Hecke eigenforms and representation numbers of arbitrary rank lattices

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Introduction. Given a totally positive quadratic form Q over a totally real number field $\mathbf{K}$, one can obtain a Hilbert modular form by restricting Q to a lattice L and forming the theta series attached to L; the Fourier coefficients of the theta series are the representation numbers of Q on L. The space of Hilbert modular forms generated by all theta series attached to lattices of the same weight, level and character is invariant under a subalgebra of the Hecke algebra, hence one can (in theory) diagonalize this space of modular forms with respect to an appropriate Hecke subalgebra and infer relations on the representation numbers of the lattices. In a previous paper the author found such relations by constructing eigenforms from theta series attached to lattices of even rank which are "nice" at dyadic primes; the purpose of this paper is to extend the previous results to all lattices.

We begin by proving a lemma (Lemma 1.1) which allows us to remove the restriction regarding dyadic primes. Then using our previous work we find that associated to any even rank lattice L is a family of lattices $\mathrm{fam}L$ which is partitioned into nuclear families (which are genera when the ground field is $\mathbf{Q}$), and the averaged representation numbers of these nuclear families satisfy arithmetic relations (Theorem 1.2).

In §2 we define "Fourier coefficients" attached to integral ideals for a half-integral weight Hilbert modular form. Then in analogy to the case $\mathbf{K} = \mathbf{Q}$, we describe the effect of the Hecke operators on these Fourier coefficients (Theorem 2.5).

In §3 we use theta series attached to odd rank lattices to construct eigenforms for the Hecke operators; the results of §2 then give us arithmetic relations on the representation numbers of the odd rank lattices. When the ground field is $\mathbf{Q}$, we may assume $Q(L) \subseteq \mathbf{Z}$ and then these relations may be stated as

$$\mathbf{r}(\mathrm{gen}L, 2p^2a) = \left(1 - p^{\frac{m-3}{2}}\chi_{_L}(p)(-1|p)^{\frac{m-1}{2}}(2a|p) + p^{m-2}\right)\mathbf{r}(\mathrm{gen}L, 2a) - p^{m-2}\mathbf{r}\left(\mathrm{gen}L, \frac{2a}{p^2}\right)$$

where $\mathbf{r}(\mathrm{gen}L, 2a)$ is the average number of times the lattices in the genus of L represent $2a$, m is the rank of L, p is a prime not dividing the level of L, and $\chi_{_L}$ is the character attached to L (Corollary 3.7).

§**1. Relations on representation numbers of lattices of even rank.** Let V be a vector space of even dimension m over $\mathbf{K}$ where $\mathbf{K}$ is a totally real number field of degree n over $\mathbf{Q}$; let Q be a totally positive quadratic form on V, L a lattice on V (so $\mathbf{K}L = V$),

$\mathcal{N}$ the level of L and $\mathbf{n}L$ the norm of L as defined in [6]. Then the theta series

$$\theta(L,\tau) = \sum_{x\in L} e^{2\pi i Tr(Q(x)\tau)}$$

is a Hilbert modular form of weight $m/2$, level $\mathcal{N}$ and quadratic character $\chi_{_L}$, and for $\mathcal{P}$ a prime ideal such that $\mathcal{P}\nmid\mathcal{N}$, either the Hecke operator $T(\mathcal{P})$ or the operator $T(\mathcal{P}^2)$ maps $\theta(L,\tau)$ to a linear combination of theta series of the same weight, level and character (see [6]; cf. [1]).

We derive relations on the representation numbers of the lattices in the "extended family" of L; essentially, the extended family of L consists of all lattices which arise when we act on the theta series attached to lattices in the genus of L with those Hecke operators known to preserve the space spanned by theta series. We begin now by giving refined definitions of a family and of an extended family; these definitions agree with those given in [8] when the lattice in question is unimodular when localized at dyadic primes.

