INDEFINITE QUADRATIC FORMS AND EISENSTEIN SERIES

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Abstract. We use geometric algebra and the theory of automorphic forms to realize the theta series attached to an indefinite quadratic form as the sum of a specific Eisenstein series and an L^2-function. From this we obtain explicit formulas for the measure of the representation of an integer by an indefinite quadratic form.

§Introduction.

In this paper we study, from the point of view of automorphic forms, the representation number of indefinite quadratic forms. In [12], the second author used similar techniques for positive definite quadratic forms. In that case, much of the local theory was similar, but the global automorphic theory was much simpler. As is typical, we start with some notation and statement of our main results.

Let V be an m-dimensional vector space over $\mathbb{Q}$ with m even and $m \geq 6$. Let L be a lattice of full rank m on V on V (so L is a $\mathbb{Z}$-module and $L \otimes \mathbb{Q} = V$). Let $B : V \times V \to \mathbb{Q}$ be a nondegenerate symmetric bilinear form whose assoicated quadratic form Q, defined by $Q(v) = B(v, v)$, is indefinite. For convenience, we assume $Q(L) \subseteq 2\mathbb{Z}$. The representation numbers

$$r(L, 2n) = \#\{\ell \in L : \; Q(\ell) = 2n \; \}$$

are typically infinite (but finite in the definite case). In fact,

$$r_t(L, 2n) = \#\{\ell \in L : \; Q(\ell) = 2n, \; \ell \text{ in a ball of radius } t \; \} = O(t^{m/2-1}),$$

and in general this bound is tight. Hence we can measure the density of solutions $\ell \in L$ to the equation $Q(\ell) = 2n$ with the quantity

$$\rho(L, 2n) = \lim_{t \to \infty} t^{1-m/2} r_t(L, 2n).$$

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Following Siegel, we call $\rho(L,2n)$ the measure of the represenation of $2n$ by L. Siegel proved that $\rho(L,2n)$ is a product of "p-adic densities" (which he did not compute); Siegel also proved that $\rho(L,2n)$ is a genus invariant and is given by the Fourier coefficients of a theta series associated to L (as defined below).

We use Siegel's theta series together with elementary geometric algebra and the theory of automorphic forms to constructively prove Siegel's results when the localization of L at 2 is unimodular. We show

$$\rho(L,2n) = \rho_{L,\infty} \prod_{p \text{ prime}} \rho_{L,p}(n)$$

where $\rho_{L,\infty}$ depends only on the signature of Q and for p prime, $e = \mathrm{ord}_p(n)$, $\epsilon = \left(\frac{2n/p^e}{p}\right)$,

$$\rho_{L,p}(n) = \left(\sum_{0\le\ell<e} \nu_\ell(0;L,p)p^{(1-m/2)\ell}\right) + \nu_e(\epsilon;L,p)p^{(1-m/2)e};$$

the summands $\nu_\ell(*;L,p)$ are given by simple formulas in terms of the invariants of $L_{(p)} = L\otimes\mathbb{Z}_p$.

We give here a brief outline of our strategy. Let R_Q be a majorant for Q; thus R_Q is a positive definite quadratic form on V and, identifying R_Q, Q with matrices, $R_Q Q^{-1} R_Q = Q$. Then for $z = x+iy$, $y>0$, Siegel's theta series is

$$\theta(L;R_Q,z) = \sum_{\ell\in L} e^{\pi i(Q(\ell)x + iR_Q(\ell)y)}.$$

In §1 we show that $\theta(L;R_Q,z)$ is an automorphic form for a congruence subgroup $\Gamma_0(N)$ with weight $((m_1-m_2)/2, m_2)$ where (m_1,m_2) is the signature of Q. (When $L_{(2)}$ is unimodular, N is necessarily odd.) We also show that $\theta(L;R_Q,z) - \theta(L';R_Q,z) \in L^2$ whenever L' is in the genus of L (relative to Q). In §2 we construct a basis $\{E_D : \ D|N\ \}$ for the space of Eisenstein series of weight (k,k') and odd, square-free level N, and we compute the Fourier coefficients of each E_D. In §3 we consider lattices K of "minimal level N and discriminant d_K". Using geometric algebra, we construct operators $T_K(q)$ for each prime $q|N$ so that

$$\theta(K;R_Q,z)\big|T_K(q) = \sum_{K'} \theta(K';R_Q,z)$$

where the K' lie in the genus of K. Thus the results of §1 imply $\theta(K;R_Q,z)$ is an approximate eigenform for $T_K(q)$; that is,

$$\theta(K;R_Q,z)\big|T_K(q) = \lambda_K(q)\theta(K;R_Q,z) + \varepsilon_q(z)$$

where $\varepsilon_q \in L^2(\Gamma_0(N)\backslash\mathcal{H})$. We also show that the simultaneous eigenspace for the $T_K(q)$ within the space of Eisenstein series is one-dimensional. We know $\theta(K; R_Q, z) = E(z) + \varepsilon(z)$ for some E in the space of Eisenstein series and some $\varepsilon \in L^2(\Gamma_0(N)\backslash\mathcal{H})$; since each $T_K(q)$ maps Eisenstein series to Eisenstein series and L^2 functions to L^2 functions, we find that

$$\theta(K; R_Q, z) = \left(\frac{1}{c_K(N) a_N(0)} \sum_{D|N} c_K(D) E_D(z) \right) + \varepsilon(z)$$

where the $c_K(D)$ are explicit constants and $a_N(0)$ is the 0th coefficient of E_N. In §4 we show that, regardless of the choice of R_Q,

$$\rho(K, 2n) = \lim_{y \to 0^+} y^{m/2-1} c_n(R_Q, y)$$

where $c_n(R_Q, y)$ is, up to constants, the nth Fourier coefficient of $\theta(K; R_Q, z)$. We show that the L^2 function ε contributes nothing to $\rho(K, 2n)$, so interpreting the Fourier coefficients of the sum of Eisenstein series as an Euler product, we find

$$\rho(K, 2n) = \rho_{K,\infty} \prod_{p \text{ prime}} \rho_{K,q}(n).$$

Finally, in §5 we consider lattices L of arbitrary odd level. Again using geometric algebra, we show L descends from a lattice K of minimal level and discriminant and

$$\theta(L; R_Q, z) = \frac{1}{\delta} \sum_{K_0 \subset K} \theta(K_0; R_Q, z) + \varepsilon'(z)$$

where the K_0 are particular sublattices of K (of which there are δ) and $\varepsilon' \in L^2(\Gamma_0(N)\backslash\mathcal{H})$. Since our construction of the K_0 allows us to count how often a vector $v \in K$ lies in a sublattice K_0, we obtain our formula for $\rho(L, 2n)$.

For basic references on the theory of automorphic forms, quadratic forms and Siegel's work in this area see [1,2,6,7,8].

§1. The theta series.

Let (m_1, m_2) be the signature of Q and, without loss of generality, assume $m_1 \geq m_2$. Let R_Q be a majorant for Q; thus $R_Q : V \otimes \mathbb{R} \to \mathbb{R}$ is a positive definite quadratic form and, identifying R_Q, Q with matrices relative to some basis for V, $R_Q Q^{-1} R_Q = Q$.

Remark. Given Q, a majorant R_Q always exists. For example, identify Q with a symmetric matrix relative to some basis for V. Then for some $S \in GL_m(\mathbb{R})$ we have $Q = S^t D S$ where $D = \operatorname{diag}\{1, \dots, 1, -1, \dots, -1\}$. (So the signature of D is that of Q.) Then $R_Q = S^t S$ is a majorant for Q.

On the other hand, suppose R_Q is a majorant for Q. Thus $R_Q = S^t S$ for some nonsingular matrix S. Set $Q' = S^{-t} Q S^{-1}$; by the Singular Value Decomposition theorem, we can write $Q' = T^t D T$ where D is diagonal and T is orthogonal. Thus $Q = (TS)^t D (TS) = T_1^t D T_1$, say, and $R_Q = (TS)^t (TS) = T_1^t T_1$. Also, since $R_Q Q^{-1} R_Q = Q$, we necessarily have $D^{-1} = D$. Hence

$$D = \mathrm{diag}\{1, \dots, 1, -1, \dots, -1\}$$

and the signature of D and Q agree.

For $h \in V$, define the inhomogeneous theta series

$$\theta(L, h; R_Q, z) = \sum_{\ell \in L} e\{Q(\ell + h)x + iR_Q(\ell + h)y\}$$

where $z = x + iy$, $y > 0$, and $e\{\alpha\} = e^{\pi i \alpha}$. For notational simplicity, let $\theta(L; R_Q, z) = \theta(L, 0; R_Q, z)$.

Let $k = (m_1 - m_2)/2$ and $k' = m_2$. Define a function (actually a generalized automorphy factor) for $z \in \mathcal{H}$ via

$$j_{k,k'}(z) = z^{-k} |z|^{-k'}.$$

Note that since m is even and $m_1 + m_2 = m$, we have $k \in \mathbb{Z}$, so there are no subtle issues about which square root to take. Let $L^{\#} = \{v \in V : B(v, L) \subseteq \mathbb{Z}\}$ denote the dual of L and let d_L denote the discriminant of L relative to Q.

Our first lemma sets the stage for the complete transformation formula for our theta function under the appropriate modular group.

Lemma 1.1 (Inversion Formula). *For $h \in V$, we have*

$$\theta(L, h; R_Q, z) = \epsilon |d_L|^{-1/2} j_{k,k'}(z) \sum_{\ell \in L^{\#}} e\{2B(\ell, h) + Q(\ell)x' + iR_Q(\ell)y'\}$$

where $-1/z = x' + iy'$ and ϵ is a 4th root of unity (independent of z and L). In particular, if $h \in L^{\#}$,

$$\theta(L, h; R_Q, z) = \epsilon |d_L|^{-1/2} j_{k,k'}(z) \sum_{\ell \in L^{\#}/L} e\{2B(\ell, h)\} \theta\left(L, \ell; R_Q, -1/z\right)$$

and

$$\theta(L; R_Q, z) = \epsilon |d_L|^{-1/2} j_{k,k'}(z) \theta\left(L^{\#}; R_Q, -1/z\right).$$

Proof. Note that the second formula is an immediate consequence of the first. It is possible to prove this directly using Poisson summation and the fact that $R_Q Q^{-1} R_Q = Q$. To avoid the grubby details of this sort of computation, we will instead identify $\theta(L,h;R_Q,z)$ with a symplectic theta function and use the corresponding inversion formula (see [3]).

Fix a $\mathbb{Z}$-basis for L and thus identify L with $\mathbb{Z}^m$. Relative to this basis, identify Q, R_Q with matrices; set $Z = Qx + iR_Qy$. Let

$$\vartheta\left(Z, \begin{pmatrix} u \\ v \end{pmatrix}\right) = \sum_{\ell\in\mathbb{Z}^m} e\{Z(\ell+v) - 2\ell^t u - v^t u\}.$$

Thus, by the referenced symplectic theta function inversion formula,

$$\begin{aligned}\theta(L,h;R_Q,z) &= \vartheta\left(Z, \begin{pmatrix} 0 \\ h \end{pmatrix}\right) \\ &= \epsilon(\det Z)^{-1/2}\vartheta\left(-Z^{-1}, \begin{pmatrix} -h \\ 0 \end{pmatrix}\right) \\ &= \epsilon(\det Z)^{-1/2}\sum_{\ell\in L} e\{Q^{-1}(\ell)x' + iR_Q^{-1}(\ell)y' + 2h^t\ell\} \\ &= \epsilon(\det Z)^{-1/2}\sum_{\ell\in Q^{-1}L} e\{Q(\ell)x' + iR_Q(\ell)y' + 2h^tQ\ell\}\end{aligned}$$

where $x' = -x/(x^2+y^2)$ and $y' = y/(x^2+y^2)$. To complete the proof, we observe that $Q^{-1}L = L^\#$ and that $\det Z = d_L \cdot j_{k,k'}(z)^2$. The latter is easily deduced from the relations $R_Q = S^tS$ and $Q = S^tDS$ where $D = \operatorname{diag}\{1,\dots,1,-1,\dots,-1\}$. $\square$

For $k, k' \in \mathbb{Z}$ as above, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, $\mathcal{H} = \{z \in \mathbb{C} : \Im m\ z > 0\}$, and a function $g : \mathcal{H} \to \mathbb{C}$, we define the slash operator with weight (k,k') by

$$\left(g\big|\gamma\right)(z) = \left(g\big|_{k,k'}\gamma\right)(z) = j_{k,k'}(cz+d)g\left(\gamma(z)\right).$$

Let N be the level of L relative to Q. In other words, if we identify Q with a (symmetric) matrix relative to a $\mathbb{Z}$-basis for L, N is the smallest positive integer such that NQ^{-1} is an integral matrix with even diagonal entries. Let $(*/*)_K = \left(\frac{*}{*}\right)_K$ denote the Kronecker symbol.

Lemma 1.2 (Transformation Formula). *For* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, $d \neq 0$,

$$\theta(L;R_Q,z)\big|\gamma = \chi(d)\theta(L;R_Q,z),$$

where χ *is a quadratic Dirichlet character modulo* N *and the weight for the slash operator is* (k,k'). *Furthermore,*

$$\chi(d) = (\operatorname{sgn} d)^k \left(\frac{(-1)^{m/2}d_L}{|d|}\right)_K.$$

If $d = 0$, *then* $N = 1$ *and* $\theta(L; R_Q, z)\big|\gamma = \theta(L; R_Q, z)$.

