SOME RELATIONS ON FOURIER COEFFICIENTS OF DEGREE 2 SIEGEL FORMS OF ARBITRARY LEVEL

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Abstract. We extend some recent work of D. McCarthy, proving relations among some Fourier coefficients of a degree 2 Siegel modular form F with arbitrary level and character, provided there are some primes q so that F is an eigenform for the Hecke operators $T(q)$ and $T_1(q^2)$.

1. Introduction

In a recent paper [3], McCarthy derives some nice results for Fourier coefficients of degree 2 Siegel modular forms of level 1, extending some classical results regarding elliptic modular forms. Most notably, McCarthy shows that with F a degree 2, level 1 Siegel modular form that is an eigenform for all the Hecke operators, and $a(T)$ denoting the Tth Fourier coefficient of F, we have:

(a) If $a(I) = 0$ then $a(mI) = 0$ for every $m \in \mathbb{Z}_+$;
(b) If $m, n \in \mathbb{Z}_+$ with $(m, n) = 1$, then $a(I)a(mnI) = a(mI)a(nI)$.

McCarthy's approach begins with some formulas from [1], which are somewhat cumbersome.

In this note we use the formulas from [2] that give the action of Hecke operators on Fourier coefficients of a Siegel modular form F, allowing for arbitrary level and character, and giving a simpler proof of McCarthy's above results (with no restriction on the level or character). We work with "Fourier coefficients" attached to lattices (as explained below), making it simpler to work with the image of F under a Hecke operator. We also relax our conditions on F, supposing it is an eigenform for at least some of the local Hecke algebras. We show that for $m, n \in \mathbb{Z}_+$ with $(m, n) = 1$, we have

$$a(mnI) = 0 \text{ if } a(mI) = 0$$

and

$$a(I)a(mnI) = a(mI)a(nI)$$

provided that n is a product of (powers of) primes q so that F is an eigenform for $T(q)$ and $T_1(q^2)$ (Theorem 3.3; note that in level 1, McCarthy's work should also yield this result). Additionally, we can show that with p an odd prime and $D = \begin{pmatrix} 1 & \\ & p \end{pmatrix}$, for $m, n \in \mathbb{Z}_+$ with $(m, n) = 1$ we have

$$a(mnD) = 0 \text{ if } a(D) = 0$$

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and

$$a(D)a(mnD) = a(mD)a(nD)$$

provided that n is a product of (powers of) odd primes q so that F is an eigenform for $T(q)$ and $T_1(q^2)$, and either $q = p$ or $\left(\frac{-p}{q}\right) = -1$ (Theorem 4.2; again, there is no restriction on the level or character).

We note that McCarthy also computes eigenvalues of the level 1 Eisenstein series with regard to the Hecke operators $T(p^r)$ (p prime); as he notes, in [5] we computed the eigenvalues of Eisenstein series of square-free levels with regard to different generators of the local Hecke algebra, namely $T(p)$ and $\widetilde{T}_1(p^2)$ (p prime).

2. Preliminaries

We will use some language and notation commonly used in quadratic forms theory: When Λ is a lattice whose quadratic form is given by the matrix T (relative to some $\mathbb{Z}$-lattice for Λ). We write $\Lambda \simeq T$. Now suppose that Λ is a lattice with $\Lambda \simeq T$ and $m \in \mathbb{Q}_+$; we write Λ^m to denote the lattice Λ "scaled" by m, meaning that $\Lambda^m \simeq mT$. Also, the discriminant of Λ is $\det T$. Finally, with Λ, Ω lattices on the same unlerlying quadratic space over $\mathbb{Q}$, we write $\{\Lambda : \Omega\}$ to denote the invariant factors of Ω in Λ.

With F a degree 2 Siegel modular form of weight k, level $\mathcal{N}$ and character χ, we can write F as a Fourier series:

$$F(\tau) = \sum_{T \geq 0} a(T) \exp(2\pi i Tr(T\tau))$$

where the sum is over 2×2 symmetric, positive semi-definite, half-integral matrices T (so the entries in T are half-integers with integers on the diagonal). Given $G \in GL_2(\mathbb{Z})$, we have $\gamma = \begin{pmatrix} G^{-1} & \\ & {}^tG \end{pmatrix} \in Sp_2(\mathbb{Z})$. Hence

$$\begin{aligned} \chi(\det G)F(\tau) &= F(\tau)|\gamma \\ &= (\det G)^k F(G^{-1}\tau\, {}^tG^{-1}) \\ &= (\det G)^k \sum_T a({}^tGTG) \exp(2\pi i Tr(T\tau)). \end{aligned}$$