Definition. A lattice L' is in the family of L, denoted fam$L$, if L' is a lattice on V^α where α is a totally positive element of $\mathbf{K}^\times$ which is relatively prime to $\mathcal{N}$, such that for all primes $\mathcal{P}|\mathcal{N}$ we have $L'_\mathcal{P}\simeq L^\alpha_\mathcal{P}$, and for all primes $\mathcal{P}\nmid\mathcal{N}$ we have $L'_\mathcal{P}\simeq L^{u_\mathcal{P}}_\mathcal{P}$ for some $u_{_\mathcal{P}}\in\mathcal{O}^\times_\mathcal{P}$. Here $L_\mathcal{P}=\mathcal{O}_\mathcal{P}L$, and V^α (resp. $L^\alpha_\mathcal{P}$) denotes the vector space V (resp. the lattice $L_\mathcal{P}$) equipped with the "scaled" quadratic form αQ. We say $L'\in$ fam$L$ is in the nuclear family of L, fam^+L, if there exists some totally positive unit u such that $L'_\mathcal{P}\simeq L^u_\mathcal{P}$ for all primes $\mathcal{P}$, and we say L' is in the extended family of L, xfam$L$, if L' is connected to L with a prime-sublattice chain as defined in §3 of [8].

For $\xi\gg 0$, we define the representation numbers $\mathbf{r}(L,\xi)$ and $\mathbf{r}(\text{xfam}L,\xi)$ by

$$\mathbf{r}(L,\xi) = \#\{x\in L:\ Q(x)=\xi\ \}$$

and

$$\mathbf{r}(\text{fam}^+L,\xi) = \sum_{L'} \frac{1}{o(L')}\ \mathbf{r}(L',\xi)$$

where $o(L')$ is the order of the orthogonal group of L' (see [4]) and the sum runs over a complete set of representatives of the isometry classes within fam^+L. Note that if $u\in\mathcal{U}=\mathcal{O}^\times$ then L^{u^2} is in the genus of L; since $\mathcal{U}^+/\mathcal{U}^2$ is finite (where $\mathcal{U}^+$ denotes the group of totally positive units and $\mathcal{U}^2$ the subgroup of squares – see §61 of [3]) and each genus has a finite number of isometry classes, it follows that fam^+L has a finite number of isometry classes.

We now show

Lemma 1.1. *The number of nucler families in* fam$L$ *is* 2^r *where* $r \in \mathbf{Z}$.

Proof: As argued in the proof of Lemma 3.1 of [8], $L_{\mathcal{P}} \simeq L_{\mathcal{P}}^{u_{\mathcal{P}}}$ for any $u_{\mathcal{P}} \in \mathcal{U}_{\mathcal{P}} = \mathcal{O}_{\mathcal{P}}^{\times}$ when $\mathcal{P}$ is a prime not dividing $2\mathcal{N}$. Thus there can only be a finite number of primes $\mathcal{Q}$ such that $L_{\mathcal{Q}} \not\simeq L_{\mathcal{Q}}^{u_{\mathcal{Q}}}$ for all $u_{\mathcal{Q}} \in \mathcal{U}_{\mathcal{Q}}$; let $\mathcal{Q}_1, \ldots, \mathcal{Q}_t$ denote these "bad" primes for L.

For each $\mathcal{Q} = \mathcal{Q}_i$ $(1 \le i \le t)$, set

$$Stab_{\mathcal{Q}}(L) = \{u \in \mathcal{U}_{\mathcal{Q}} : \; L_{\mathcal{Q}}^{u} \simeq L_{\mathcal{Q}} \}.$$