Remark. With N odd, another description of N is as follows. A prime p does not divide N if and only if $L_{(p)} = L \otimes \mathbb{Z}_p$ is unimodular. Also, $\chi(p) = 1$ if and only if $L_{(p)}$ is an orthogonal sum of hyperbolic planes $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. (So for $p = 2$, either $L_{(2)} \simeq \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \perp \cdots \perp \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ or $L_{(2)} \simeq \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \perp \cdots \perp \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$). For $q^r \| N$,

$$L_{(q)} \simeq \big\langle 1, \dots, 1, \epsilon_0 \big\rangle \perp q\big\langle 1, \dots, 1, \epsilon_1 \big\rangle \perp \cdots \perp q^r\big\langle 1, \dots, 1, \epsilon_r \big\rangle$$

with $\epsilon_i \in \mathbb{Z}_q^\times$.

Proof. We prove the last statement of the lemma first. If $d = 0$ then clearly $N = 1$, and $\gamma = \pm \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus, by Lemma 1.1, $\theta(L; R_Q, z)\big|\gamma = \theta(L; R_Q, z)$.

Now assume $d \neq 0$, write $z' = (c + d/z)^{-1} = x' + iy'$, $-1/z = x'' + iy''$ and observe that

$$\frac{az+b}{cz+d} = \frac{b}{d} + \frac{1}{d(c+d/z)}.$$

So, by definition,

$$\begin{aligned}
&\theta(L; R_Q, z)\big|\gamma \\
&\quad = j_{k,k'}(cz+d) \sum_{\ell \in L} e\left\{\frac{b}{d}Q(\ell)\right\} e\left\{Q(\ell)\frac{x'}{d} + iR_Q(\ell)\frac{y'}{d}\right\} \\
&\quad = j_{k,k'}(cz+d) \sum_{h \in L/dL} e\left\{\frac{b}{d}Q(h)\right\} \theta\left(dL, h; R_Q, \frac{z'}{d}\right) \\
&\quad = j_{k,k'}(cz+d) \sum_{h \in L/dL} e\left\{\frac{b}{d}Q(h)\right\} \theta\left(L, \frac{h}{d}; R_Q, dz'\right)
\end{aligned}$$

and by the Inversion Formula of Lemma 1.1, this is

$$\begin{aligned}
&= \epsilon |d_L|^{-1/2} j_{k,k'}(cz+d) j_{k,k'}(dz') \\
&\quad \cdot \sum_{\substack{h \in L/dL \\ \ell \in L^{\#}}} e\left\{\frac{b}{d}Q(h) + \frac{2}{d}B(\ell, h) - \frac{c}{d}Q(\ell)\right\} e\{Q(\ell)x'' + iR_Q(\ell)y''\}.
\end{aligned} \tag{1.1}$$

For $h \in L$, $\ell \in L^{\#}$,

$$\frac{b}{d}Q(h - c\ell) \equiv \frac{b}{d}Q(h) + \frac{2}{d}B(h, \ell) - \frac{c}{d}Q(\ell) \pmod 2,$$

since $bc = ad - 1$ and also, $c \in N\mathbb{Z}$ so $cQ(\ell) \in 2\mathbb{Z}$. For any $\ell \in L^{\#}$, $h - c\ell$ varies over L/dL as h does. Therefore

$$\begin{aligned}
&\theta(L; R_Q, z)\big|\gamma \\
&\quad = \epsilon |d_L|^{-1/2} j_{k,k'}(cz+d) j_{k,k'}(dz') \sum_{h \in L/dL} e\left\{\frac{b}{d}Q(h)\right\} \cdot \theta\left(L^{\#}; R_Q, -\frac{1}{z}\right)
\end{aligned}$$

and again using the Inversion Formula, this is

$$\begin{aligned}
&= j_{k,k'}(cz+d) j_{k,k'}(dz') j_{k,k'}(z)^{-1} \sum_{h \in L/dL} e\left\{\frac{b}{d}Q(h)\right\} \cdot \theta\left(L; R_Q, z\right) \\
&= (\operatorname{sgn} d)^k |d|^{-m/2} \sum_{h \in L/dL} e\left\{\frac{b}{d}Q(h)\right\} \cdot \theta\left(L; R_Q, z\right).
\end{aligned}$$

To analyze the exponential sum, write $d = \pm p_1^{e_1} \cdots p_r^{e_r}$ where the p_i are primes. Then by the Chinese Remainder Theorem,

$$L/dL \approx dp_1^{-e_1} L/dL \oplus \cdots \oplus dp_r^{-e_r} L/dL,$$

and for $\ell_i \in dp_i^{-e_i} L$,

$$Q(\ell_1 + \cdots + \ell_r) \equiv Q(\ell_1) + \cdots Q(\ell_r) \pmod{2d}.$$

Hence

$$\sum_{\ell \in L/dL} e\left\{\frac{b}{d}Q(\ell)\right\} = \prod_{p^e \| d} \sum_{\ell \in dp^{-e} L/dL} e\left\{\frac{b}{d}Q(\ell)\right\}.$$

Now, for $p^e \parallel d$,

$$\sum_{\ell \in dp^{-e} L/dL} e\left\{\frac{b}{d}Q(\ell)\right\} = \sum_{L/p^e L} e\left\{\frac{b'}{p^e}Q(\ell)\right\},$$

where $b' = bd^2/p^{2e}$. By Proposition 3.2 of [10],

$$\sum_{L/p^e L} e\left\{\frac{b'}{p^e}Q(\ell)\right\} = \begin{cases} p^{me/2} & \text{if } 2|e, \\ p^{m(e-1)/2} \sum\limits_{L/pL} e\left\{\dfrac{b'}{p}Q(\ell)\right\} & \text{otherwise.} \end{cases}$$

Note that $L/pL \approx L_{(p)}/pL_{(p)}$, and when p is odd with $p \nmid N$,

$$L_{(p)} \simeq \langle 1, \ldots, 1, d_L \rangle.$$

Hence

$$\sum_{L/pL} e\left\{\frac{b'}{p}Q(\ell)\right\} = \left(\frac{b'}{p}\right)^m \left(\frac{d_L}{p}\right) \left(\sum_{a\in\mathbb{Z}/p\mathbb{Z}} e\left\{\frac{2a^2}{p}\right\}\right)^m = p^{m/2}\left(\frac{(-1)^{m/2}d_L}{p}\right)_K.$$

Also, since $2 \nmid N$, we have

$$L_{(2)} \simeq \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } (-1)^{m/2}d_L \equiv 1 \pmod 8, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} & \text{if } (-1)^{m/2}d_L \equiv 5 \pmod 8. \end{cases}$$

(See, for example [4].) So

$$\sum_{L/2L} e\left\{\frac{b'}{2}Q(\ell)\right\} = 2^{m/2}\left(\frac{(-1)^{m/2}d_L}{2}\right)_K.$$

It remains to show that the mapping $d \mapsto (\operatorname{sgn} d)^k \left(\frac{(-1)^{m/2}d_L}{|d|}\right)_K$ is a character modulo N. We do this as follows. Write $(-1)^{m/2}d_L = N_0 N_1^2$ where N_0 is square-free; so the Kronecker symbol in the above expression is $(N_0/|d|)_K$. For $0 < d < N$, $(d, N) = 1$, set $\chi(d) = (N_0/d)_K$ and extend χ to a character modulo N. Thus for any $d > 0$ with $(d, N) = 1$, $\chi(d) = (N_0/d)_K$ and $\chi(-d) = \chi(-1)\chi(d)$. So far we have shown that the mapping above agrees with the character χ if d is positive. To complete the proof, we need to show that $\chi(-1) = (-1)^k$.

Choose a prime p such that $p \equiv -1 \pmod N$ and $p \equiv 1 \pmod 4$. So by quadratic reciprocity and the fact that $N_0 \equiv 1 \pmod 4$, we have

$$\chi(-1) = \chi(p) = \left(\frac{N_0}{p}\right) = \left(\frac{p}{|N_0|}\right) = \left(\frac{-1}{|N_0|}\right) = \operatorname{sgn} N_0.$$

But, by construction, $N_0|N$, $N_0 \equiv 1 \pmod 4$, and

$$\operatorname{sgn} N_0 = (-1)^{m/2}\operatorname{sgn} d_L = (-1)^{m/2}\operatorname{sgn} d_V = (-1)^k,$$

where d_V denotes the discriminant of V. This completes the proof. $\square$

Up to this point, we have concentrated on the transformation properties of our theta functions. Now we investigate some analytic properties.

Lemma 1.3. *There exist positive integers M and $M^{\#}$, dependent only on L and R_Q, such that*

$$|\theta(L; R_Q, z) - 1| \ll e^{-y/M}$$

and for $h \in L^{\#}$, $h \notin L$,

$$|\theta(L, h; R_Q, z)| \ll e^{-y/M^{\#}}$$

uniformly in x as $y \to \infty$.

Proof. Since R_Q is positive definite, we know that

$$\#\{\ell \in L : \ R_Q(\ell) = n\} = O(n^{m/2-1}),$$

and that $R(\ell) = 0$ if and only if $\ell = 0$. Thus, uniformly in x,

$$|\theta(L; R_Q, z) - 1| \leq \sum_{\substack{\ell \in L \\ \ell \neq 0}} e^{-R(\ell)y} \ll \sum_{n \geq 1} n^{m/2-1} e^{-ny/M} \ll e^{-y/M},$$

where $M \in \mathbb{Z}_+$ so that $R(L) \subseteq M^{-1}\mathbb{Z}$.

Now for $h \in L^{\#} - L$, $L + h \subseteq L^{\#}$ but $0 \notin L + h$, so, by a similar argument, $|\theta(L, h; R_Q, z)| \leq \sum_{\ell \in L^{\#}, \ell \neq 0} e^{-R(\ell)y} \ll e^{-y/M^{\#}}$. $\square$

We now define define the relevant analytic space. Let $L^2(\Gamma_0(N)\backslash\mathcal{H})$ be the space of functions $f : \mathcal{H} \to \mathbb{C}$ such that for $\gamma \in \Gamma_0(N)$, we have $f|\gamma = \chi(d)f$ and

$$\int_{\Gamma_0(N)\backslash\mathcal{H}} |f(z)|^2 y^{m/2} \frac{dx\,dy}{y^2} < \infty.$$

Note that the integrand is invariant under the action of $\Gamma_0(N)$.

Proposition 1.4. *Suppose $L' \in \operatorname{gen}_Q L$, the genus of L relative to Q. Then for all $\gamma \in \Gamma_0(N)$, $\theta(L'; R_Q, z)\big|\gamma = \chi(d)\theta(L'; R_Q, z)$ and $\theta(L; R_Q, z) - \theta(L'; R_Q, z) \in L^2(\Gamma_0(N)\backslash\mathcal{H})$.*

Proof. The first part of the proposition is immediate from Lemma 1.2 because both N and χ in the transformation formula depend only on the genus of L and not on which lattice we take from $\operatorname{gen}_Q L$.

Next, take a general $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and $z \in \mathcal{H}$. Continuing the calculation in (1.1) by partitioning the sum over $\ell \in L^{\#}$, we have

$$\begin{aligned} &\theta(L; R_Q, z)\big|\gamma \\ &\quad = \epsilon |d_L|^{-1/2} j_{k,k'}(dz) \\ &\qquad \cdot \sum_{\substack{h \in L/dL \\ \ell \in L^{\#}/dL}} e\left\{\frac{b}{d}Q(h) + \frac{2}{d}B(\ell, h) - \frac{c}{d}Q(\ell)\right\} \theta(dL, \ell; R_Q, -1/z) \end{aligned}$$

which again by the inversion formula (noting that $\ell \in L^{\#}$ implies $\ell \in (dL)^{\#}$) equals

$$= \epsilon |d_L|^{-1/2} j_{k,k'}(dz) |d_{dL}|^{-1/2} j_{k,k'}(-1/z)$$
$$\cdot \sum_{\substack{h \in L/dL \\ \ell \in L^{\#}/dL \\ \ell' \in (dL)^{\#}/dL}} e\left\{ \frac{b}{d} Q(h) + \frac{2}{d} B(\ell, h) - \frac{c}{d} Q(\ell) + 2B(\ell', \ell) \right\} \theta(dL, \ell'; R_Q, z).$$

By Lemma 1.3, if $\ell' \notin dL$, then $\theta(dL, \ell'; R_Q, z)$ decays exponentially as $z \to i\infty$. For the one term $\ell' \in dL$ in the sum, $\theta(dL, \ell'; R_Q, z) - 1$ decays exponentially. Consequently, as $z \to i\infty$, the above expression tends a constant depending only on d and d_L times

$$\sum_{\substack{h \in L/dL \\ \ell \in L^{\#}/dL}} e\left\{ \frac{b}{d} Q(h) + \frac{2}{d} B(\ell, h) - \frac{c}{d} Q(\ell) \right\}.$$

This then is the constant term in the Fourier expansion of $\theta(L; R_Q, z)$ at the cusp a/c. As this depends only on the genus of L, we deduce that the difference $\theta(L; R_Q, z) - \theta(L'; R_Q, z)$ decays exponentially at every cusp. By a compactness argument, we deduce that this difference is in $L^2(\Gamma_0(N) \backslash \mathcal{H})$, as claimed. $\square$

§2. Eisenstein series for odd, square-free level.