Thus $a({}^tGTG) = \chi(\det G)(\det G)^k a(T)$. So we can also write F as a "Fourier series" supported on isometry classes of rank 2 even integral, positive semi-definite lattices (so each of these lattices is equipped with an even integral, positive semi-definite quadratic form):

$$F(\tau) = \sum_{\mathrm{cls}\,\Lambda} c(\Lambda)\, \mathrm{e}^*\{\Lambda\tau\}$$

where each lattice Λ is oriented if $\chi(-1)(-1)^k = -1$,

$$\mathrm{e}^*\{\Lambda\tau\} = \sum_G \exp(\pi i Tr({}^tGT_\Lambda G\tau))$$

with T_Λ an even integral matrix representing the quadratic form on Λ, G varying over $O(\Lambda)\backslash GL_2(\mathbb{Z})$ when $\chi(-1)(-1)^k = 1$ and over $O^+(\Lambda)\backslash SL_2(\mathbb{Z})$ otherwise. (Here $O(\Lambda)$ denotes the orthogonal group of Λ, and $O^+(\Lambda) =$

$O(\Lambda) \cap SL_2(\mathbb{Z})$.) Note that if Λ is an oriented lattice and Ω is a sublattice Λ, Ω inherits from Λ an orientation.

As discussed in [2], for each prime p we have a local Hecke algebra generated by the operators $T(p)$ and $T_1(p^2)$. Equivalently, this local Hecke algebra is generated by $T(p)$ and $\widetilde{T}_1(p^2)$ where

$$\widetilde{T}_1(p^2) = p^{k-3}T_1(p^2) + \chi(p)p^{k-3}(p+1).$$

Theorem 6.1 of [2] gives us the following.

Theorem 2.1. *Let F be a degree 2 Siegel modular form of level $\mathcal{N}$, character χ, and lattice coefficients $c(\Lambda)$. Then for any even integral lattice Λ, the Λth coefficient of $F|T(p)$ is*

$$\chi(p^2)p^{2k-3}c(\Lambda^{1/p}) + \chi(p)p^{k-2} \cdot \sum_{\{\Lambda:\Omega\}=(1,p)} c(\Omega^{1/p}) + c(\Lambda^p),$$

and the Λth coefficient of $F|\widetilde{T}_1(p^2)$ is

$$\chi(p^2)p^{2k-3} \cdot \sum_{\{\Lambda:\Omega\}=(1/p,1)} c(\Omega) + \chi(p)p^{k-1}\alpha_1(\Lambda;p)c(\Lambda) + \sum_{\{\Lambda:\Omega\}=(1,p)} c(\Omega)$$

where $\alpha_1(\Lambda;p)$ is the number of isotropic lines in the quadratic space $\Lambda/p\Lambda$.

Note that with p a prime and $m \in \mathbb{Z}_+$ so that $p \nmid m$, for any even integral rank 2 lattice Λ we have $\alpha_1(\Lambda;p) = \alpha_1(\Lambda^m;p)$ since scaling by m does not change whether a line is isotropic modulo p.

3. Proof of main result

The next lemma is pivotal in our proof of our main result; when this lemma generalizes, we can generalize our main result (as seen in Theorem 4.2).

Lemma 3.1. *Suppose that F is a degree 2 Siegel modular form of weight k, level $\mathcal{N}$, and character χ, and let $c(\Lambda)$ denote the Λth coefficient of F. With $\Delta \simeq 2I$, p prime and $m \in \mathbb{Z}_+$ so that $p \nmid m$, we have*

$$\sum_{\{\Delta:\Omega\}=(1/p,1)} c(\Omega^{pm}) = \sum_{\{\Delta:\Omega\}=(1,p)} c(\Omega^{m/p}) = \eta(p)c(\Delta^m)$$

where

$$\eta(p) = \begin{cases} 1+\chi(-1)(-1)^k & \text{if } p \equiv 1\ (4), \\ 0 & \text{if } p \equiv 3\ (4), \\ 1 & \text{if } p = 2. \end{cases}$$

Proof. Suppose that $\{\Delta : \Omega\} = (1/p, 1)$. Then $\{\Delta : p\Omega\} = (1,p)$; also, with T a matrix so that $\Omega^{m/p} \simeq \frac{m}{p}T$, we have $p\Omega^{m/p} \simeq pmT$. This proves that

$$\sum_{\{\Delta:\Omega\}=(1/p,1)} c(\Omega^{pm}) = \sum_{\{\Delta:\Omega\}=(1,p)} c(\Omega^{m/p}).$$