Clearly $Stab_{\mathcal{Q}}(L)$ is a multiplicative subgroup of $\mathcal{U}_{\mathcal{Q}}$, and $\mathcal{U}_{\mathcal{Q}}^2 = \{u^2 : \; u \in \mathcal{U}_{\mathcal{Q}} \} \subseteq Stab_{\mathcal{Q}}(L)$. Now, since $[\mathcal{U}_{\mathcal{Q}} : \mathcal{U}_{\mathcal{Q}}^2]$ is a power of 2 (see 63:9 of [4]) it follows that $[\mathcal{U}_{\mathcal{Q}} : Stab_{\mathcal{Q}}(L)]$ is also a power of 2. Thus $\prod_{i=1}^{t} \mathcal{U}_{\mathcal{Q}_i}/Stab_{\mathcal{Q}_i}(L)$ is a group of order 2^s for some $s \in \mathbf{Z}$. We associate each nuclear family fam^+L' within fam$L$ to an element of $\prod_{i=1}^{t} \mathcal{U}_{\mathcal{Q}_i}/Stab_{\mathcal{Q}_i}(L)$ as follows. For $L' \in$ fam$L$ we know L' is a lattice on V^{α} for some $\alpha \in \mathbf{K}^{\times}$ with $\alpha \in \mathcal{U}_{\mathcal{Q}_i}$ and $L'_{\mathcal{Q}_i} \simeq L_{\mathcal{Q}_i}^{\alpha}$ $(1 \le i \le t)$; associate fam^+L' with $(\ldots, \alpha \cdot Stab_{\mathcal{Q}_i}(L), \ldots)$. It is easily seen that this map is well-defined and injective. The techniques used to prove Lemma 3.1 of [8] show that the nuclear families within fam$L$ are associated with a multiplicatively closed subset of the product $\prod_{i=1}^{t} \mathcal{U}_{\mathcal{Q}_i}/Stab_{\mathcal{Q}_i}(L)$; since this product is a finite group, it follows that the nuclear families within fam$L$ are associated with a subgroup of $\prod_{i=1}^{t} \mathcal{U}_{\mathcal{Q}_i}/Stab_{\mathcal{Q}_i}(L)$. The order of $\prod_{i=1}^{t} \mathcal{U}_{\mathcal{Q}_i}/Stab_{\mathcal{Q}_i}(L)$ is 2^s, so there must be 2^r nuclear families in fam$L$ where $r \in \mathbf{Z}$. **q.e.d.**

For a prime $\mathcal{P} \nmid 2\mathcal{N}$, define

$$\varepsilon_L(\mathcal{P}) = \begin{cases} 1 & \text{if } L/\mathcal{P}L \text{ is hyperbolic,} \\ -1 & \text{otherwise;} \end{cases}$$

define

$$\lambda(\mathcal{P}) = N(\mathcal{P})^{\frac{k}{2}}(N(\mathcal{P})^{k-1}+1) \text{ if } \varepsilon_L(\mathcal{P}) = 1, \text{ and}$$

$$\lambda(\mathcal{P}^2) = N(\mathcal{P})^{k}(N(\mathcal{P})^{k-1}-1)^2 \text{ if } \varepsilon_L(\mathcal{P}) = -1.$$

For $\mathcal{A} \subseteq \mathcal{O}$ such that $\mathrm{ord}_{\mathcal{P}}(\mathcal{A})$ is even whenever $\varepsilon_L(\mathcal{P}) = -1$, set $\varepsilon_L(\mathcal{A}) = \prod_{\mathcal{P}|\mathcal{A}} \varepsilon_L(\mathcal{P})^{\mathrm{ord}_{\mathcal{P}}\mathcal{A}}$, and set

$$\lambda(\mathcal{P}^a)\lambda(\mathcal{P}^b) = \sum_{c=0}^{\min\{a,b\}} N(\mathcal{P})^{c(2k-1)}\lambda(\mathcal{P}^{a+b-2c})$$

and $\lambda(\mathcal{A}) = \prod_{\mathcal{P}|\mathcal{A}} \lambda\left(\mathcal{P}^{\mathrm{ord}_{\mathcal{P}}(\mathcal{A})}\right)$. Now the arguments of [8] can be used to extend Theorem 3.9 of [8] to include any even rank lattice L, giving us

Theorem 1.2. *Let L be any lattice on V where* $\dim V = 2k$ *($k \in \mathbf{Z}_+$). Take $\xi \in \mathbf{n}L$, $\xi \gg 0$, and write $\xi(\mathbf{n}L)^{-1} = \mathcal{M}\mathcal{M}'$ where $\mathcal{M}$ and $\mathcal{M}'$ are integral ideals such that $(\mathcal{M}, 2\mathcal{N}) = 1$ and* $\mathrm{ord}_{\mathcal{P}}\mathcal{M}$ *is even whenever $\mathcal{P}$ is a prime such that $\varepsilon_L(\mathcal{P}) = -1$. Then*

$$\mathbf{r}(\mathrm{fam}^+ L, 2\xi) = \lambda(\mathcal{M}) N_{K/Q}(\mathcal{M})^{-k/2}\ \mathbf{r}(\mathrm{fam}^+ L', 2\xi) - \sum_{\substack{\mathcal{A} \supseteq \mathcal{M}+\mathcal{M}' \\ \mathcal{A} \neq \mathcal{O}}} \varepsilon_L(\mathcal{A}) N_{K/Q}(\mathcal{A})^{k-1}\ \mathbf{r}(\mathrm{fam}^+ \mathcal{A}L, 2\xi)$$

where $\mathbf{n}L' = \mathcal{M} \cdot \mathbf{n}L$ and L' is connected to L by a prime-sublattice chain.