Fix an odd, square-free positive integer N and a Dirichlet character χ mod N. For $D|N$, let χ_D denote the D-part of χ. Let N' be the conductor of χ so that $N'|N$. For $c, d \in \mathbb{Z}$, set

$$G_{c,d}(z) = G_{c,d}^{k,k'}(z; N) = \sideset{}{'}\sum_{\substack{u \equiv c \ (N) \\ v \equiv d \ (N)}} j_{k,k'}(uz + v),$$

where the prime indicates omission of the $u = v = 0$ term (if it might occur in spite of the congruence conditions).

For $D|N$, $D > 0$, set

$$E_D(z) = E_D^{k,k'}(z) = \sum_{\substack{a \bmod N \\ b \bmod N/D \\ c \bmod D}} \bar{\chi}_{N/D}(b) \chi_D(c) e\left\{ \frac{-2ac}{D} \right\} G_{bD,a}(z).$$

Our first goal in this section is to compute the constant term for the Fourier expansion for $E_D(z)$ at each cusp. Later we will give the complete Fourier expansion at the cusp at infinity. To these ends, we begin with the Fourier expansion for $G_{c,d}$ at the cusp at infinity. These computations are similar to Ogg [7], so we leave out the details. The only serious difference is that since Ogg deals only with holomorphic Eisenstein series, the zero-th Fourier coefficient is easier to handle.

Proposition 2.1. *We have*

$$\begin{aligned}G_{c,d}(z) &= a_{c,d}(0) + a'_{c,d}(0)W_{k,k'}(0)y^{1-m/2}\\ &\quad + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} a_{c,d}(n)y^{1-m/2}W_{k,k'}(ny/N)e\{2nx/N\},\end{aligned}$$

where

$$\begin{aligned}a_{c,d}(0) &= \begin{cases} 0 & \text{if } c\not\equiv 0\ (N)\\ \zeta_{k,k'}(d,N) & \text{if } c\equiv 0\ (N),\end{cases}\\ a'_{c,d}(0) &= \frac{1}{N}\zeta_{k,k'-1}(c,N),\end{aligned}$$

for $n \neq 0$,

$$a_{c,d}(n) = \frac{(\operatorname{sgn} n)^k|n|^{1-m/2}}{N}\sum_{\substack{v|n\\ n/v\equiv c\ (N)}}(\operatorname{sgn} v)^k|v|^{m/2-1}e\{2vd/N\},$$

and

$$\begin{aligned}\zeta_{k,k'}(d,N) &= \sideset{}{'}\sum_{v\equiv d\ (N)}(\operatorname{sgn} v)^k|v|^{-m/2},\\ W_{k,k'}(y) &= \int_{-\infty}^{\infty} j_{k,k'}(t+i)e\{-2yt\}dt.\end{aligned}$$

Our next simple proposition tells us something about how these G-Eisenstein series permute under the action of $SL(2,\mathbb{Z})$.

Lemma 2.2. *The functions $G_{c,d}$ over all pairs $(c,d) \bmod N$ form $SL(2,\mathbb{Z})$-equivalence classes. The equivalence classes are identified by the value of* $\gcd(c,d) \bmod N$.

Proof. Let $\gamma = \left(\begin{smallmatrix} s & t\\ u & v\end{smallmatrix}\right)$ be an element of $SL(2,\mathbb{Z})$. We see, by an easy calculation, that

$$G_{c,d}(\gamma z) = j_{k,k'}^{-1}(uz+v)G_{cs+du,ct+dv}(z),$$

(in other words, $G_{c,d}|\gamma = G_{(c,d)\gamma}$). We first deduce that $\gcd(c,d) = \gcd(cs+du, ct+dv)$. Furthermore, given any two pairs (c_1,d_1) and (c_2,d_2) with equal greatest common divisors, there is a matrix $\gamma \in SL(2,\mathbb{Z})$ with the property that $c_1s + d_1u = c_2$ and $c_1t + d_1v = d_2$. To see this, note that without loss of generality we can assume that the common gcd is equal to one. Let γ_i be any matrix in $SL(2,\mathbb{Z})$ whose first column is (c_i, d_i), for $i = 1, 2$. Then $\gamma = \gamma_2\gamma_1^{-1}$ is in $SL(2,\mathbb{Z})$ and this solves the problem. $\square$

Our next goal is to show that the set of functions E_D, with D ranging over the divisors of N have the property that their constant terms (the constant part of the zero-th Fourier coefficient) can support all functions which don't vanish at the cusps (e.g., our theta function). In other words, given any modular function F satisfying $F\big|\gamma = \chi(d)F$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, there exists a linear combination E of these functions E_D so that $F - E$ vanishes at every cusp and is in fact L^2 on the fundamental domain. (This is not to say that this difference is cuspidal — it is possible, and likely, that the non-constant part of the zero-th Fourier coefficient does not vanish.)

The set of $\Gamma_0(N)$ inequivalent cusps can be parameterized by the numbers $-1/M$, for $M > 0$ and $M|N$. (Note, the cusp at infinity is equivalent to the cusp $-1/N$.) The next proposition proves our claim and much more.

Recall that N' is the conductor of χ.

Proposition 2.3. *The Eisenstein series E_D has non-zero constant term at the cusp $-1/M$ if and only if $M|D$ and $\gcd(D/M, N') = 1$. Under these circumstances, the constant term is*

$$2\chi(-1)\varphi(D/M)(M/D)^{m/2} g_{\chi_{M'}} \mu(M'') L(m/2, \bar{\chi}_{N/D}\chi_M) \prod_{q|M''} L(m/2, \bar{\chi}_{N/D}\chi_{M'}, q)$$

where for any Dirichle character ψ, $L(s,\psi)$ is the classical Dirichlet L-function, and

$$L(s,\psi,q) = \frac{1-\psi(q)q^{1-s}}{1-\psi(q)q^{-s}},$$

$M' = \gcd(M, N')$, $M'' = M/M' = \gcd(M, N/N')$ and $g_{\chi_{M'}}$ is the standard Gauss sum.

Proof. This proposition follows by straigthforward though quite tedious calculations, so we leave them out for the sake of brevity. We note only that key to the calculations are the facts that N is square-free and that $\chi(-1) = (-1)^k$. □

Parts of the next proposition are a corollary of the last proposition. In the above, we only computed the constant term. In the next proposition we deal with all the other Fourier coefficients, but only at the cusp at infinity.

It is important to note here that even though the $G_{c,d}$ are defined on the group $\Gamma(N)$, it will turn out that E_D is defined on $\Gamma_0(N)$. In particular, we find that, though $G_{c,d}$ are translation invariant in $z \to z+N$, E_D is invariant under $z \to z+1$.

Proposition 2.4. *With the notation above, we have*

$$E_D(z) = a_D(0) + a'_D(0)y^{1-m/2}W_{k,k'}(0) + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} a_D(n)y^{1-m/2}W_{k,k'}(ny)e\{2nx\},$$

where

$$a_D(0) = \begin{cases} 0 & \textit{if } D \neq N \\ 2\chi(-1) g_{\chi_{N'}} \mu(N/N') L(m/2, \chi) \displaystyle\prod_{\substack{q \mid N/N' \\ q \text{ prime}}} L(m/2, \chi_{N'}, q) & \textit{if } D = N, \end{cases}$$

$$a'_D(0) = \begin{cases} 0 & \textit{if } D \neq 1 \\ 2L(m/2-1, \bar{\chi}) & \textit{if } D = 1, \end{cases}$$

and for $n \neq 0$,

$$a_D(n) = 2\chi_D(\operatorname{sgn} n) D^{1-m/2} \sum_{\substack{d \mid n \\ d>0}} d^{1-m/2} \chi_D(|n|/d) \bar{\chi}_{N/D}(d).$$

Also, for $n|N$, $a_D(n^2) = 2D^{1-m/2}(n, D)^{2-m}$.

Remark. Note that the zero-th Fourier coefficients at infinity of E_D are all zero unless $D = 1$ or $D = N$. In the first case, only the non-constant part survives; in the latter case, only the constant term is non-zero.

Proof. Again, the proof is just a calculation so we leave it out. □

§3. Lattices of minimal level and discriminant.

In this section we restrict our attention to lattices K of minimal level and discriminant at an odd prime q (as defined below). We design operators of Hecke type for which $\theta(K; R_Q, z)$ is an approximate eigenform. By an approximate eigenform, we mean a form whose image under the operator differs from a scalar multiple of itself by an $L^2(\Gamma_0(N)\backslash\mathcal{H})$ form. We then apply these operators to the Eisenstein series and show the subspace of simultaneous eigenforms is 1-dimensional. In §4 we use these results to obtain formulas for the measures of representation of K. In §5 we extend the formulas to include all odd levels.

Definition. We say a lattice of even rank m has minimal level and discriminant at an odd prime q if, locally, K has one of the following shapes

$$K_{(q)} \simeq \begin{cases} \langle 1, \ldots, 1, \eta \rangle \\ \langle 1, \ldots, 1, \eta \rangle \perp q\langle \eta' \rangle \\ \langle 1, \ldots, 1, \eta \rangle \perp q\langle 1, \eta' \rangle \end{cases}$$

where $\eta, \eta' \in \mathbb{Z}_q^\times$ and in the last case the Legendre symbols $((-1)^{m/2-1}\eta/q)$ and $(-\eta'/q)$ both equal -1. Throughout the rest of the paper, we will use the shorthand notation

$$\psi_q(\eta) = \left(\frac{(-1)^{m/2-1}\eta}{q} \right).$$

If K has odd level and minimal level and discriminant at all odd primes then we simply say K has minimal (odd) level and minimal discriminant.

Recall that if $K_{(q)} \simeq \langle 1, \dots, 1, \eta \rangle$ for an odd prime q with $\eta \in \mathbb{Z}_q^\times$, then q does not divide the level of K. So when K has minimal level and discriminant at q with q dividing the level, either $q \| d_K$ or $q^2 \| d_K$. Recall that $N = \prod_{p|d_K} p$.

For the first part of this section, fix an odd prime q and let K be a lattice which at q has minimal odd level $N \neq 1$ and minimal discriminant d_K. Let R_Q be a majorant for Q.

We need some more notation. Let q be an odd prime dividing N. For any $t \in \mathbb{Z}_+$, we have

$$K \simeq \begin{cases} \langle 1, \dots, 1, \eta \rangle \perp q\langle \eta' \rangle & \pmod{q^t} \quad \text{if } q \| d_K, \\ \langle 1, \dots, 1, \eta \rangle \perp q\langle 1, \eta' \rangle & \pmod{q^t} \quad \text{if } q^2 \| d_K, \end{cases}$$

where $\psi_q(\eta) = (-\eta/q) = -1$. Let p be an odd prime such that $p \nmid N$ and

$$\chi_q(p) = \chi_q\left((-1)^{m/2-1}\eta\eta'\right), \qquad \left(\frac{p}{q'}\right) = \left(\frac{q}{q'}\right) \tag{3.1}$$

for all primes $q'|N$, $q' \neq q$. We refer to p as a prime associated to q. Finally, let

$$P = \text{preimage in } K \text{ of } \operatorname{rad} K/qK,$$

so that, for example in the case $q^2 \| d_K$, $P = qK + q\langle 1, \eta' \rangle$.

We first define a Hecke-type operator relative to the quotient K/P and q as follows. Let

$$T_{K/P}(q) = (1 + q^{1-m/2}) T_p^* B_q - q^{1-m/2} U_q B_q,$$

where $T_p^* = (p^{m/2-1}+1)^{-1} T_p$ is the classical Hecke operator for weight $m/2 = k+k'$ (with a special normalization) and B_q and U_q are the classical Hecke operators defined by the following $(q|N)$:

$$B_q = q^{-m/4}\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}, \quad U_q = q^{m/4-1}\sum_{b=1}^{q}\begin{pmatrix} 1 & b \\ 0 & q \end{pmatrix}.$$

Notice that for any lattice L, $\theta(L;\tau)|B_q^2 = \theta(qL;\tau)$. We will see in the proof that our operator $T_{K/P}(q)$ does not depend on our choice of prime p associated to q.

Here and in the sequel, we will use the notation

$$f(z) \overset{L^2}{\equiv} g(z)$$

to mean that $f - g \in L^2(\Gamma_0(N)\backslash\mathcal{H})$.

Proposition 3.1. *With K, q, P, and p as above, we have $\chi(p) = 1$ (where $\chi = \chi_K$ is the character which arises in the transformation formula for $\theta(K; R_Q, z)$) and*

$$\theta(K; R_Q, z)\big|T_{K/P}(q) \overset{L^2}{\equiv} \theta(P; R_Q, z).$$

Remark. When $q \nmid d_K$, $P = qK$ and so $\theta(P; R_Q, z) = \theta(K; R_Q, z)\big|B_q^2$. Also, this proposition extends easily to the case d_K even by imposing the extra condition $p \equiv q \pmod 8$.