With $\{x,y\}$ a basis for Δ relative to which $\Delta\simeq 2I$, the lattices with invariant factors $(1,p)$ in Δ are

$$\mathbb{Z}(x+\beta y)\oplus\mathbb{Z}py\simeq 2\begin{pmatrix}1+\beta^2 & p\beta\\ p\beta & p^2\end{pmatrix}$$

where β varies modulo p, and

$$\mathbb{Z}px\oplus\mathbb{Z}y\simeq 2\begin{pmatrix}p^2 & \\ & 1\end{pmatrix}.$$

Suppose that Ω is one of these lattices. Then $\Omega^{m/p}$ is even integral if and only if $\Omega=\mathbb{Z}(x+\beta y)\oplus\mathbb{Z}py$ with $\beta^2\equiv -1\ (p)$. If $p\equiv 3\ (4)$, there are no such β. Suppose that $p\equiv 1\ (4)$, and fix γ so that $\gamma^2\equiv -1\ (p)$. Set $\Omega_+=\mathbb{Z}(x+\gamma y)\oplus\mathbb{Z}py$ and $\Omega_-=\mathbb{Z}(x-\gamma y)\oplus\mathbb{Z}py$. With $T=\begin{pmatrix}1+\gamma^2 & p\gamma\\ p\gamma & p^2\end{pmatrix}$, we have

$$\Omega_+\simeq 2T \text{ and } \Omega_-\simeq 2\begin{pmatrix}-1 & \\ & 1\end{pmatrix}T\begin{pmatrix}-1 & \\ & 1\end{pmatrix}.$$

Hence $c(\Omega_-^{m/p})=\chi(-1)(-1)^k c(\Omega_+^{m/p})$. Also, $\frac{1}{p}T$ is integral with $\det T=1$. Thus (by an algorithmic process, or by Exercise 5, p. 77 [4]), there is some $G\in GL_2(\mathbb{Z})$ so that ${}^tGTG=I$. Therefore $c(\Omega_+^{m/p})=\chi(\det G)(\det G)^k c(\Delta^m)$. When $p=2$, the lattices with invariant factors $(1,2)$ in Δ are

$$\mathbb{Z}x\oplus\mathbb{Z}2y\simeq 2\begin{pmatrix}1 & \\ & 4\end{pmatrix},\ \mathbb{Z}2x\oplus\mathbb{Z}y\simeq 2\begin{pmatrix}4 & \\ & 1\end{pmatrix},$$

and

$$\mathbb{Z}(x+y)\oplus\mathbb{Z}2y\simeq 4\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}.$$

When scaled by $m/2$, these first two lattices are not even integral, but the last one is and is isometric to Δ^m. Thus when $p=2$, the sum on Ω is

$$c(\Omega^{m/2})=c(\Delta^m).$$

$\square$

We use this lemma to prove a relation between $c(\Delta^{p^rm})$ and $c(\Delta^m)$ where $\Delta\simeq 2I$ and $m\in\mathbb{Z}_+$ so that $p\nmid m$. The argument is inductive, and it requires a relation between certain sums, and so we include this in the statement of the proposition.

Proposition 3.2. *Suppose that F is a degree 2 Siegel modular form of weight k, level $\mathcal{N}$, and character χ, and let $c(\Lambda)$ denote the Λth coefficient of F. Also suppose that that F is an eigenform for $T(p)$ and $\widetilde{T}_1(p^2)$ where p is a prime. Then with $\Delta\simeq 2I$ and $m\in\mathbb{Z}_+$ so that $p\nmid m$, for every integer $\ell\ge 0$ we have*

$$\sum_{\{\Delta:\Omega\}=(1/p,1)} c(\Omega^{p^\ell m})=\sum_{\{\Delta:\Omega\}=(1,p)} c(\Omega^{p^{\ell-2}m})=\eta(p^\ell)c(\Lambda^m)$$

and $c(\Delta^{p^\ell m})=\kappa(p^\ell)c(\Delta^m)$ where $\eta(p^\ell)$, $\kappa(p^\ell)$ depend only on $p,\lambda(p),\widetilde{\lambda}_1(p^2)$.

Proof. Let $\lambda(p)$ and $\widetilde{\lambda}_1(p^2)$ be the eigenvalues of F under $T(p)$ and $\widetilde{T}_1(p^2)$. Note that the first equality in the displayed equation is easily verified by replacing Ω by $p\Omega$.