§**2. Hecke operators on forms of half-integral weight.** In this section we develop some of the theory of half-integral weight Hilbert modular forms. To read about the general theory of Hilbert modular forms, see [2].

Let $\mathcal{N}$ be an integral ideal such that $4\mathcal{O} \subseteq \mathcal{N}$, and let $\mathcal{I}$ be a fractional ideal; then as in [8] we define

$$\Gamma_0(\mathcal{N}, \mathcal{I}^2) = \left\{ A \in \begin{pmatrix} \mathcal{O} & \mathcal{I}^{-2}\partial^{-1} \\ \mathcal{N}\mathcal{I}^2\partial & \mathcal{O} \end{pmatrix} :\ \det A \in \mathcal{U} = \mathcal{O}^\times,\ \det A \gg 0 \right\}.$$

We also define

$$\widetilde{\Gamma}_0(\mathcal{N}, \mathcal{I}^2) = \left\{ \widetilde{A} = \left[A, \frac{\theta(\mathcal{I}, A\tau)}{\theta(\mathcal{I}, \tau)} \right] :\ A \in \Gamma_0(\mathcal{N}, \mathcal{I}^2),\ \det A \in \mathcal{U}^2 \right\}$$

where $\theta(\mathcal{I}, \tau) = \sum_{\alpha \in \mathcal{I}} e(2\alpha^2\tau)$ with $e(\beta\tau) = \mathrm{e}^{\pi i Tr(\beta\tau)}$, and $\mathcal{U}^2 = \{u^2 :\ u \in \mathcal{U} = \mathcal{O}^\times\}$. As shown in §3 of [6], when $A \in \Gamma_0(\mathcal{N}, \mathcal{I}^2)$ and $\det A = 1$, $\theta(\mathcal{I}, A\tau)/\theta(\mathcal{I}, \tau)$ is a well-defined automorphy factor for A, and it is easily seen that for $u \in \mathcal{U}$, $\theta(\mathcal{I}, u^2\tau) = \theta(\mathcal{I}, \tau)$. Thus we can define a group action of $\widetilde{\Gamma}_0(\mathcal{N}, \mathcal{I}^2)$ on $f : \mathcal{H}^n \to \mathbf{C}$ by

$$f\big|_{\frac{m}{2}} \widetilde{A}(\tau) = f\big|\widetilde{A}(\tau) = \left(\frac{\theta(\mathcal{I}, A\tau)}{\theta(\mathcal{I}, \tau)} \right)^{-m} f(A\tau).$$

(Here $\mathcal{H}$ denotes the complex upper half-plane.) For $\chi_{\mathcal{N}}$ a numerical character modulo the ideal $\mathcal{N}$ and m an odd integer, we let $\mathcal{M}_{\frac{m}{2}}\left(\widetilde{\Gamma}_0(\mathcal{N}, \mathcal{I}^2), \chi_{\mathcal{N}}\right)$ denote the space of Hilbert modular forms f which satisfy

$$f\big|_{\frac{m}{2}} \widetilde{A}(\tau) = \chi_{\mathcal{N}}(a)\ f(\tau)$$

for all $\widetilde{A}=\widetilde{\begin{pmatrix} a & b\\ c & d\end{pmatrix}}\in\widetilde{\Gamma}_0(\mathcal{N},\mathcal{I}^2)$. Notice that by definition, $f\big|\widetilde{\begin{pmatrix} u & 0\\ 0 & u^{-1}\end{pmatrix}}(\tau)=f(u^2\tau)=$ $f\big|\widetilde{\begin{pmatrix} u^2 & 0\\ 0 & 1\end{pmatrix}}(\tau)$ for any $u\in\mathcal{U}$, so $\mathcal{M}_{\frac{m}{2}}\left(\widetilde{\Gamma}_0(\mathcal{N},\mathcal{I}^2),\chi_{_\mathcal{N}}\right)=\{0\}$ unless $\chi_{_\mathcal{N}}(u)=1$ for all $u\in\mathcal{U}$. For $\mathcal{P}$ a prime, $\mathcal{P}\not|\,\mathcal{N}$, we define the Hecke operator

