Quiz 1, 2019

1. Define $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ by $f((m, n)) = (3, 2n)$. Let
$$A = \{(n, 0) : \ n \in \mathbb{Z}\ \}.$$
Find $f(A)$.
(a) $f(A) = (3, 0)$.
(b) $f(A) = \{(n, n) : \ n \in \mathbb{Z}\ \}$.
(c) $f(A) = \{(3, 2n) : \ n \in \mathbb{Z}\ \}$.
(d) $f(A) = \{(3, 0)\}$.
(e) $f(A) = (n, n)$.
[Answer: (d). Recall that $f(A)$ is a subset of the codomain of f.]

2. Let $X = \{\alpha, \beta, \omega\}$ and $Y = \{a, b, c\}$. What is the Cartesian product of X and Y?
(a) $X \times Y = \{(\alpha, a),\ (\beta, b),\ (\omega, c)\}$.
(b) $X \times Y = (\alpha \cdot a, \beta \cdot b, \omega \cdot c)$.
(c) $X \times Y = \{(\alpha, a),\ (\alpha, b),\ (\alpha, c),\ (\beta, a),\ (\beta, b),\ (\beta, c),\ (\omega, a),\ (\omega, b),\ (\omega, c)\}$.
(d) $X \times Y = \alpha \cdot a + \beta \cdot b + \omega \cdot c$.
[Answer: (c). Recall that $X \times Y = \{(x, y) : x \in X,\ y \in Y\ \}$, the set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$.]

3. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d\}$, where a, b, c, d are all distinct elements. Define $f : X \to Y$ by $f = \{(1, b),\ (2, a),\ (3, b),\ (4, c)\}$. Please match the following sentences.
(i) The domain of f is:
(ii) The co-domain of f is:
(iii) The range [or image] of f is:
All answer choices:
A. $\{1, 2, 3, 4\}$.
B. $\{a, b, c\}$.
C. $\{a, b, c, d\}$.
[Answers: in order, the answers are A, C, B. With $f : X \to Y$, the domain of f is X, the codomain of f is Y, and the range (or image) of f is $f(X) = \{f(x) : \ x \in X\ \} = \{f(1), f(2), f(3), f(4)\} = \{b, a, b, c\} = \{a, b, c\}$.]

4. Define $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ and $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ by $f((a, b)) = (a + b, 3a - b)$ and $g((a, b)) = (a - b, 2a + b)$. Find $(g \circ f)((a, b))$.
(a) $(g \circ f)((a, b)) = \{(3a, a - 4b)\}$.
(b) $(g \circ f)((a, b)) = (3a, a - 4b)$.
(c) $(g \circ f)((a, b)) = \{(2b - 2a, 5a + b)\}$.
(d) $(g \circ f)((a, b)) = (2b - 2a, 5a + b)$.
[Answer: (d). Recall that $(g \circ F)(x) = g(f(x))$.]

5. Let $X = \mathbb{R}_+$ and $Y = \{x \in \mathbb{R} : \ x > -2\ \}$. Define $f : X \to Y$ and $g : Y \to X$ by $f(x) = 7x - 2$ and $g(x) = \frac{x+2}{7}$. Which of the following proofs correctly shows that f and g are inverses of one another?

(a) Let $x \in X$. Then

$$(g \circ f)(x) = 7g(x) - 2 = 7\left(\frac{x+2}{7}\right) - 2 = x + 2 - 2 = x.$$

Now, take $x \in Y$. Then

$$(f \circ g)(x) = \frac{f(x)+2}{7} = \frac{7x-2+2}{7} = \frac{7x}{7} = x.$$

Hence, $g \circ f$ is the identity functio on X and $f \circ g$ is the identity function on Y. Therefore, f and g are inverses of one another.

(b) Let $x \in X$. Then

$$(g \circ f)(x) = \frac{f(x)+2}{7} = \frac{7x-2+2}{7} = \frac{7x}{7} = x,$$

so $g \circ f$ is the identity function on X. Therefore, f and g are inverses of one another.

(c) Let $x \in X$. Then

$$(f \circ g)(x) = afracf(x) + 27 = \frac{7x-2+2}{7} = \frac{7x}{7} = x.$$

Now take $x \in Y$. Then

$$(g \circ f)(x) = 7g(x) - 2 = 7\left(\frac{x+2}{7}\right) - 2 = x + 2 - 2 = x.$$

Hence, $f \circ g$ is the identity function on X and $g \circ f$ is the identity function on Y. Therefore, f and g are inverses of one another.

(d) Let $x \in X$. Then

$$(g \circ f)(x) = afracf(x) + 27 = \frac{7x-2+2}{7} = \frac{7x}{7} = x.$$

Now take $x \in Y$. Then

$$(f \circ g)(x) = 7g(x) - 2 = 7\left(\frac{x+2}{7}\right) - 2 = x + 2 - 2 = x.$$

Hence, $f \circ g$ is the identity function on X and $g \circ f$ is the identity function on Y. Therefore, f and g are inverses of one another.

[Answer: (d). Recall that f and g are inverses of each other if $g \circ f$ is the identity function on X and $f \circ g$ is the identity function on Y.]

6. Let $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3, 4\}$. Which of the following subsets of $X \times Y$ are functions from X to Y?
   (a) $\{(a, 1),\ (b, 2),\ (c, 4),\ (a, 3),\ (d, 2)\}$.
   (b) $\{(a, 3),\ (b, 4),\ (c, 3),\ (d, 3)\}$.
   (c) $\{(a, 2),\ (b, 4)\}$.
   (d) $\{(a, 1),\ (b, 1),\ (c, 1),\ (d, 1)\}$.

   [Answers: (b) and (c). Recall that, formally, a function from X to Y is a subset f of $X \times Y$ so that for all $x \in X$, there is exactly one element of f with first coordinate x.]