(Case $\ell = 0$:) It is easy to see that $\kappa(1) = 1$. So suppose that we have $\{\Delta : \Omega\} = (1, p)$. Then $\operatorname{disc} \Omega^{p^{-2}m} = 4m^2/p^2$. Hence when $p \neq 2$, $\Omega^{m/p}$ cannot be integral, so $c(\Omega^{m/p}) = 0$. When $p = 2$, we see from the discussion at the end of the proof of Lemma 3.1 that $\Omega^{m/4}$ is not even integral for any Ω with $\{\Delta : \Omega\} = (1, 2)$. Thus $\eta(1) = 0$.

(Case $\ell = 1$:) By Lemma 3.1,

$$\sum_{\{\Delta:\Omega\}=(1,p)} c(\Omega^{m/p}) = \eta(p)c(\Delta^m)$$

where $\eta(p)$ is dependent only on p. By Theorem 2.1 we have

$$\lambda(p)c(\Delta^m) = \chi(p^2)p^{2k-3}c(\Delta^{m/p}) + \chi(p)p^{k-2} \sum_{\{\Delta:\Omega\}=(1,p)} c(\Omega^{m/p}) + c(\Delta^{pm}).$$

We see that $\Delta^{m/p}$ is not even integral, so $c(\Delta^{m/p}) = 0$. Thus with Lemma 3.2, we have

$$c(\Delta^{pm}) = \kappa(p)c(\Delta^m)$$

where $\kappa(p) = \lambda(p) - \eta(p)$, and so $\kappa(p)$ depends only on p and $\lambda(p)$.

(Induction step:) Suppose that $r \geq 2$ and that the proposition holds for all ℓ with $0 \leq \ell < r$. First, from Theorem 2.1 we have

$$\begin{aligned}\widetilde{\lambda}_1(p^2)c(\Delta^{p^{r-2}m}) =& \chi(p^2)p^{2k-3} \sum_{\{\Delta:\Omega\}=(1/p,1)} c(\Omega^{p^{r-2}m}) \\ &+ \chi(p)p^{k-2}\alpha_1(\Delta^{p^{r-2}m}; p)c(\Delta^{p^{r-2}m}) + \sum_{\{\Delta:\Omega\}=(1,p)} c(\Omega^{p^{r-2}m}).\end{aligned}$$

Thus

$$\sum_{\{\Delta:\Omega\}=(1,p)} c(\Omega^{p^{r-2}m}) = \sum_{\{\Delta:\Omega\}=(1/p,1)} c(\Omega^{p^r m}) = \eta(p^r)c(\Delta^m)$$

where

$$\eta(p^r) = \widetilde{\lambda}_1(p^2)\kappa(p^{r-2}) - \chi(p^2)p^{2k-3}\eta(p^{r-2}) - \chi(p)p^{k-2}\alpha_1(\Delta^{p^{r-2}}; p)\kappa(p^{r-2}).$$

(Recall that at the end of Sec. 2 we discussed why $\alpha_1(\Lambda^{p^\ell m}; p) = \alpha_1(\Lambda^{p^\ell m}; p)$.) Note that for $r \geq 3$, $\alpha_1(\Delta^{p^{r-2}}; p) = p + 1$, the number of lines in $\Delta/p\Delta$. For $r = 2$,

$$\alpha_1(\Delta^{p^{r-2}}; p) = \begin{cases} 2 & \text{if } p \equiv 1 \ (4), \\ 0 & \text{if } p \equiv 3 \ (4), \\ 1 & \text{if } p = 2. \end{cases}$$

In particular, this means that $\eta(p^r)$ depends only on p, $\lambda(p)$, and $\widetilde{\lambda}_1(p^2)$.

Also from Theorem 2.1 we have

$$\begin{aligned}\lambda(p)c(\Delta^{p^{r-1}m}) =& \chi(p^2)p^{2k-3}c(\Delta^{p^{r-2}m}) \\ &+ \chi(p)p^{k-2} \sum_{\{\Delta:\Omega\}=(1.p)} c(\Omega^{p^{r-2}m}) + c(\Delta^{p^r m}).\end{aligned}$$

Therefore $c(\Delta^{p^r m}) = \kappa(p^r)c(\Delta^m)$ where

$$\kappa(p^r) = \lambda(p)\kappa(p^{r-1}) - \chi(p^2)p^{2k-3}\kappa(p^{r-2}) - \chi(p)p^{k-2}\eta(p^r).$$

Thus by the induction hypothesis and the result from the previous paragraph, $\kappa(p^r)$ depends only on p, $\lambda(p)$, and $\widetilde{\lambda}_1(p^2)$. $\square$

Our main result is now easy to prove.