$$T(\mathcal{P}^2):\mathcal{M}_{\frac{m}{2}}\left(\widetilde{\Gamma}_0(\mathcal{N},\mathcal{I}^2),\chi_{_\mathcal{N}}\right)\to\mathcal{M}_{\frac{m}{2}}\left(\widetilde{\Gamma}_0(\mathcal{N},\mathcal{P}^2\mathcal{I}^2),\chi_{_\mathcal{N}}\right)$$

as follows. Let $\left\{\widetilde{A}_j\right\}$ be a complete set of coset representatives for

$$\left(\widetilde{\Gamma}_1(\mathcal{N},\mathcal{I}^2)\cap\widetilde{\Gamma}_1(\mathcal{N},\mathcal{P}^2\mathcal{I}^2)\right)\Big\backslash\widetilde{\Gamma}_1(\mathcal{N},\mathcal{P}^2\mathcal{I}^2)$$

where

$$\widetilde{\Gamma}_1(\mathcal{N},\mathcal{I}^2)=\left\{\widetilde{\begin{pmatrix} a & b\\ c & d\end{pmatrix}}\in\widetilde{\Gamma}_0(\mathcal{N},\mathcal{I}^2):\ a\equiv 1\ (\mathrm{mod}\ \mathcal{N})\right\}.$$

Then for $f\in\mathcal{M}_{\frac{m}{2}}\left(\widetilde{\Gamma}_0(\mathcal{N},\mathcal{I}^2),\chi_{_\mathcal{N}}\right)$, define

$$f\big|T(\mathcal{P}^2)=N(\mathcal{P})^{\frac{m}{2}-2}\sum_j f\big|\widetilde{A}_j.$$

Clearly $T(\mathcal{P}^2)$ is well-defined and $f\big|T(\mathcal{P}^2)\in\mathcal{M}_{\frac{m}{2}}\left(\widetilde{\Gamma}_0(\mathcal{N},\mathcal{P}^2\mathcal{I}^2),\chi_{_\mathcal{N}}\right)$. Similar to the case of integral weight, we also define operators

$$S(\mathcal{P}):\mathcal{M}_{\frac{m}{2}}\left(\widetilde{\Gamma}_0(\mathcal{N},\mathcal{I}^2),\chi_{_\mathcal{N}}\right)\to\mathcal{M}_{\frac{m}{2}}\left(\widetilde{\Gamma}_0(\mathcal{N},\mathcal{P}^2\mathcal{I}^2),\chi_{_\mathcal{N}}\right)$$

by

$$f\big|S(\mathcal{P})=f\big|\left[C,N(\mathcal{P})^{-\frac{1}{2}}\frac{\theta(\mathcal{I},C\tau)}{\theta(\mathcal{P}\mathcal{I},\tau)}\right]$$

where $C\in\begin{pmatrix}\mathcal{P} & \mathcal{P}^{-1}\mathcal{I}^{-2}\partial^{-1}\\ \mathcal{N}\mathcal{P}\mathcal{I}^2\partial & \mathcal{O}\end{pmatrix}$, $\det C=1$, and $a_C\equiv 1\pmod{\mathcal{N}}$. The proof of Proposition 6.1 of [6] shows that $N(\mathcal{P})^{-\frac{1}{2}}\theta(\mathcal{I},C\tau)/\theta(\mathcal{P}\mathcal{I},\tau)$ is a well-defined automorphy factor for C, and it is easy to check that $S(\mathcal{P})$ is well-defined and that $f\big|S(\mathcal{P})\in\mathcal{M}_{\frac{m}{2}}(\widetilde{\Gamma}_0(\mathcal{N},\mathcal{P}^2\mathcal{I}^2),\chi_{_\mathcal{N}})$. (Note that the restrictions on d in Proposition 6.1 of [6] are unnecessary, but one must then use the extended transformation formula from §4 of [7].) In fact, $S(\mathcal{P})$ is an isomorphism, so by setting $S(\mathcal{P}^{-1})=S(\mathcal{P})^{-1}$ and $S(\mathcal{J}_1)S(\mathcal{J}_2)=S(\mathcal{J}_1\mathcal{J}_2)$, we can inductively define $S(\mathcal{J})$ for any fractional ideal J relatively prime to $\mathcal{N}$.

