Quiz 5, 2019

1. Suppose that $a, b \in \mathbb{Z}_+$ and $6 = \text{hcf}(a, b)$. Is it true or false that there could be $u, v \in \mathbb{Z}$ so that $au + bv = 1$?

   [Answer: false. We have $6|a$ and $6|b$, so for any $u, v \in \mathbb{Z}$ we have $6|au + bv$. Thus $au + bv$ cannot equal 1, as 6 does not divide 1.]

2. Suppose that $a, b, s, t \in \mathbb{Z}$ with $as + bt = 1$. Is it true or false that $5as \equiv 5 \pmod{b}$?

   [Answer: true: Since $as + bt = 1$, we have $as \equiv 1 \pmod{b}$. Thus $5as \equiv 5 \cdot 1 \pmod{b}$.]

3. Suppose X and Y are sets and $m, n \in \mathbb{Z}_+$ with $|X| = m$ and $|Y| = n$. Then the number of functions from X to Y is
   (a) mn
   (b) $m + n$
   (c) n^m
   (d) m^n

   [Answer: (c). To define $f : X \to Y$, for each element of X, we need to associate exactly one element of Y. So for each element of X, there are n choices for $f(x)$. So the number of choices for defining f is the product of n, m times.]

4. Suppose X is a countable set and $A \subseteq X$ with A an infinite set. Choose all of the following that are true.
   (a) The set $X \setminus A$ is a finite set.
   (b) The set A is a countable set.
   (c) There is an injective map $f : X \to \mathbb{Z}_+$ that is not surjective.
   (d) There is a surjective map $g : \mathbb{Z}_+ \to A$.

   [Answers: (b), (c), (d). Suggestion: review the definitions and results in Section 8. Recall that a bijective function has an inverse, which is also bijective. Also, if $f : X \to \mathbb{Z}_+$ is bijective, the map $g : X \to \mathbb{Z}_+$ defined by $g(x) = f(x)+1$ is injective but not surjective (and proving this is a good exercise!).]

5. Is it true or false that there are sets X and Y with $X \in Y$?

   [Answer: true. Suggestion: review the definition of the power set of a set X.]

6. Given any inifinite set X, is it true or false that there is a set Y with $|X| < |Y|$?

   [Answer: true. Suggestion: review the results in Section 9.]

7. Identify which of the following sets are countable. (Here $\mathcal{P}(\mathbb{Z})$ denotes the power set of $\mathbb{Z}$.)
   (a) $\mathbb{Z}$
   (b) $\mathbb{Z} \times \mathbb{Z}$
   (c) $\mathbb{Q} \times \mathbb{Q}$
   (d) $\mathbb{R} \times \mathbb{R}$

(e) $\mathcal{P}(\mathbb{Z})$

[Answers: (a), (b), (c). Suggestion: review the results of Sections 8 and 9.]

8. Suppose X is a set with $|X| = 3$. Identify which of the following is correct.
   (a) $|\mathcal{P}(X)| = 3$.
   (b) $|\mathcal{P}(X)| = 4$.
   (c) $|\mathcal{P}(X)| = 6$.
   (d) $|\mathcal{P}(X)| = 8$.
   (e) $|\mathcal{P}(X)| = 10$.

   [Answer: (d). Suggestion: review Theorem 9.3.]