Theorem 3.3. *Suppose that F is a degree 2 Siegel modular form of level $\mathcal{N}$ and character χ with Fourier expansion*

$$F(\tau) = \sum_T a(T)\exp(2\pi i Tr(T\tau)).$$

Let $\mathcal{S}$ be the set of primes so that for $p \in \mathcal{S}$, F is an eigenform for $T(p)$ and $\widetilde{T}_1(p^2)$. Let n be a product of powers of primes in $\mathcal{S}$. Then for any $m \in \mathbb{Z}_+$ with $(m,n) = 1$, we have

$$a(mnI) = 0 \text{ if } a(mI) = 0,$$

and in any case we have

$$a(I)a(mnI) = a(mI)a(nI).$$

Proof. Write $n = p_1^{e_1} \cdots p_t^{e_t}$ where $p_1, \ldots, p_t$ are distinct elements of $\mathcal{S}$. For any $m' \in \mathbb{Z}_+$ with $(n, m') = 1$, repeated applications of Proposition 3.2 gives us

$$a(m'nI) = \kappa(p_1^{e_1}) \cdots \kappa(p_t^{e_t})a(m'I).$$

Thus (taking $m' = m$) we have $a(mnI) = 0$ if $a(mI) = 0$. Further (taking $m' = 1$), we have

$$a(nI) = \kappa(p_1^{e_1}) \cdots \kappa(p_t^{e_t})a(I)$$

and hence $a(I)a(mnI) = a(mI)a(nI)$.

$\square$

4. A modest generalization

As previously noted, the key to proving our main result is Lemma 3.1. We can extend this lemma to some extent, as follows.

Lemma 4.1. *Suppose that F is a degree 2 Siegel modular form of weight k, level $\mathcal{N}$, and character χ, and let $c(\Lambda)$ denote the Λth coefficient of F. Suppose that p is an odd prime and $\Delta \simeq 2\begin{pmatrix}1 & \\ & p\end{pmatrix}$. For $m \in \mathbb{Z}_+$ with $p \nmid m$, we have*

$$\sum_{\{\Delta:\Omega\}=(1/p,1)} c(\Omega^{pm}) = \chi(-1)(-1)^k c(\Delta^m).$$

For q an odd prime with $\left(\frac{-p}{q}\right) = -1$ we have

$$\sum_{\{\Delta:\Omega\}=(1/q,1)} c(\Omega^{qm}) = 0.$$

Proof. Let $\{x, y\}$ be a $\mathbb{Z}$-basis for Δ relative to which $\Delta \simeq \begin{pmatrix} 2 & \\ & 2p \end{pmatrix}$. Then the only lattice Ω so that $\{\Delta : \Omega\} = (1, p)$ and $\Omega^{m/p}$ is even integral is

$$\Omega = \mathbb{Z}px \oplus \mathbb{Z}y \simeq 2p \begin{pmatrix} p & \\ & 1 \end{pmatrix} = 2p \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p & \\ & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence

$$\sum_{\{\Delta:\Omega\}=(1/p,1)} c(\Omega^{pm}) = \sum_{\{\Delta:\Omega\}=(1,p)} c(\Omega^{m/p}) = \chi(-1)(-1)^k c(\Delta^m).$$

With q an odd prime with $\left(\frac{-p}{q}\right) = -1$, there is no lattice Ω so that $\{\Delta : \Omega\} = (1, q)$ and $\Omega^{m/q}$ is even integral, and hence

$$\sum_{\{\Delta:\Omega\}=(1/q,1)} c(\Omega^{qm}) = 0.$$

□

From this, one mimics the proofs of Proposition 3.2 and Theorem 3.3 to obtain the following.

Theorem 4.2. *Suppose that F is a degree 2 Siegel modular form of level $\mathcal{N}$ and character χ with Fourier expansion*

$$F(\tau) = \sum_T a(T) \exp(2\pi i Tr(T\tau)).$$

Suppose that p is an odd prime, and set $D = \begin{pmatrix} 1 & \\ & p \end{pmatrix}$. Let $\mathcal{S}$ be the set of odd primes so that for $q \in \mathcal{S}$, F is an eigenform for $T(q)$ and $\widetilde{T}_1(q^2)$, and either $q = p$ or $\left(\frac{-p}{q}\right) = -1$. Let n be a product of powers of primes in $\mathcal{S}$. Then for any $m \in \mathbb{Z}_+$ so that $(m, n) = 1$, we have

$$a(mnD) = 0 \text{ if } a(mD) = 0,$$

and in any case we have

$$a(D)a(mnD) = a(mD)a(nD).$$

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