Proof. We follow the proof of Proposition 2.1 of [12]. Let $\overline{C}$ be a maximal totally isotropic subspace of K/qK (so $\operatorname{rad} K/qK \subseteq \overline{C}$), and let

$$K' = \text{ preimage in } K \text{ of } \overline{C},$$

so $\overline{C} = K'/qK$. By Proposition 1.2 of [12],

$$K' \simeq \begin{cases} q\big\langle 1, \dots, 1, (-1)^{m/2-1}\eta'\big\rangle \perp q^2\big\langle (-1)^{m/2-1}\eta\big\rangle & \pmod{q^t} \quad \text{if } q\| d_K, \\ q\big\langle 1, \dots, 1, \eta\big\rangle \perp q^2\big\langle 1, \eta'\big\rangle & \pmod{q^t} \quad \text{if } q^2\| d_K \end{cases}$$

for any $t \in \mathbb{Z}_+$. Also, for any prime $q' \neq q$, $K'_{(q')} = K_{(q')}$.

Clearly these sublattices K' are in one-to-one correspondence with these subspaces $\overline{C}$. Using the formulas from p. 146 of [1] (cf. Proposition 7.2 of [10]), we find there are (defining β by this expression)

$$(q^{m/2-1} + 1)\beta = \begin{cases} (q^{m/2-1} + 1)(q^{m/2-2} + 1) \cdots (q + 1) & \text{if } q\| d_K, \\ (q^{m/2-1} + 1)(q^{m/2-2} + 1) \cdots (q^2 + 1) & \text{if } q^2\| d_K \end{cases}$$

ways to choose $\overline{C}$. If $q \nmid Q(x)$, then $x \notin K'$ for any K'. If $q|Q(x)$ and $x \in K - P$, then $x \in K'$ for exactly β of the K'. If $q|Q(x)$ and $x \in P$, then $x \in K'$ for all $(q^{m/2-1} + 1)\beta$ of these K'. Recognizing that $\theta(K; R_Q, z)\big|U_q B_q$ is the subsum of $\theta(K; R_Q, z)$ containing only those terms x for which $q|Q(x)$, we see that

$$\theta(K; R_Q, z)\big|U_q B_q \; + \; q^{m/2-1}\theta(P; R_Q, z) = \frac{1}{\beta} \sum_{K'} \theta(K'; R_Q, z)$$

where K' varies over all the sublattices constructed as above. Also, since the K' lie in the same genus and $\theta(K'; R_Q, z)$ is a modular form on $\Gamma_0(N)$, Proposition 1.4 implies that we have for any fixed K',

$$\theta(K; R_Q, z)\big|U_q B_q + q^{m/2-1}\theta(P; R_Q, z) \overset{L^2}{\equiv} (q^{m/2-1} + 1)\theta(K'; R_Q, z). \tag{3.2}$$

If $q' \nmid \operatorname{cond}\chi$ then $K^q_{(q')} \simeq K_{(q')} \simeq K'_{(q')}$, but if $q' | \operatorname{cond}\chi$ then $K^q_{(q')} \simeq K'_{(q')}$ only if $(q/q') = 1$. We claim that

$$\theta(K; R_Q, z)\big|T_p^* B_q \overset{L^2}{\equiv} \theta(K'; R_Q, z).$$

To verify this claim, first note that

$$\chi(p) = \left(\frac{(-1)^{m/2} d_K}{p}\right) = \left(\frac{(-1)^{m/2} q_1 \cdots q_h N_0^2}{p}\right)$$

where $q_1, \dots, q_h$ are distinct primes and $N_0 \in \mathbb{Z}_+$. Since by assumption χ is a character of odd level N, we must have $(-1)^{m/2} q_1 \cdots q_h \equiv 1 \pmod 4$ and $\operatorname{cond}\chi = q_1 \cdots q_h$. Hence our constraints on p and quadratic reciprocity imply that $\chi(p) = 1$. Thus Lemmas 5.2 of [10], 3.3 of [11] and 1.3 of §1 imply that

$$\theta(K; R_Q, z)\big|T_p^* \overset{L^2}{\equiv} \theta(M; R_Q, z)$$

where M is a lattice on $V^{1/p}$, $M_{(p)} \simeq K_{(p)}$, for all primes $q' \neq p$, $M_{(q')} \simeq K^p_{(q')}$. So for $q' \neq p$, our constraints on p imply

$$M^q_{(q')} \simeq K^{pq}_{(q')} \simeq K_{(q')} \simeq K'_{(q')}.$$

Also, since $p \nmid d_K$, $M^q_{(p)} \simeq K^q_{(p)} \simeq K_{(p)} \simeq K'_{(p)}$. Hence $M^q \in \operatorname{gen} K'$, and

$$\theta(K; R_Q, z)\big|T_p^* B_q \overset{L^2}{\equiv} \theta(K'; R_Q, z).$$

(We see here also that the effect of T_p^* is independent of p up to $\operatorname{gen} K'$.) The proposition now follows by solving (3.2) for $\theta(P; R_Q, z)$. $\square$

Assume still that q is an odd prime dividing N. Let p be a prime associated to q as above. Let X_q be the twist (by the quadratic character modulo q) operator and define

$$T_K(q) = \begin{cases} U_q^2 - q^{m/2-1} U_q B_q + (q^{m-2} + q^{m/2-1}) T_p^* B_q + \psi_q(\eta) \left(\dfrac{2}{q}\right) q^{m/2-1} X_q & \\ & \text{if } q \| d_K \\ U_q^2 - (q^{m/2} + q^{m/2-1}) U_q B_q + (q^{m-2} + q^{m/2-1}) T_p^* B_q & \\ & \text{if } q^2 \| d_K, \end{cases}$$

and set

$$\lambda_K(q) = \begin{cases} q^{m-2} + 1 & \text{if } q \| d_K. \\ q^{m-2} - q^{m/2} + 1 & \text{if } q^2 \| d_K, \end{cases} \tag{3.3}$$

Proposition 3.2. *With notation as above,*

$$\theta(\operatorname{gen} K; R_Q, z)\big|T_K(q) \overset{L^2}{\equiv} \lambda_K(q)\theta(K; R_Q, z).$$

Proof. The arguments used to prove Proposition 2.2 of [12] show that

$$\theta(K; R_Q, z)\big|T_K(q)B_q^2 = \sum_{K''} \theta(K''; R_Q, z)$$

where the sum is over certain sublattices K'' of K, all of which lie in $\operatorname{gen} qK$. Thus by applying Proposition 1.4 we obtain our result. □

For q an odd prime dividing N, the level of K, let $\mathcal{C}_K(q)$ denote the subspace of Eisenstein series E of level N, weight (k, k') (as defined in §2), character χ, such that $E\big|T_K(q) = \lambda_K(q)E$. As in the proof of Lemmw 2.3 of [12], examining the action of $T_K(q)$ on the Fourier coefficients of the E_D $(D|N)$ gives us

Lemma 3.3. *For any prime $q|N$,* $\operatorname{span}\{E_D : \ D|N/q\ \} \cap \mathcal{C}_K(q) = \{0\}$ *where the E_D are the Eisenstein series with character χ, level N and weight (k, k').*

Set

$$c_K(q) = \begin{cases} \psi_q(\eta)\left(\dfrac{2}{q}\right) = \psi_q(2\eta) & \text{if } q\|d_K, \\[2ex] \dfrac{q^{m/2}-1}{q^{m/2-1}-q} & \text{if } q^2\|d_K, \end{cases}$$

and extend $c_K(*)$ multiplicatively to the divisors of N.

Proposition 3.4. $\mathcal{C}_K(q) = \operatorname{span}\{E_D + c_K(q)E_{Dq} : \ D|N/q\}$.

Proof. By looking at Fourier coefficients one easily verifies that $E_D + c_K(q)E_{Dq} \in \mathcal{C}_K(q)$ for all $D|N/q$. The proposition now follows from the preceding lemma. □

Theorem 3.5. *Let $E = \sum_{D|N} c_K(D)E_D$. Then $\bigcap_{q|N} \mathcal{C}_K(q) = \mathbb{C}E$*

Proof. Write $N = q_1 \cdots q_\ell$. Using induction on $r \le \ell$, we argue that

$$\bigcap_{1\le i\le r} \mathcal{C}_K(q_i) = \operatorname{span}\left\{ \sum_{d|q_1\cdots q_r} c_K(d)E_{Dd} : \ D|N/q_1\cdots q_r \right\}.$$

This is clearly true for $r = 0$. Take $r \ge 0$ and $f \in \bigcap_{1\le i\le r+1} \mathcal{C}_K(q_i)$. The induction hypothesis tells us that for some $\alpha_D \in \mathbb{C}$,

$$f = \sum_{D|N/q_1\cdots q_r} \alpha_D \left(\sum_{d|q_1\cdots q_r} c_K(d)E_{Dd} \right) = \sum_{\substack{D|N/q_1\cdots q_{r+1} \\ d|q_1\cdots q_r}} c_K(d)\left(\alpha_D E_{Dd} + \alpha_{Dq_{r+1}} E_{Ddq_{r+1}}\right).$$

Since $f \in \mathcal{C}_K(q_{r+1})$, Proposition 3.4 implies $\alpha_{Dq_{r+1}} = c_K(q_{r+1})\alpha_D$. Hence

$$f = \sum_{D|N/q_1\cdots q_{r+1}} \alpha_D \left(\sum_{d|q_1\cdots q_{r+1}} c_K(d) E_{Dd} \right),$$

as claimed. $\square$

We have from Proposition 2.4 that E has zeroth Fourier coefficient (at infinity) equal to $\chi(2)c_K(N)a_N(0) \neq 0$. Let

$$E_K(z) = \frac{1}{c_K(N)a_N(0)} \cdot E(z),$$

so that the zeroth Fourier coefficient of E_K is 1 (agreeing now with that for $\theta(K; R_Q, z)$).

Corollary 3.6. *With K and E as above, then*

$$\theta(K; R_Q, z) \overset{L^2}{\equiv} E_K(z).$$

Proof. There are three ingredients to this argument. First, $T_K(q)$ takes Eisenstein series into Eisenstein series, though *a priori* the level may increase by a factor of q or q^2. Second, $T_K(q)$ maps $L_2(\Gamma_0(N)\backslash\mathcal{H})$ into itself. Both of these are straightforward calculations. Third, by Proposition 2.3 there exist $\alpha_D \in \mathbb{C}$ so that

$$\theta(K; R_Q, z) = \sum_{D|N} \alpha_D E_D(z) + \varepsilon_1(z),$$

where (as always) $\varepsilon_1 \in L_2(\Gamma_0(N)\backslash\mathcal{H})$. Let S be the sum of Eisenstein series on the right. By Proposition 3.2,

$$\begin{aligned} S\big|T_K(q) &= (\theta(K) - \varepsilon_1)\big|T_K(q) \\ &= \lambda_K(q)\theta(K) + \varepsilon_2 - \varepsilon_1\big|T_K(q) \\ &= \lambda_K(q)S + \lambda_K(q)\varepsilon_1 + \varepsilon_2 - \varepsilon_1\big|T_K(q) \\ &= \lambda_K(q)S + \varepsilon_3, \end{aligned}$$

say, where by our second remark $\varepsilon_3 \in L_2(\Gamma_0(N)\backslash\mathcal{H})$. By our first remark, $\varepsilon_3 \equiv 0$ and by Theorem 3.5, $S = \alpha E_K$ for some $\alpha \neq 0$. But our choice of normalization for E_K implies that $\alpha = 1$ and the Corollary. $\square$

§4. Relationship between Fourier coefficient and representation number.

In the previous section, we showed how to construct an Eisenstein series E as a linear combination of the Eisenstein series E_D in such a way that the constant terms of E exactly match those of the theta function at *every* cusp. In this section, we use this information to describe the Fourier coefficients of the theta function explicitly in terms of the more arithmetic Fourier coefficients of the Eisenstein series. We also show how asymptotic knowledge of the Fourier coefficient (as a function of $y \to 0$) determines the representation number for n, according to Siegel's definition. Siegel's representation number is defined in terms of the quantity

$$r_t(K, 2n) = \#\left\{\ell \in K : Q(\ell) = 2n, R_Q(\ell) \le t\right\}.$$

(Note, this makes explicit the definition in the introduction. Namely, our "ball of radius t" is determined by the positive definite quadratic form determined by R_Q.) We assume that $n \ge 1$. We first need a couple of lemmas.

Lemma 4.1. *If $k' > 0$, then*

$$W_{k,k'}(y) = \begin{cases} W_{k,k'}(0) + O(y) & \text{as } y \to 0 \\ O\left(y^{-k'/2+m/2-1}e^{-2\pi y}\right) & \text{as } y \to +\infty. \end{cases}$$

Furthermore, $W_{k,k'}(0) \neq 0$.

Proof. If k' is even, then an explicit formula for this integral can be derived via a standard contour integration argument. Namely, the value of the integral is determined by the residue at the pole $t = -i$ of the integrand. This residue is $P_{k+k'/2-1}(y)e^{-2\pi y}$, where $P_{k+k'/2-1}(y)$ is a polynomial of degree $k + k'/2 - 1$ with non-zero constant term. From here, all statements in the lemma are immediate. (Note, $k + k'/2 - 1 = m_1/2 - 1 = -k'/2 + m/2 - 1$.)

