 &= 0.a_{1,1}a_{1,2}a_{1,3}a_{1,4}a_{1,5}\cdots \\
\alpha_2 &= 0.a_{2,1}a_{2,2}a_{2,3}a_{2,4}a_{2,5}\cdots \\
\alpha_3 &= 0.a_{3,1}a_{3,2}a_{3,3}a_{3,4}a_{3,5}\cdots \\
\alpha_4 &= 0.a_{4,1}a_{4,2}a_{4,3}a_{4,4}a_{4,5}\cdots \\
\alpha_5 &= 0.a_{5,1}a_{5,2}a_{5,3}a_{5,4}a_{5,5}\cdots \\
&\vdots
\end{aligned}$$

We set $\beta = 0.b_1 b_2 b_3 b_4 b_5 \cdots$ where for each counting number i,

$$b_i = \begin{cases} 1 & \text{if } a_{i,i} \neq 1, \\ 2 & \text{if } a_{i,i} = 1. \end{cases}$$

We know that β is not equal to any α_i since for each counting number i, the ith digit in the decimal expansion of β is not equal to the ith digit in the decimal expansion of α_i.

This is our **contradiction**:

We *supposed* that the set of real numbers between 0 and 1 was countable, meaning that we could enumerate all of them as $\alpha_1, \alpha_2, \alpha_3, \cdots$, and then we *constructed* a real number β between 0 and 1 that is not in this list.

Hence it **cannot** be the case that the set of real numbers between 0 and 1 is countable.

We know there are infinitely many real numbers between 0 and 1, and the smallest infinite set is countable in magnitude, so...
the magnitude of the set of real numbers between 0 and 1 is **larger** than that of the counting numbers.

Unknown: Is there a magnitude of infinity **between** that of the counting numbers and that of the real numbers between 0 and 1?

We can also prove that there are **infinitely many magnitudes of infinity**!

We can do this using power sets: given a set X, the power set of X is the set of all subsets of X.

So if $X = \{1, 2, 3\}$, the power set of X is

$$\mathcal{P}(X) = \Big\{ \{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \Big\}.$$

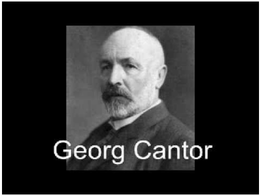
When X is an infinite set, the magnitude of $\mathcal{P}(X)$ is larger than that of X.

The proof of this is very similar to the following paradox:

The barber is the man in town who shaves all the men who do not shave themselves.

Who shaves the barber?

A bit of history:



Georg Cantor, 1845 – 1918

Cantor introduced and studied these notions of magnitudes of infinity, which he believed came directly from God.

Although Cantor's work is now fundamental in mathematics, initially there were mathematicians and philosophers who objected vehemently to Cantor's claims and methods of argument, calling him a *scientific charlatan* and a *corrupter of youth*, whose work was *laughable, wrong* and *contradicted the uniqueness of the absolute infinity in the nature of God*.

Alice Walker on infinity, life, and love:

Those who love the entire cosmos rather than their own tiny country, city or farm will be shown the unbroken web of life and the meaning of infinity.

By Virginia DeBolt (Alice Walker speaks) [CC BY-SA 2.0 (https://creativecommons.org/licenses/by-sa/2.0)], via Wikimedia Commons