Quiz 3, 2019

1. Let X be a nonempty set and let A and B be nonempty subsets of X. Please select **all** statements which are true.
   (a) We have $A \cap A^c = A$.
   (b) Suppose $x \in X$. We have $[x \notin A \cap B] \implies [(x \notin A) \vee (s \notin B)]$.
   (c) We have $(B^c)^c = B$.
   (d) When $A \subseteq B$, we have $B \setminus A = \emptyset$.

   [Answers: (b) and (c). First, note that $x \in A \cap A^c \implies (x \in A \wedge x \notin A)$, so $A \cap A^c = \emptyset$. Next, recall that with P, Q propositions, $\neg(P \wedge Q) \implies (\neg P \vee \neg Q)$, so (b) holds, and $\neg(\neg P) \iff P$, so (c) holds. When $A \subseteq B$ with $A \neq B$, there must be an element in $B \setminus A$.]

2. Let X be a nonempty set. For $x \in X$, let $P(x)$ and $Q(x)$ be propositions that involve x. Also, take $A \subseteq X$. Please select **all** statements that are true. (Note that notation such as $\exists\, x \in X,\, P(x) \implies Q(x)$ can be read as $\exists\, x \in X$ so that $P(x) \implies Q(x)$.)
   (a)
$$[\forall\, x \in X,\, P(x) \implies Q(x)] \iff [\exists\, x \in X,\, \neg Q(x) \implies \neg P(x)].$$
   (b)
$$[\exists\, x \in X,\, P(x) \implies Q(x)] \iff [\exists\, x \in X,\, Q(x) \implies P(x)].$$
   (c)
$$[\forall\, x \in X,\, P(x) \implies Q(x)] \iff [\forall\, x \in X,\, \neg Q(x) \implies \neg P(x)].$$
   (d)
$$[\neg(\forall\, x \in A,\, P(x) \implies Q(x))] \iff [\exists\, x \in A^c,\, \neg(P(x) \implies Q(x))].$$

   [Answer: (c). First, $P(x) \implies Q(x)$ is equivalent to $\neg Q(x) \implies \neg P(x)$, but $\forall x \in X$ is not equivalent to $\exists x \in X$; so (a) is not true but (c) is true. From truth tables, one sees that $P(x) \implies Q(x)$ is not equivalent to $Q(x) \implies P(x)$, so (b) is not true. We have
$$[\neg(\forall x \in A,\ P(x) \implies Q(x))] \iff [\exists x \in A,\ \neg(P(x) \implies Q(x))],$$
   so (d) is not true.]

3. For $i \in \mathbb{Z}_+$, let
$$A_i = \left\{ 2n :\ n \in \mathbb{Z}_+,\ n \geq \frac{i}{2} \right\}$$
   and
$$B_i = \left\{ 2n :\ n \in \mathbb{Z}_+,\ n \geq \frac{i+1}{2} \right\}.$$
   Match each of the items in List 1 with an item in List 2.
   List 1: (i) $\cup_{i \in \mathbb{Z}_+} A_i$, (ii) $\cap_{i \in \mathbb{Z}_+} B_i$, (iii) $\cup_{i \in \mathbb{Z}_+} (A_i \cup B_i)$.
   List 2: A. $\emptyset$, B. $\{2n - 1 :\ n \in \mathbb{Z}_+\ \}$, C. $\{2n :\ n \in \mathbb{Z}_+\ \}$, D. Z_+.
   [Answers: in order, the answer to (i) is B, the answer to (ii) is C, and the answer to (iii) is A. We have that $\cup_{i \in \mathbb{Z}_+} A_i$ is a subset of the

positive even integers, and for any $k \in \mathbb{Z}_+$, $(k \geq \frac{i}{2}) \iff (2k \geq i)$. Thus $2k \in A_k$, and $A_k = \{2k, 2k+2, 2k+4, 2k+6, \ldots\}$. Somewhat similarly, with $k \in \mathbb{Z}_+$, we have $(k \geq \frac{i+1}{2}) \iff (2k-1 \geq i)$, so $2k-1 \in B_k$ and $B_k = \{2k-1, 2k+1, 2k+3, 2k+5, \ldots\}$.]

4. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x$. Let $V = \{x \in \mathbb{R} : \ 5 < x \leq 100 \ \}$. Identify which of the below statements is correct, where $f^{-1}(V)$ denotes the inverse image of V.
   (a) $f^{-1}(V) = \{(5, 100]\}$
   (b) $f^{-1}(V) = \left\{ \left(\frac{5}{2}, 50 \right] \right\}$
   (c) $f^{-1}(V) = (5, 100]$
   (d) $f^{-1}(V) = \left(\frac{5}{2}, 50 \right]$

   [Answer: (d). Suggestion: review the definition of an inverse image. Also note that $\left(\frac{5}{2}, 50 \right] = \left\{ x \in \mathbb{R} : \ \frac{5}{2} < x \leq 50 \ \right\}$, which is a subset of the domain of f, whereas $\left\{ \left(\frac{5}{2}, 50 \right] \right\}$ is not (as it is the set of a subset of $\mathbb{R}$).]

5. Let $X = \{x \in \mathbb{R} : \ x > 1 \ \}$ and $Y = \{y \in \mathbb{R} : \ y > 0 \ \}$. Define $f : X \to Y$ by $f(x) = \dfrac{x-1}{x+1}$. Identify all arguments below that are correct.
   (a) We prove that f is injective. Take $x, x' \in X$ and suppose that $x = x'$. Then $x + 1 = x' + 1$, and $x - 1 = x' - 1$, so
   $$f(x) = \frac{x-1}{x+1} = \frac{x'-1}{x'+1} = f(x').$$
   Hence, we have that $f(x) = f(x')$. This argument holds for all $x, x' \in X$, and so f is injective.
   (b) We prove that f is injective. Take $x, x' \in X$ and suppose that $f(x) = f(x')$. Then
   $$\frac{x-1}{x+1} = \frac{x'-1}{x'+1},$$
   so
   $$(x-1)(x'+1) = (x'-1)(x+1).$$
   Hence
   $$xx' - x' + x - 1 = x'x - x + x' - 1$$
   so $-x' + x = -x + x'$. From this we get $2x = 2x'$, and so $x = x'$. This argument holds for all $x, x' \in X$, and so f is injective.
   (c) We prove that f is injective. Take $x, x' \in X$ and suppose that $f(x) \neq f(x')$. Then
   $$\frac{x-1}{x+1} \neq \frac{x'-1}{x'+1},$$
   so
   $$(x-1)(x'+1) \neq (x'-1)(x+1).$$
   Hence
   $$xx' - x' + x - 1 \neq x'x - x + x' - 1$$

so $-x' + x \neq -x + x'$. From this we get $2x \neq 2x'$, and so $x \neq x'$. This argument holds for all $x, x' \in X$, and so f is injective.

(d) We prove that f is injective. Take $x, x' \in X$ and suppose that $f(x) = f(x')$. Then

$$\frac{x-1}{x+1} = \frac{x'-1}{x'+1},$$

so

$$(x-1)(x'+1) = (x'-1)(x+1).$$

Hence

$$xx' - x' + x - 1 = x'x - x + x' - 1$$

so $-x' + x = -x + x'$. From this we get $x - x' = -(x - x')$, and so $1 = -1$, which is clearly not true. Hence we cannot have $f(x) = f(x')$, showing f is injective.

[Answer: (b). Suggestion: review Theorem 3.1.]