For k' odd, a similar but more complex contour integration method is possible. In this case, the contour has to be a "key" contour looping around the singularity at $t = -i$. By a careful calculation, one finds that $W_{k,k'}(y)$ is a non-zero multiple of the function

$$e^{-2\pi y}\int_0^\infty t^{-1/2}(2+t)^{-k/2-k'+1/2}e^{-2\pi yt}P_{k+(k'-1)/2}(y(2+t))dt,$$

where $P_{k+(k'-1)/2}$ is again a polynomial of degree $k + (k' - 1)/2$ and non-zero constant term. From here, the first and last statements are evident. The asymptotic as $y \to +\infty$ can be deduced by a change of variables from t to ty. (This asymptotic bound can also be deduced from formulas 3.384.9, 9.232.1 and 9.227 of Gradshteyn and Ryzhik [5].)

Though we don't need it here, a similar proof shows that as $y \to -\infty$, $W_{k,k'}(y) = O\left(|y|^{-k-k'/2+m/2-1}e^{2\pi y}\right)$. Also, if $k' = 0$ (when the quadratic form is positive definite and the Eisenstein series is holomorphic), this function is identically zero for $y < 0$, and equals a constant times $y^{m/2-1}e^{-2\pi y}$ for $y > 0$. Note that the behavior as $y \to \infty$ is the same for $k' = 0$ and $k' > 0$, but the behavior as $y \to 0$ differs dramatically. □

Lemma 4.2. *If $c_n(y)$ denotes the nth Fourier coefficient at infinity of $E_K - \theta$, then for $y > 0$, we have*

$$c_n(y) = O(y^{-m/4}).$$

Proof. First, we note that by Corollary 3.6, at every cusp either $E_K - \theta$ has no zero-th Fourier coefficient, or has a zero-th coefficient consisting only of a term of the form $cy^{1-m/2}$, for some constant c. By the above lemma, at every cusp all the other Fourier coefficients decay as $y^{-k'/2}e^{-2\pi y}$ toward the cusp. This shows that $|E_K(z) - \theta(K; R_Q, z)|^2 y^{m/2}$ is uniformly (in x and y) bounded on the fundamental domain. Since it is invariant under $\Gamma_0(N)$, it is bounded on the entire upper half-plane. Thus

$$c_n(y) = \int_0^1 \left(E_K(z) - \theta(K; R_Q, z)\right) e\{-2nx\} dx \ll y^{-m/4}.$$

This completes the proof of the lemma. □

Let $A_K(n)$ be the scalar in the nth Fourier coefficient of E_K at infinity, so that by the results of the previous section,

$$A_K(n) = \frac{1}{c_K(N)a_N(0)} \sum_{D|N} c_K(D)a_D(n). \tag{4.1}$$

Let $T_n(y)$ be the nth Fourier coefficient of $\theta(K; R_Q, z)$ at infinity and $a(n, n')$ be the number of simultaneous solutions of $Q(\ell) = 2n$ and $R_Q(\ell) = n'$ for lattice points $\ell \in K$. With this definition, we have

$$T_n(y) = \sum_{n'} a(n, n')e^{-2\pi n' y}.$$

Also $r_t(K, 2n) = \sum_{n' \le t} a(n, n')$. By the two lemmas above we have

$$\begin{aligned} T_n(y) &= A_K(n)y^{1-m/2}W_{k,k'}(ny) + c_n(y) \\ &= A_K(n)y^{1-m/2}W_{k,k'}(ny) + O(y^{-m/4}) \\ &= A_K(n)y^{1-m/2}W_{k,k'}(0) + O(ny^{2-m/2}). \end{aligned}$$

Note that the leading term here is not zero.

Our goal is to prove the following proposition.

Proposition 4.3. *We have for fixed $n \geq 1$,*

$$\rho(K, 2n) = \frac{(2\pi)^{m/2-1}}{\Gamma(m/2)} A_K(n) W_{k,k'}(0).$$

Proof. Let $S_n(t) = r_t(K, 2n)$ and $\hat{S}_n(s) = \int_0^\infty t^{-s} dS_n(t)$, the Mellin transform of the Riemann-Stieljtes measure $dS_n(t)$. This clearly is analytic in some half-plane $\sigma = \Re\mathrm{e}\, s > \sigma_0$ because $S_n(t)$ is easily estimated as having polynomial growth in t. We next make the trivial observation that

$$T_n(y) = \int_0^\infty e^{-2\pi t y} dS_n(t).$$

From these two formulas, it is easy to see that for s in some right half-plane

$$\begin{aligned} \hat{S}_n(s) &= \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty T_n(y) y^{s-1} dy, \\ &= \frac{(2\pi)^s}{\Gamma(s)} \left\{ \int_0^1 + \int_1^\infty \right\} T_n(y) y^{s-1} dy \\ &= I_1(s) + I_2(s), \end{aligned}$$

say. Now $T_n(y)$ decays exponentially as y tends to infinity (because $R_Q(\ell)$ is positive definite, or by the formula above relating T_n and S_n and realizing that S_n is non-negative, monotonic and of polynomial growth) so $I_2(s)$ is entire. Furthermore, from the above asymptotic formula for $T_n(y)$ as $y \to 0$, we get, for $\sigma > m/2 - 3/2$, say,

$$I_1(s) = \frac{(2\pi)^s}{\Gamma(s)(s - m/2 + 1)} A_K(n) W_{k,k'}(0) + I_1'(s),$$

where $I_1'(s)$ is analytic in this region. We now appeal to the classical Wiener-Ikehara theorem [13].

The assumptions of that theorem are that the function $S_n(t)$ be monotone, non-decreasing, locally bounded variation, vanish to the left of 1 and right continuous and that the function $\hat{S}_n(s)$ be continuous in a closed half-plane, except for a simple pole (one-sided) on the real axis (in our case, at $s = m/2 - 1$). Under these circumstances, one determines the asymptotic behavior at infinity of the function $S_n(t)$. Clearly, we have established all (and more) of these hypotheses. Consequently, we deduce

$$S_n(t) \sim \frac{(2\pi)^{m/2-1}}{\Gamma(m/2)} A_K(n) W_{k,k'}(0) t^{m/2-1}.$$

Dividing by $t^{m/2-1}$ letting $t \to \infty$ we derive the proposition. $\square$

By Proposition 2.4 and (4.1), we have

$$\rho(K,2n) = \frac{2\epsilon\,\rho_{K,\infty}}{c_K(N)a_N(0)} \sum_{\substack{D|N\\ d|n}} c_K(D)D^{1-m/2}\chi_D(n/d)\chi_{N/D}(d)d^{1-m/2}, \tag{4.2}$$

where $\epsilon = \mathrm{sgn}(W_{k,k'}(0))$,

$$\rho_{K,\infty} = \frac{(2\pi)^{m/2-1}|W_{k,k'}(0)|}{\Gamma(m/2)}, \tag{4.3}$$

and as in §3 for a prime $q|d_K$,

$$c_K(q) = \begin{cases} \psi_q(2\eta) & \text{if } q\|d_K, \\ \dfrac{q^{m/2}-1}{q^{m/2-1}-q} & \text{if } q^2\|d_K, \end{cases}$$

with $N' = \mathrm{cond}\,\chi$,

$$L(m/2,\chi_{N'},q) = \frac{1-\chi_{N'}(q)q^{1-m/2}}{1-\chi_{N'}(q)q^{-m/2}}$$

and $a_N(0)$ is defined by

$$a_N(0) = 2\chi(-1)g_{\chi_{N'}}\mu(N/N')L(m/2,\chi) \prod_{\substack{q|N/N'\\ q \text{ prime}}} L(m/2,\chi_{N'},q).$$

To write (4.2) as a product over (finite) primes, we define pairs of local factors. One part of each pair will come from the factor $c_K(N)a_N(0)$ and the other from the summation. Fixed a prime p, and set $e = \mathrm{ord}_p(n)$. Define

$$\rho'_{K,p}(n) = \begin{cases} \dfrac{1-(\chi(p)p^{1-m/2})^{e+1}}{1-\chi(p)p^{1-m/2}} & \text{if } p|n,\ p\nmid N, \\ 1+c_K(p)\chi_p(n/p^e)\chi_{N/p}(p^e)p^{(e+1)(1-m/2)} & \text{if } p|N \end{cases} \tag{4.4}$$

and

$$\rho''_{K,p} = \begin{cases} 1-\chi(p)p^{-m/2} & p\nmid N, \\ \dfrac{1-p^{2-m/2}}{p-p^{2-m/2}} & \text{if } p|N/N' \\ p^{-1/2} & \text{if } p|N' \end{cases}$$

and set $\rho_{K,p}(n) = \rho'_{K,p}(n)\rho''_{K,p}$.

Note that $\rho'_{K,p}(n)$ depends on n only via e and $\left(\frac{n/p^e}{p}\right)$ and that each of these local factors is positive.

Corollary 4.4. *With K and $\rho_{K,p}(n)$ as above,*

$$\rho(K,2n)=\rho_{K,\infty}\prod_{p\text{ prime}}\rho_{K,p}(n).$$

Proof. Let $S(n)$ be the sum on the right hand side of (4.2). An arbitrary integer n can be written as a product of two integers, one which is relative prime to N and one all of whose prime factors come from N (i.e., which divides N^∞). Let $(n',N)=1$ and $n|N^\infty$. We first develop a product formula for $S(nn')$. So,

$$S(nn')=\sum_{\substack{D|N\\ d'|n'\\ d|n}} c_K(D)D^{1-m/2}\chi_D(nn'/dd')\chi_{N/D}(dd')(dd')^{1-m/2}.$$

Now

$$\begin{aligned}\chi_D(nn'/dd')\chi_{N/D}(dd')&=\chi_D(d')\chi_{N/D}(d')\chi_D(nn'/d)\chi_{N/D}(d)\\ &=\chi(d')\chi_D(nn'/d)\chi_{N/D}(d).\end{aligned}$$

Thus

$$\begin{aligned}S(nn')&=\sum_{d'|n'}\chi(d'){d'}^{1-m/2}\sum_{\substack{D|N\\ d|n}}c_K(D)D^{1-m/2}\chi_D(nn'/d)\chi_{N/D}(d)d^{1-m/2}\\ &=S'(n')S(n,N),\end{aligned}$$

say, where the dependence on N in $S(n,N)$ is *only* in the range of summation and not in the character $\chi_{N/D}$. The first sum is easily seen to be

$$S'(n')=\prod_{p|n'}\rho'_{K,p}(n')=\prod_{p|nn',\ p\nmid N}\rho'_{K,p}(nn')$$

as defined above, since $\rho'_{K,p}(n')=\rho'_{K,p}(nn')$ for $p\nmid N$.

For $S(n,N)$, we have to work a bit harder. Note that a term in the d sum is non-zero only if $(n/d,D)=(d,N/D)=1$ and this can happen for only one term, namely $d=(n,D^\infty)$. For simplicity, let $n_D=(n,D^\infty)$. Then for any fixed prime $q|N$,

$$\begin{aligned}S(n,N)&=\sum_{D|N}c_K(D)D^{1-m/2}\chi_D(nn'/n_D)\chi_{N/D}(n_D)n_D^{1-m/2}\\ &=\sum_{D|N/q}\Big\{c_K(D)D^{1-m/2}\chi_D(nn'/n_D)\chi_{N/D}(n_D)n_D^{1-m/2}\\ &\qquad+c_K(Dq)(Dq)^{1-m/2}\chi_{Dq}(nn'/n_{Dq})\chi_{N/Dq}(n_{Dq})n_{Dq}^{1-m/2}\Big\}.\end{aligned}$$

Let $q^e \| n$. Then $n_{Dq} = q^e n_D$. Also, for $D | N/q$,

$$\begin{aligned}
&\chi_{Dq}(nn'/n_{Dq})\chi_{N/Dq}(n_{Dq}) \\
&\quad = \chi_D(q^e)\chi_D(nn'/n_D)\chi_q(nn'/q^e n_D) \cdot \chi_{N/Dq}(q^e)\chi_{N/D}(n_D)\chi_q(n_D) \\
&\quad = \chi_D(nn'/n_D)\chi_{N/D}(n_D) \cdot \chi_D(q^e)\chi_{N/Dq}(q^e) \cdot \chi_q(nn'/q^e n_D)\chi_q(n_D) \\
&\quad = \chi_D(nn'/n_D)\chi_{N/D}(n_D) \cdot \chi_{N/q}(q^e) \cdot \chi_q(nn'/q^e).
\end{aligned}$$

Thus, since $c_K(Dq) = c_K(D)c_K(q)$,

$$\begin{aligned}
S(n, N) &= \sum_{D|N/q} c_K(D) D^{1-m/2} \chi_D(nn'/n_D)\chi_{N/D}(n_D) n_D^{1-m/2} \\
&\qquad \left\{1 + c_K(q)\chi_q(n/q^e)\chi_{N/q}(q^e) q^{(e+1)(1-m/2)}\right\} \\
&= \rho'_{K,q}(n) S(n, N/q).
\end{aligned}$$

Arguing inductively on the primes dividing N, we have

$$S(n, N) = \prod_{q|N} \rho'_{K,q}(n).$$

Since all primes $p \nmid nN$ contribute nothing to the product,

$$S(nn') = \prod_{p \text{ prime}} \rho'_{K,p}(n).$$

To complete the proof, we must show

$$\frac{2\epsilon}{c_K(N) a_N(0)} = \prod_{p \text{ prime}} \rho''_{K,p}.$$

Since $\rho(K, 2n) \geq 0$, $\rho_{K,\infty} > 0$ and $\rho_{K,p}(n) > 0$ for any $n \in \mathbb{Z}$, we must have $\frac{2\epsilon}{c_K(N) a_N(0)} > 0$. Because of these, we can ignore the sign contributions from any of the factors (e.g., the Gauss sum and ϵ). Also, for a prime $p | N/N'$, we know

$$K_{(p)} \simeq \langle 1, \ldots, 1, \eta \rangle \perp p\langle 1, \eta' \rangle$$

where $\left(\frac{(-1)^{m/2-1}\eta}{p}\right) = \left(\frac{(-1)^{m/2-1}\eta'}{p}\right) = -1$ and $d_K = p^2 \eta\eta' u^2$ where $u \in \mathbb{Z}_p^\times$. So for $p | N/N'$,

$$\chi_{N'}(p) = \left(\frac{(-1)^{m/2} N'}{p}\right) = \left(\frac{(-1)^{m/2} d_K/p^2}{p}\right) = \left(\frac{(-1)^{m/2}\eta\eta'}{p}\right) = 1$$

and thus $L(m/2, \chi_{N'}, p) = \frac{1-p^{1-m/2}}{1-p^{-m/2}}$. (Note, for $p|N$, we have $p|N/N'$ if and only $p^2 \| d_K$.) Hence, factoring the Gauss sum and the Dirichlet L-function and collecting the terms we have

$$\frac{2\epsilon}{c_K(N) a_N(0)} = \prod_{p \text{ prime}} \rho''_{K,p}$$

as claimed. □

§5. Lattices of descent.

