Quiz 4, 2019

1. Let $X = \{1, 2, 3, 4\}$ and let

$$R = \{(1,1),\ (2,2),\ (3,3),\ (4,4),\ (1,4),\ (4,1),\ (2,4),\ (4,2),\ (1,2),\ (2,1)\}$$

be a relation on X. **Fact:** R is an equivalence relation on X. Which of the following sets is the equivalence class of 2?
(a) $\{2\}$
(b) $\{3\}$
(c) $\{1, 4\}$
(d) $\{1, 2, 4\}$
(e) $\{1, 2, 3, 4\}$

[Answer: (d). From the definition of R, we that $2, 4$, and 1 are related to 2. (Since R is an equivalence relation, we also see that 2 is related to $2, 4$, and 1.)]

2. Let $A = a, b, c, d, e, f$ where a, b, c, d, e, f are distinct elements of A. Which of the following are partitions of A? Select **all** that apply.
(a) $\{\{a, b, e\},\ \{c\},\ \{d, f\}\}$.
(b) $\{\{a, e\},\ \{b, c, d\},\ \emptyset,\ \{f\}\}$.
(c) $\{\{a, e, f\},\ \{c\},\ \{b, e\}\}$.
(d) $\{\{a, f\},\ \{b, d\},\ \{e\}\}$.
(e) $\{a\},\ \{e, f\},\ \{b\},\ \{c, d\}\}$.

[Answers: (a) and (e). Suggestion: review the definition of a partition.]

3. Let $f : X \to Y$ and let $\{V_i :\ i \in \mathcal{I}\}$ be a collection of nonempty subsets of Y, where $\mathcal{I}$ is some indexing set. Which one of the following proofs is correct?
(a) We have

$$\begin{aligned} x \in f^{-1}\left(\cup_{i\in\mathcal{I}} V_i\right) &\iff f(x) \in \left(\cup_{i\in\mathcal{I}} V_i\right) \\ &\iff \forall i \in \mathcal{I},\ f(x) \in V_i \\ &\iff \forall i \in \mathcal{I},\ x \in f^{-1}(V_i) \\ &\iff x \in \cup_{i\in\mathcal{I}} f^{-1}(V_i). \end{aligned}$$

(b) We have

$$\begin{aligned} x \in f^{-1}\left(\cup_{i\in\mathcal{I}} V_i\right) &\implies f(x) \in \left(\cup_{i\in\mathcal{I}} V_i\right) \\ &\implies \exists i \in \mathcal{I},\ f(x) \in V_i \\ &\implies \exists i \in \mathcal{I},\ x \in f^{-1}(V_i) \\ &\implies x \in \cup_{i\in\mathcal{I}} f^{-1}(V_i). \end{aligned}$$

(c) We will show that if f is surjective then

$$f^{-1}\left(\cup_{i\in\mathcal{I}} V_i\right) \subseteq \cup_{i\in\mathcal{I}} f^{-1}(V_i).$$

Let $y \in Y$. Since f is surjective, there there exists $x \in X$ such that $f(x) = y$ Hence, we have that

$$\begin{aligned} y \in f^{-1}\left(\cup_{i\in\mathcal{I}} V_i\right) &\implies f(x) \in f^{-1}\left(\cup_{i\in\mathcal{I}} V_i\right) \\ &\implies x \in \cup_{i\in\mathcal{I}} V_i \\ &\implies \exists i \in \mathcal{I} \text{ such that } x \in V_i \\ &\implies \exists i \in \mathcal{I} \text{ such that } f(x) \in f^{-1}(V_i) \\ &\implies f(x) \in \cup_{i\in\mathcal{I}} f^{-1}(V_i) \\ &\implies y \in \cup_{i\in\mathcal{I}} f^{-1}(V_i). \end{aligned}$$

It follows that $f^{-1}\left(\cup_{i\in\mathcal{I}} V_i\right) \subseteq \cup_{i\in\mathcal{I}} f^{-1}(V_i)$.

(d) We will show that if f is surjective then

$$f^{-1}\left(\cup_{i\in\mathcal{I}} V_i\right) = \cup_{i\in\mathcal{I}} f^{-1}(V_i).$$

Let $y \in Y$. Since f is surjective, there there exists $x \in X$ such that $f(x) = y$ Hence, we have that

$$\begin{aligned} y \in f^{-1}\left(\cup_{i\in\mathcal{I}} V_i\right) &\iff f(x) \in f^{-1}\left(\cup_{i\in\mathcal{I}} V_i\right) \\ &\iff x \in \cup_{i\in\mathcal{I}} V_i \\ &\iff \forall i \in \mathcal{I} \text{ such that } x \in V_i \\ &\iff \forall i \in \mathcal{I} \text{ such that } f(x) \in f^{-1}(V_i) \\ &\iff f(x) \in \cup_{i\in\mathcal{I}} f^{-1}(V_i) \\ &\iff y \in \cup_{i\in\mathcal{I}} f^{-1}(V_i). \end{aligned}$$

Hence, $y \in f^{-1}\left(\cup_{i\in\mathcal{I}} V_i\right)$ if and only if $y \in \cup_{i\in\mathcal{I}} f^{-1}(V_i)$. It follows that $f^{-1}\left(\cup_{i\in\mathcal{I}} V_i\right) = \cup_{i\in\mathcal{I}} f^{-1}(V_i)$.

[Answer: (b). Suggestion: review the definition of $f^{-1}(V)$, and the definitions of a union as well as of the symbols $\implies$ and $\iff$.]

4. Suppose $a, b \in \mathbb{Z}_+$. Is it true or false that there are infinitely many $q \in \mathbb{Z}$ so that $\exists r \in \mathbb{Z}$ with $b = aq + r$?

   [Answer: true. By Prop. 6.2, there exist unique integers q, r so that $b = aq + r$ with $0 \leq r < a$. However, removing these constraints on r, we see that for any integer $q \in \mathbb{Z}$, we can take $r = b - aq$ and have $b = aq_r$.]

5. Suppose $a, b \in \mathbb{Z}_+$ with $hcf(a, b) = 1$. Is it true or false that there exist $u, v \in \mathbb{Z}$ so that $au + bv = 3$?

   [Answer: true. We know there are $s, t \in \mathbb{Z}$ so that $as + bt = 1$. Then taking $u = 3s$, $v = 3t$, we get $au + bv = 3$.]

6. Suppose $x \in \mathbb{Z}$ with $x \equiv 8 \pmod 9$. What can we therefore say x is equivalent to modulo 3? Your answer should be in the range $[0, 2]$.

   [Answer: 2. Suggestion: review the definition of a congruence (Section 5).]

7. Is it true or false that there is some $x \in \mathbb{Z}$ so that $x \equiv 2 \pmod 9$ and $x \equiv 7 \pmod{15}$?

   [Answer: false. If there were such x, then we would have $x \equiv 2 \pmod 3$ and $x \equiv 1 \pmod 3$, yielding a contradiction.]