Now we fix an integral $\mathbb{Z}$-lattice L of even rank m and odd level N'. For convenience, assume L is scaled so that $Q(L) \subseteq 2\mathbb{Z}$, $Q(L) \not\subseteq 2n\mathbb{Z}$ for any $n > 1$. Using local constructions, we show L descends from a lattice of minimal level and discriminant, then we construct chains of lattices from the minimal lattice to lattices K_0 in $\operatorname{gen} L$; by counting how often an element of the minimal lattice lies in these lattices K_0 we obtain formulas for $\rho(\operatorname{gen} L, 2n)$.

Notation. Fix a prime $q|N'$ and set $s = s(L,q) = [\operatorname{ord}_q N'/2]$. Fix $t > 2s+1$; then by Lemma 1.2 of [12]

$$L = L_0 \oplus \cdots \oplus L_{2s+1} \simeq \big\langle 1, \ldots, 1, \epsilon_0 \big\rangle \perp \cdots \perp q^{2s+1}\big\langle 1, \ldots, 1, \epsilon_{2s+1}\big\rangle \pmod{q^t}$$

where the $\epsilon_i \in \mathbb{Z} - q\mathbb{Z}$, and the ith component, $L_i \simeq q^i \big\langle 1, \ldots, 1, \epsilon_i \big\rangle$, has rank $m_i \geq 0$. Let $H_{2i} = q^{-i} L_{2i}$, $H_{2i+1} = q^{-i} L_{2i+1}$. Thus

$$L = H_0 \oplus H_1 \oplus qH_2 \oplus qH_3 \oplus \cdots \oplus q^s H_{2s} \oplus q^s H_{2s+1}$$

where the H_{2i} are unimodular (mod q^t) and the H_{2i+1} are q-modular (mod q^t). Let

$$\widetilde{H}_i = \bigoplus_{\substack{0 \leq \ell \leq i \\ \ell \equiv i \pmod 2}} H_\ell, \qquad r_i = r_i(L,q) = \operatorname{rank} \widetilde{H}_i, \qquad \eta_{2i} = \eta_{2i}(L,q) = \operatorname{disc} \widetilde{H}_{2i},$$

$$q^{r_{2i+1}} \eta_{2i+1} = q^{r_{2i+1}} \eta_{2i+1}(L,q) = \operatorname{disc} \widetilde{H}_{2i+1}, \qquad \mu_i = \mu_i(L,q) = \left(\frac{(-1)^{\ell_i} \eta_i}{q} \right)$$

where $\ell_i = [r_i/2]$. (When $r_i = 0$, set $\mu_i = 1$.) Note that s, r_i, μ_i are invariants of $\operatorname{gen} L$, and when r_i is even, $\mu_i = 1$ exactly when $\widetilde{H}_i$ is hyperbolic modulo q (here $\widetilde{H}_i$ is scaled by $1/q$ when i is odd).

Convention. When a lattice J has the property that the first Jordan component of $J_{(q)}$ is q^k-modular, we use the quadratic form $q^{-k}Q$ on the $\mathbb{Z}/qZ$-space J/qJ.

Since the local constructions used in [12] are independent of whether Q is positive definite or indefinite, we have the following three lemmas.

Lemma 5.1. *Fix a prime q dividing the level of L and let μ_j, r_j be as above.*

(a) If r_{2s} is odd or $\mu_{2s} = -1$, then there is a lattice K on V with $q^{s+1}K \subseteq L \subseteq K$, $K_{(p)} \simeq L_{(p)}$ for all primes $p \neq q$, and K has minimal level and discriminant at q.

(b) If r_{2s} is even and $\mu_{2s} = 1$, then there is a lattice K^q on V so that $q^{s+1}K^q \subseteq L \subseteq K^q$, $K_{(p)} \simeq L^q_{(p)}$ for all primes $p \neq q$, and K^q has minimal level and discriminant at q.

Furthermore, if r_{2s} is even, $\mu_{2s} = -1$ and $\mu_{2s+1} = 1$ then

$$K_{(q)} \simeq \langle 1, \dots, 1, \epsilon_K \rangle, \qquad \psi_q(-\epsilon_K) = -1.$$

If r_{2s} is even, $\mu_{2s} = -1 = \mu_{2s+1}$ then

$$K_{(q)} \simeq \langle 1, \dots, 1, \epsilon_K \rangle \perp q\langle 1, \epsilon'_K \rangle, \qquad \psi_q(\epsilon_K) = (-\epsilon'_K/q) = -1.$$

If r_{2s} is even and $\mu_{2s} = 1$ then

$$K^q_{(q)} \simeq \langle 1, \dots, 1, \epsilon_K \rangle, \qquad \psi_q(-\epsilon_K) = \mu_{2s+1}.$$

Finally, $\operatorname{gen} K$ *is determined by* $\operatorname{gen} L$.

Remark. Since d_L, d_K and d^q_K differ by squares, their theta series are associated with the same character χ (although the modulus may differ between the theta series).

Proof. See Lemma 3.1 of [12].

We construct descending chains of lattices $K, K_{2s}, \dots, K_0$ such that $K_i \in \operatorname{gen} L_i$.

More notation. For q fixed and r_i, μ_i as above, set $\mu = \mu_{2s}$, $\mu' = \mu_{2s+1}$,

$$d = \begin{cases} r_{2s}/2 & \text{if } 2|r_{2s},\ \mu = 1, \\ r_{2s+1}/2 & \text{if } 2|r_{2s},\ \mu = -1,\ \mu' = 1, \\ (r_{2s+1}-1)/2 & \text{if } 2 \nmid r_{2s}, \\ r_{2s+1}/2 - 1 & \text{if } 2|r_{2s},\ \mu = \mu' = -1; \end{cases}$$

$$\alpha = \alpha_{L,q} = \begin{cases} (q^d-1)/[(q^{m/2-1}+1)(q^{m/2-2}-1)] & \text{if } 2|r_{2s},\ \mu = \mu' = -1, \\ (q^d-1)/(q^{m-2}-1) & \text{if } 2 \nmid r_{2s}, \\ (q^d-1)/[(q^{m/2}-\mu\mu')(q^{m/2-1}+\mu\mu')] & \text{otherwise;} \end{cases}$$

$$\beta = \beta_{L,q} = \begin{cases} q^d(q^{m/2-d-1}+1)(q^{m/2-d-2}-1)/[(q^{m/2-1}+1)(q^{m/2-2}-1)] & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } 2|r_{2s},\ \mu = \mu' = -1, \\ q^d(q^{m-2d-2}-1)/(q^{m-2}-1) & \text{if } 2 \nmid r_{2s}, \\ q^d(q^{m/2-d}-\mu\mu')(q^{m/2-d-1}+\mu\mu')/[(q^{m/2}-\mu\mu')(q^{m/2-1}+\mu\mu')] & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise;} \end{cases}$$

and for $\omega = \pm 1$,

$$\gamma(\omega) = \gamma_{L,q}(\omega) = \begin{cases} (q^{m/2-d-1}+1)/(q^{m/2-1}+1) & \text{if } 2|r_{2s},\ \mu = \mu' = -1, \\ (q^{m/2-d-1}+\omega\mu)/(q^{m/2-1}+\omega\mu) & \text{if } 2 \nmid r_{2s}, \\ (q^{m/2-d}-\mu\mu')/(q^{m/2}-\mu\mu') & \text{otherwise.} \end{cases}$$

Lemma 5.2. *Let the notation be as above, and let χ denote the (primitive) character associated to $\theta(K;\tau)$ and to $\theta(L;\tau)$. There exist sublattices K_{2s} of K such that $qK \subseteq K_{2s} \subseteq K$ and for all $t \in \mathbb{Z}_+$,*

$$K_{2s} \simeq \langle 1,\dots,1,\eta_{2s}\rangle \perp q\langle 1,\dots,1,\eta_{2s+1}\rangle \qquad (\mathrm{mod}\ q^t),$$

where the first component has rank r_{2s}, and the second has rank r_{2s+1}. In particular, $K_{2s} \in \operatorname{gen} L_{2s}$. Let P = preimage in K of $\operatorname{rad} K/qK$, P_{2s} = preimage in K_{2s} of $\operatorname{rad} K_{2s}/qK_{2s}$. (Scale P_{2s} by $1/q$ in the case $2|r_{2s}$, $\mu_{2s}=1$.) Let $x \in K - P$.

(a) Suppose r_{2s} is odd or $\mu_{2s}=-1$. If $q \nmid Q(x)$ then $x \notin P_{2s}$ and the proportion of K_{2s} such that $x \in K_{2s}-P_{2s}$ is $\gamma(\omega)$ where $\omega=\chi_q(Q(x))$. If $q|Q(x)$ then the proportion of K_{2s} such that $x \in P_{2s}$ is α, and the proportion of K_{2s} such that $x \in K_{2s}-P_{2s}$ is β.

(b) Suppose r_{2s} is even, $\mu_{2s}=1$. If $q \nmid Q(x)$ then x is in none of the lattices K_{2s}, and the proportion of K_{2s} such that $qx \in P_{2s}-qK_{2s}$ is $\gamma(\pm 1)$. If $q|Q(x)$ then the proportion of K_{2s} such that $x \in K_{2s}-P_{2s}$ is α.

Finally, if $x \in K_{2s}$ then necessarily $qx \in P_{2s}$.

Proof. See the proof of Lemma 3.2 of [12].

Lemma 5.3. *For $0 \le j \le 2s+1$, $\omega = 0,\pm 1$, let*

$$r'_j = \begin{cases} r_{2s}+m & \text{if } j \text{ is even,}\\ r_{2s+1}+m & \text{if } j \text{ is odd.}\end{cases}$$

Set

$$\nu_j(\omega)=\nu_j(\omega;L,q)=\begin{cases} q^{(r_j-r'_j)/2}(q^{r_j/2}-\mu_j) & \text{if } 2|r_j,\ \omega\neq 0,\\ q^{(r_j-r'_j+1)/2}(q^{(r_j-1)/2}+\omega\mu_j) & \text{if } 2\nmid r_j,\ \omega\neq 0,\\ q^{1-r'_j/2}(q^{r_j/2}-\mu_j)(q^{r_j/2-1}+\mu_j) & \text{if } 2|r_j,\ \omega=0,\\ q^{1-r'_j/2}(q^{r_j-1}-1) & \text{if } 2\nmid r_j,\ \omega=0.\end{cases}$$

For $j>s$, let $\nu_{2j}(\omega)=\nu_{2s}(\omega)$, $\nu_{2j+1}(\omega)=\nu_{2s+1}(\omega)$. Let $x \in K_{2s}-qK_{2s}$; fix $\ell \ge 0$.

(a) Suppose $x \in P_{2s}$ and set $\omega=\chi_q(Q(x)/q)$. Then the proportion of chains $K_{2s},\dots,K_0$ such that $q^\ell x \in K_0$ is $\frac{\nu_{2\ell+1}(\omega)}{\nu_{2s+1}(\omega)}$.

(b) Suppose $x \notin P_{2s}$ and set $\omega=\chi_q(Q(x))$. Then the proportion of chains $K_{2s},\dots,K_0$ such that $q^\ell x \in K_0$ is $\frac{\nu_{2\ell}(\omega)}{\nu_{2s}(\omega)}$.

Proof. See the proof of Lemma 3.6 in [12].

Now we can prove

Theorem 5.4. *Suppose $m = \operatorname{rank} L$ is even, $m \geq 6$, and $(-1)^{m/2} d_L \equiv 1 \pmod 4$. For $n \in \mathbb{Z}_+$, the measure of representation is $\rho(L, 2n) = \rho_{L,\infty} \prod_q \rho_{L,q}(n)$ where the product is over all finite primes q and $\rho_{L,\infty} = \rho_{K,\infty}$ is defined by (4.3) for a minimal lattice K and for fixed $q \neq \infty$, $\rho_{L,q}(n)$ is defined as follows:*

Let $\chi(*) = (\operatorname{sgn} *)^k \left(\frac{(-1)^{m/2} d_L}{|*|}\right)$, the character of the theta transformation formula which has (odd) conductor. Let $\nu_j(\omega) = \nu_j(\omega; L, q)$ be as defined in Lemma 5.3, and let $\mu = \mu_{2s}(L, q)$, $\mu' = \mu_{2s+1}(L, q)$. Then for $e = \operatorname{ord}_q(n)$, $\epsilon = \left(\frac{n/q^e}{q}\right)$,

$$\rho_{L,q}(n) = \nu_t(\epsilon; L, q) q^{(1-m/2)e} \; + \sum_{0 \leq \ell < e} q^{(1-m/2)\ell} \nu_\ell(0; L, q).$$

Proof. We argue by induction on the number h of primes q such that L does not have minimal level and discriminant at q. If $h = 0$, that is if L has minimal level N and discriminant then the statement of the theorem follows immediately from Corollary 4.4 (after deciphering the notation).

Our induction hypothesis has two parts. The first is that the theorem holds for all lattices which have fewer than h primes at which the lattice does not have minimal level and discriminant. Let L be a lattice with h such primes. Fix such a prime q and let K be as in Lemma 5.1. Note that $\rho_{L,\infty} = \rho_{K,\infty}$ since this factor only reflects the signature of Q. If r_{2s} is odd or $\mu = -1$ then K has minimal level and discriminant at q, and the induction hypothesis (and the fact that the local structure of K and L agree for primes $p \neq q$) implies

$$\rho(K, 2n) = \rho_{K,q}(n) \cdot \rho_{L,\infty} \prod_{p \neq q} \rho_{L,p}(n)$$

where $\rho_{K,q}(n)$ is as in (4.4) and $\rho_{L,p}(n)$ is as in the statement of the theorem to be proved. If r_{2s} is even and $\mu = 1$ then K^q has minimal level and discriminant at q, and the induction hypothesis again implies

$$\rho(K^q, 2n) = \rho_{K^q,q}(n) \cdot \rho_{L,\infty} \prod_{p \neq q} \rho_{L,p}(qn) = \chi_N(q) \rho_{K^q,q}(n) \cdot \rho_{L,\infty} \prod_{p \neq q} \rho_{L,p}(n)$$

where $\rho_{K^q,q}(n)$ is defined as in (4.4) and $\rho_{L,p}(n)$ is as in the theorem to be proved.

The second part of our induction hypothesis is that for all lattices J which have fewer than h primes at which the lattice does not have minimal level and discriminant there is an approximate Fourier series expansion in the form

$$\begin{aligned} \theta(J; R, z) &\overset{L^2}{\equiv} 1 + A'_J(0) y^{1-m/w} W_{k,k'}(0) \\ &\quad + \frac{\Gamma(m/2)}{(2\pi)^{m/2-1} W_{k,k'}(0)} \sum_{n \neq 0} \rho(J, 2n) y^{1-m/2} W_{k,k'}(ny) e\{2nx\}. \end{aligned} \tag{5.1}$$

By Corollary 3.6, this is true for $J = L$ if L has minimal level and discriminant (and so starts the induction for this part of the hypothesis). This is essentially equivalent to the theta function being L^2-equivalent to an Eisenstein series.

For notational simplicity, let $\epsilon_0(v) = \left(\frac{Q(v)}{q}\right)$, $\epsilon_1(v) = \left(\frac{Q(v)/q}{q}\right)$ and $\epsilon_{-1}(v) = \left(\frac{qQ(v)}{q}\right)$.

Case 1. First consider the case that either $2 \nmid r_{2s}$ or $\mu = -1$; so K, L lie in the same quadratic space. Assume $s \geq 1$. For $\epsilon = \pm 1$, $\omega = 0, \pm 1$, set

$$\begin{aligned} A_\ell(\omega) &= \alpha \frac{\nu_{2\ell+1}(\omega)}{\nu_{2s+1}(\omega)} + \beta \frac{\nu_{2\ell}(0)}{\nu_{2s}(0)} + (1 - \alpha - \beta) \frac{\nu_{2\ell-1}(0)}{\nu_{2s+1}(0)}, \\ B_\ell(\epsilon) &= \gamma(\epsilon) \frac{\nu_{2\ell}(\epsilon)}{\nu_{2s}(\epsilon)} + (1 - \gamma(\epsilon)) \frac{\nu_{2\ell-1}(0)}{\nu_{2s+1}(0)}. \end{aligned}$$

(Here the notation is as in Lemmas 5.2 and 5.3; we take $\nu_{-1}(*) = 0$.) Fix $\ell \geq 0$, and let $P =$ preimage in K of $\operatorname{rad} K/qK$. Then from Lemmas 5.2 and 5.3we have:

(a) For $v \in K - P$ and $q|Q(v)$, the proportion of K_0 in K containing $q^\ell v$ is $A_\ell(\epsilon_1(v))$.

(b) For $v \in K - P$ and $q \nmid Q(v)$, the proportion of K_0 in K containing $q^\ell v$ is $B_\ell(\epsilon_0(v))$.

(c) For $v \in P - qK$ and $q|Q(v)$, the proportion of K_0 in K containing $q^\ell v$ is $\frac{\nu_{2\ell+1}(\epsilon_1(v))}{\nu_{2s+1}(\epsilon_1(v))}$.

Since K satisfies our induction hypothesis, $\theta(K; R_Q, z)$ has an approximate Fourier expansion of the type in (5.1). But then this must also be true for any theta function derived from $\theta(K; R_Q, z)$ via the action of any of the Hecke operators since they act on Fourier series by definition and also map L^2 to L^2. Consequently, by Proposition 3.1, $\theta(P; R_Q, z)$ has such an expansion.

Furthermore, if J is any lattice which has such a Fourier expansion, then the constant term in the nth Fourier coefficient of $\theta(J; R_Q, z)\big|B_q$ is given by $q^{1-m/2}\rho(J, 2n/q)$. Since $\theta(J; R_Q, z)\big|B_q^{2\ell} = \theta(q^\ell J; R_Q, z)$, we have

$$\rho(q^\ell J, 2n) = q^{(2-m)\ell} \rho(J, 2n/q^{2\ell}). \tag{5.2}$$

In particular, this formula holds for $J = K$ and $J = P$.

Let δ denote the number of K_0 constructed via Lemma 5.3from K. Each K_0 lies in $\operatorname{gen} L$, so by Proposition 1.4,

$$\theta(L; R_Q, z) \overset{L^2}{\equiv} \frac{1}{\delta} \sum_{K_0 \subseteq K} \theta(K_0; R_Q, z).$$

On the other hand, we have by Lemma 5.2 and Lemma 5.3and cases (a)-(c) above,

$$\begin{aligned}\frac{1}{\delta}\sum_{K_0\subseteq K}\theta(K_0;R_Q,z) = \theta(q^{s+1}K;R_Q,z)\\ &+\sum_{0\le\ell<s+1}\sum_{\substack{v\in K-P\\ q|Q(v)}} A_\ell\left(\epsilon_1(v)\right)e\{Q(q^\ell v)x+iR_Q(q^\ell v)y\}\\ &+\sum_{0\le\ell<s+1}\sum_{\substack{v\in K\\ q\nmid Q(v)}} B_\ell\left(\epsilon_0(v)\right)e\{Q(q^\ell v)x+iR_Q(q^\ell v)y\}\\ &+\sum_{0\le\ell<s+1}\sum_{v\in P-qK}\frac{\nu_{2\ell+1}\left(\epsilon_1(v)\right)}{\nu_{2s+1}\left(\epsilon_1(v)\right)}e\{Q(q^\ell v)x+iR_Q(q^\ell v)y\}\end{aligned}$$

For any $\ell > s$, $A_\ell(*) = B_\ell(*) = 1$ and $\nu_{2\ell+1}(*) = \nu_{2s+1}(*)$. This means that the above formula holds with $s+1$ replaced by any larger integer s'. In fact we can let such a s' tend to infinity and derive

(5.3)

$$\begin{aligned}\theta(L;R_Q,z) \overset{L^2}{\equiv} 1 &+\sum_{\ell\ge0}\sum_{\substack{v\in K-P\\ q|Q(v)}} A_\ell\left(\epsilon_1(v)\right)e\{Q(v)q^{2\ell}x+iR_Q(v)q^{2\ell}y\}\\ &+\sum_{\ell\ge0}\sum_{\substack{v\in K\\ q\nmid Q(v)}} B_\ell\left(\epsilon_0(v)\right)e\{Q(v)q^{2\ell}x+iR_Q(v)q^{2\ell}y\}\\ &+\sum_{\ell\ge0}\sum_{v\in P-qK}\frac{\nu_{2\ell+1}\left(\epsilon_1(v)\right)}{\nu_{2s+1}\left(\epsilon_1(v)\right)}e\{Q(v)q^{2\ell}x+iR_Q(v)q^{2\ell}y\}.\end{aligned}$$

For each fixed ℓ, each of the summands on the right of the above formula can be written as $\theta(K;R_Q,z)$ and/or $\theta(P;R_Q,z)$ acted on by an appropriate algebraic combination of the standard Hecke operators $U_q, B_q, X_q, T^*_{q'}$. For example, if $T_\pm(\ell) = \frac{1}{2}U_q\left(\pm\left(\frac{2}{q}\right)X_q + 1 - U_qB_q\right)B_q^{2\ell+1}$, then

$$\sum_{\substack{v\in K-P\\ q|Q(v)\\ \epsilon_1(v)=\pm1}} e\{Q(v)q^{2\ell}x+iR_Q(v)q^{2\ell}y\} = \left(\theta(K;R_Q,z)-\theta(P;R_Q,z)\right)\big|T_\pm(\ell).$$

By the second part of the induction hypothesis, observing how the corresponding Fourier expansions and L^2 functions change under these operators, we can replace the right hand side of (5.3) above by an approximate Fourier series involving the representation numbers for K and P. This shows that L satisfies the second part of our induction hypothesis. More explicitly, if $e = \operatorname{ord}_q(n)$ and $\epsilon = \left(\frac{2n/q^e}{q}\right)$, Lemma 4.2 implies

$$\rho(L,2n) = \begin{cases} A_t(\epsilon)\left(\rho(q^tK,2n) - \rho(q^tP,2n)\right) \\ \qquad + \frac{\nu_{2t+1}(\epsilon)}{\nu_{2s+1}(\epsilon)}\left(\rho(q^tP,2n) - \rho(q^{t+1}K,2n)\right) \\ \qquad + \sum_{0\le \ell<t} A_\ell(0)\left(\rho(q^\ell K,2n) - \rho(q^\ell P,2n)\right) \\ \qquad + \sum_{0\le \ell<t} \frac{\nu_{2\ell+1}(0)}{\nu_{2s+1}(0)}\left(\rho(q^\ell P,2n) - \rho(q^{\ell+1}K,2n)\right) & \text{if } e = 2t+1, \\ B_t(\epsilon)\rho(q^tK,2n) \\ \qquad + \sum_{0\le \ell<t} A_\ell(0)\left(\rho(q^\ell K,2n) - \rho(q^\ell P,2n)\right) \\ \qquad + \sum_{0\le \ell<t} \frac{\nu_{2\ell+1}(0)}{\nu_{2s+1}(0)}\left(\rho(q^\ell P,2n) - \rho(q^{\ell+1}K,2n)\right) & \text{if } e = 2t. \end{cases}$$

Our next step is to simplify this formula.

First note that for $e = 2t+1$, $\rho(q^{t+1}K,2n) = 0$ and so its term in the above formula can be ignored. Next, suppose the $q \nmid N$ so that $P = qK$ and

$$\rho(q^\ell P,2n) = \rho(q^{\ell+1}K,2n) \tag{5.4}$$

for $0 \le \ell \le t$. This implies that the summations involving the $\nu_{2\ell+1}$ terms vanish identically.

It turns out that the same thing happens for $q|N$. If $e \le 1$ then these sums are empty. We will show that if $e \ge 2$, and $0 \le \ell < t$, then (5.4) holds and again the sums vanish. To see this, let q' be a prime associated to q as in (3.1). Then, as discussed in the proof of Proposition 3.1, we know that

$$\theta(K;R_Q,z)\big|T^*_{q'} \overset{L^2}{\equiv} \theta(M;R_Q,z)$$

where $M_{(p)} \simeq K^{q'}_{(p)}$ for all primes $p \ne q'$, $M_{(q')} \simeq K_{(q')}$. Since $\theta(M;R_Q,z)$ is derived from K via a Hecke operator, the remark above concerning the approximate Fourier expansion for $\theta(M;R_Q,z)$ and Corollary 4.4 imply that $\rho(M,2n) = \rho_{M,\infty}\prod_p \rho_{M,p}(n)$ where our conditions on q' give us

$$\rho_{M,p}(n) = \begin{cases} \rho_{K,p}(n) & \text{for } p \nmid N \\ \rho_{K,p}(q'n) & \text{for } p|N \end{cases}$$

$$= \begin{cases} \rho_{K,p}(qn) & \text{for } p \ne q, \\ \rho_{K,q}(q'n) & \text{for } p = q. \end{cases}$$

We have the constant part of the nth Fourier coefficient of $\theta(M; R_Q, z)\big|B_q$ is given by $q^{1-m/2}\rho(M, 2n/q)$. Because

$$\begin{aligned}\theta(P; R_Q, z) &\overset{L^2}{\equiv} \theta(K; R_Q, z)\Big|\Big((1+q^{1-m/2})T^*_{q'}B_q - q^{1-m/2}U_qB_q\Big)\\ &\overset{L^2}{\equiv} (1+q^{1-m/2})\theta(M; R_Q, z)\big|B_q - q^{1-m/2}\theta(K; R_Q, z)\big|U_qB_q,\end{aligned}$$

we have

$$\rho(P, 2n) = (1+q^{1-m/2})q^{1-m/2}\rho(M, 2n/q) - q^{1-m/2}\rho(K, 2n)$$

if $q|n$ and is zero otherwise.

Consequently, for $e \geq 1$, Proposition 3.1 and (4.4) give us

$$\begin{aligned}\rho'_{P,q}(n) =& q^{2-m} + c_K(q)\chi_q(n/q^e)\chi_{N/q}(q^e)q^{(1-m/2)(e+1)}\\ &\cdot \Big((1+q^{1-m/2})\chi_q(q')\chi_{N_K/q}(q) - q^{1-m/2}\Big).\end{aligned}$$

Again using the conditions on q', we find that

$$\chi_q(q')\chi_{N_K/q}(q) = \chi_q(q')\chi_{N_K/q}(q') = \chi(q') = 1.$$

So, for $q|N$ and $e = \mathrm{ord}_q(n)$,

$$\rho'_{P,q}(n) = \begin{cases} 0 & \text{if } e = 0,\\ q^{2-m}\Big(1 + c_K(q)\chi_q(n/q^e)\chi_{N/q}(q^e)q^{(1-m/2)(e-1)}\Big) & \text{if } e \geq 1.\end{cases}$$

Thus, if $e \geq 2$, $\rho'_{P,q}(n) = q^{2-m}\rho'_{K,q}(n/q^2)$ from which we see that $\rho(P, 2n) = q^{2-m}\rho(K, 2n/q^2)$. This and (5.2) imply that if $e \geq 2$ then (5.4) holds for $0 \leq \ell < t$ as claimed.

Using this information to simplify our formula for $\rho(L, 2n)$, we get

$$\rho(L, 2n) = \begin{cases} A_t(\epsilon)q^{(2-m)t}\left(\rho(K, 2n/q^{2t}) - \rho(P, 2n/q^{2t})\right)\\ + \quad \frac{\nu_{2t+1}(\epsilon)}{\nu_{2s+1}(\epsilon)}q^{(2-m)t}\rho(P, 2n/q^{2t})\\ + \sum_{0\leq\ell<t} A_\ell(0)q^{(2-m)\ell}\left(\rho(K, 2n/q^{2\ell}) - q^{2-m}\rho(K, 2n/q^{2\ell+2})\right)\\ \hfill \text{if } e = 2t+1,\\ B_t(\epsilon)q^{(2-m)t}\rho(K, 2n/q^{2t})\\ + \sum_{0\leq\ell<t} A_\ell(0)q^{(2-m)\ell}\left(\rho(K, 2n/q^{2\ell}) - q^{2-m}\rho(K, 2n/q^{2\ell+2})\right)\\ \hfill \text{if } e = 2t.\end{cases}$$

As before, $e = \mathrm{ord}_q(n)$ and $\epsilon = \left(\frac{2n/q^e}{q}\right)$.

Extracting the q local factor, we have

$$\rho(L, 2n) = \left(\rho_{K,\infty} \prod_{\substack{p \text{ prime} \\ p \neq q}} \rho_{K,p}(n)\right) \rho_{L,q}(n),$$

where

$$\rho_{L,q}(n) = \begin{cases} A_t(\epsilon) q^{(2-m)t} \left(\rho_{K,q}(n/q^{2t}) - \rho_{P,q}(n/q^{2t})\right) \\ + \quad \frac{\nu_{2t+1}(\epsilon)}{\nu_{2s+1}(\epsilon)} q^{(2-m)t} \rho_{P,q}(n/q^{2t}) \\ + \sum_{0 \le \ell < t} A_\ell(\epsilon) q^{(2-m)\ell} \left(\rho_{K,q}(n/q^{2\ell}) - q^{2-m} \rho_{K,q}(n/q^{2\ell+2})\right) \\ \hfill \text{if } e = 2t+1, \\ B_t(\epsilon) q^{(2-m)t} \rho_{K,q}(n/q^{2t}) \\ + \sum_{0 \le \ell < t} A_\ell(0) q^{(2-m)\ell} \left(\rho_{K,q}(n/q^{2\ell}) - q^{2-m} \rho_{K,q}(n/q^{2\ell+2})\right) \\ \hfill \text{if } e = 2t. \end{cases}$$

To complete the proof in this case, we need to separate a couple of cases. When $q \nmid d_K$, we have r_{2s} is even, $\mu = -1$,$\mu' = 1$, $\chi_q = 1$ and $\chi_N(q) = \chi(q) = \mu\mu' = -1$; also, $P = qK$. So for $e = 2t+1$, $\rho_{P,q}(n/q^{2t}) = 0$. When $q \| d_K$, we have r_{2s} is odd, $\chi_q(*) = (*/q)$,

$$\chi_{N/q}(q) = \left(\frac{q}{N_0/q}\right) = \psi_q(N_0) = \psi_q(d_K/q) = \mu\mu',$$

and $c_K(q) = \mu_j(2/q)$; so for $e = 2t+1$,

$$\alpha\rho_{K,q}(n/q^{2t}) + (1-\alpha)\rho_{P,q}(n/q^{2t}) = q^{2-m}(q^d + \epsilon\mu')\rho''_{K,q} = q^{3/2-m}(q^d + \epsilon\mu'),$$

where $d = (r_{2s+1} - 1)/2$. When $q^2 \| d_K$, we have r_{2s} is even, $\mu = \mu' = -1$, $\chi_q = 1$,

$$\chi_{N/q}(q) = \left(\frac{q}{N_0}\right) = \psi_q(-N_0) = \psi_q(-d_K/q^2) = \mu\mu' = 1$$

and $c_K(q) = (q^{m/2} - 1)/(q^{m/2-1} - q)$; so for $e = 2t+1$,

$$\begin{aligned} \alpha\rho_{K,q}(n/q^{2t}) + (1-\alpha)\rho_{P,q}(n/q^{2t}) &= \frac{q^{1-m}(q^d + 1)(q^{m/2-1} - 1)}{q^{m/2-2} - 1}\rho''_{K,q} \\ &= q^{1-m}(q^{d-1} + 1), \end{aligned}$$

where $d = r_{2s+1}/2 - 1$.

Using these observations and the formulas for A_ℓ and B_ℓ, one performs straightforward computations to show $\rho_{L,q}(n)$ is as claimed in the theorem.

Case 2. Now suppose r_{2s} is even and $\mu = 1$; so L, K_0 and K^q lie on V. While K is not integral, K^q is integral and $K^q_{(q)}$ is unimodular. Thus, our induction hypotheses apply to K^q, a lattice on V^q. Set

$$A_\ell(\omega) = \alpha\frac{\nu_{2\ell}(\omega)}{\nu_{2s}(\omega)} + \beta\frac{\nu_{2\ell-1}(0)}{\nu_{2s+1}(0)} + (1-\alpha-\beta)\frac{\nu_{2\ell-2}(0)}{\nu_{2s}(0)},$$

and

$$B_\ell(\omega) = \gamma\frac{\nu_{2\ell-1}(\omega)}{\nu_{2s+1}(\omega)} + (1-\gamma)\frac{\nu_{2\ell-2}(0)}{\nu_{2s}(0)}.$$

As in the preceding case, for $\ell \geq 0$ Lemmas 5.2 and 5.3give us:

(a) For $v \in K^q - qK^q$, $q|qQ(v)$, the proportion of K^q_0 in K^q containing $q^\ell v$ is $A_\ell(\epsilon_0(v))$.

(b) For $v \in K^q - qK^q$, $q \nmid qQ(v)$, the proportion of K^q_0 in K^q containing $q^\ell v$ is 0 if $\ell = 0$ and $B_\ell(\epsilon_{-1}(v))$.

Thus

$$\begin{aligned}\frac{1}{\delta}\sum_{K_0\subseteq K}\theta(K_0^q; R_Q, z) &= \theta(q^{s+1}K^q; R_Q, z)\\ &\quad + \sum_{\substack{v\in K-qK\\ q|qQ(v)}}\sum_{0\leq\ell\leq s} A_\ell\left(\epsilon_0(v)\right) e\{qQ(q^\ell v)x + iqR_Q(q^\ell v)y\}\\ &\quad + \sum_{\substack{v\in K-qK\\ q\nmid qQ(v)}}\sum_{1\leq\ell\leq s} B_\ell\left(\epsilon_{-1}(v)\right) e\{qQ(q^\ell v)x + iqR_Q(q^\ell v)y\}\end{aligned}$$

where δ is the number of K_0 in K^q. We have $L \in \operatorname{gen} K_0$ so an argument similar to that when r_{2s} is odd or $\mu = -1$ gives us:

$$\rho(L^q, 2n) = \begin{cases} A_t(\epsilon)q^{(2-m)t}\rho(K^q, 2n/q^{2t}) \\ \qquad + \sum\limits_{0\leq\ell<t} A_\ell(0)q^{(2-m)\ell}\left(\rho(K^q, 2n/q^{2\ell}) - q^{2-m}\rho(K^q, 2n/q^{2\ell+2})\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } q^{2t+1}\|n, \\ B_t(\epsilon)q^{(2-m)t}\rho(K^q, 2n/q^{2t}) \\ \qquad + \sum\limits_{0\leq\ell<t} A_\ell(0)q^{(2-m)\ell}\left(\rho(K^q, 2n/q^{2\ell}) - q^{2-m}\rho(K^q, 2n/q^{2\ell+2})\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } q^{2t}\|n,\ t\geq 1, \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } q\nmid n \end{cases}$$

where $e = \operatorname{ord}_q(n)$ and $\epsilon = \left(\frac{2n/q^e}{q}\right)$. By hypothesis,

$$\rho(K^q, 2n) = \chi_N(q)\rho_{K^q,q}(n) \cdot \rho_{L,\infty} \prod_{p \neq q} \rho_{L,p}(n/q)$$

and hence (as in Case 1)

$$\rho(L^q, 2qn) = \rho_{L,\infty} \prod_{p} \rho_{L^q,p}(qn)$$

where

$$\rho_{L^q,q}(qn) = \begin{cases} A_t(\epsilon) q^{(2-m)t} \rho_{K^q,q}(qn/q^{2t}) & \\ \qquad + \sum\limits_{0 \le \ell < t} A_\ell(0) q^{(2-m)\ell} \left(\rho_{K^q,q}(qn/q^{2\ell}) - q^{2-m} \rho_{K^q,q}(qn/q^{2\ell+2}) \right) & \\ & \text{if } q^{2t} \| n, \\ B_t(\epsilon) q^{(2-m)t} \rho_{K^q,q}(qn/q^{2t}) & \\ \qquad + \sum\limits_{0 \le \ell < t} A_\ell(0) q^{(2-m)\ell} \left(\rho_{K^q,q}(qn/q^{2\ell}) - q^{2-m} \rho_{K^q,q}(qn/q^{2\ell+2}) \right) & \\ & \text{if } q^{2t-1} \| n,\ t \ge 1, \\ 0 & \text{otherwise} \end{cases}$$

We know that

$$\rho'_{K^q,q}(n) = \frac{1 - (\chi_N(q) q^{1-m/2})^{e+1}}{1 - \chi_N(q) q^{1-m/2}}$$

and

$$\rho_{K^q,q}(n/q^{2\ell}) - q^{2-m} \rho_{K^q,q}(n/q^{2\ell+2}) = 1 + \chi_N(q) q^{1-m/2}$$

where $\chi_N(q) = \mu\mu' = \mu'$. Also, from the definition of the measure of a representation, we find that

$$\rho(L^q, 2qn) = q^{1-m/2} \rho(L, 2n).$$

Thus

$$\begin{aligned} \rho(L, 2n) &= q^{m/2-1} \rho(L^q, 2qn) \\ &= q^{m/2-1} \rho_{L^q,q}(qn) \cdot \rho_{L,\infty} \prod_{p \neq q} \rho_{L^q,p}(qn) \\ &= q^{m/2-1} \chi_N(q) \rho_{L^q,q}(qn) \cdot \rho_{L,\infty}(n) \prod_{p \neq q} \rho_{L,p}(n) \\ &= \rho_{L,\infty} \prod_{p} \rho_{L,p}(n), \end{aligned}$$

where $\rho_{L,q}(n)$ is as claimed in the theorem. □

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