Lemma 2.1. *Suppose* $A \in \begin{pmatrix} \mathcal{P} & \mathcal{P}^{-1}\mathcal{I}^{-2}\partial^{-1} \\ \mathcal{N}\mathcal{P}\mathcal{I}^2\partial & \mathcal{P}^{-1} \end{pmatrix}$ *such that* $\det A = 1$ *and* $a_A \equiv 1 \pmod{\mathcal{N}}$. *Then for* $f \in \mathcal{M}_{\frac{m}{2}}\left(\widetilde{\Gamma}_0(\mathcal{N},\mathcal{I}^2), \chi_{\mathcal{N}}\right)$,

$$f\big|\left[A, N(\mathcal{P})^{-\frac{1}{2}}\frac{\theta(\mathcal{I},A\tau)}{\theta(\mathcal{P}\mathcal{I},\tau)}\right] = f\big|S(\mathcal{P}).$$

Proof: Let C be a matrix as in the definition of $S(\mathcal{P})$; so

$$\begin{aligned} &f\big|\left[A, N(\mathcal{P})^{-\frac{1}{2}}\frac{\theta(\mathcal{I},A\tau)}{\theta(\mathcal{P}\mathcal{I},\tau)}\right]\big|S(\mathcal{P})^{-1} \\ &= f\big|\left[A, N(\mathcal{P})^{-\frac{1}{2}}\frac{\theta(\mathcal{I},A\tau)}{\theta(\mathcal{P}\mathcal{I},\tau)}\right]\big|\left[C^{-1}, N(\mathcal{P})^{\frac{1}{2}}\frac{\theta(\mathcal{P}\mathcal{I},C^{-1}\tau)}{\theta(\mathcal{I},\tau)}\right] \\ &= f\big|\left[AC^{-1}, \frac{\theta(\mathcal{I},AC^{-1}\tau)}{\theta(\mathcal{I},\tau)}\right] \\ &= f \end{aligned}$$

since $\left[AC^{-1}, \theta(\mathcal{I},AC^{-1}\tau)/\theta(\mathcal{I},\tau)\right] \in \widetilde{\Gamma}_1(\mathcal{N},\mathcal{I}^2)$. **q.e.d.**

We now use this lemma to give us a useful description of $T(\mathcal{P}^2)$ when $\mathcal{P}\nmid\mathcal{N}$.

Lemma 2.2. *For* $\mathcal{P}$ *a prime,* $\mathcal{P}\nmid\mathcal{N}$, *and* $f \in \mathcal{M}_{\frac{m}{2}}\left(\widetilde{\Gamma}_0(\mathcal{N},\mathcal{I}^2), \chi_{\mathcal{N}}\right)$ *we have*

$$\begin{aligned} N(\mathcal{P})^{2-\frac{m}{2}}\ f\big|T(\mathcal{P}^2) = &\sum_b f\big|\left[\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, 1\right] \\ &+ \sum_\beta f\big|S(\mathcal{P})\big|\left[\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, N(\mathcal{P})^{\frac{1}{2}}\left(\sum_{\alpha\in\mathcal{P}\mathcal{I}/\mathcal{P}^2\mathcal{I}} e(-2\beta\alpha^2)\right)^{-1}\right] \\ &+ f\big|S(\mathcal{P}^2) \end{aligned}$$

where b *runs over* $\mathcal{P}^{-2}\mathcal{I}^{-2}\partial^{-1}/\mathcal{I}^{-2}\partial^{-1}$ *and* β *runs over* $(\mathcal{P}^{-3}\mathcal{I}^{-2}\partial^{-1}/\mathcal{P}^{-2}\mathcal{I}^{-2}\partial^{-1})^\times$.

Proof: Since for $\alpha \in \mathbf{K}^\times$ the mapping $f \mapsto f\big|\left[\begin{pmatrix} \alpha^{-2} & 0 \\ 0 & 1 \end{pmatrix}, N(\alpha^2)^{\frac{1}{4}}\right]$ is an isomorphism from the space $\mathcal{M}_{\frac{m}{2}}\left(\widetilde{\Gamma}_0(\mathcal{N},\mathcal{I}^2), \chi_{\mathcal{N}}\right)$ onto $\mathcal{M}_{\frac{m}{2}}\left(\widetilde{\Gamma}_0(\mathcal{N},\alpha^2\mathcal{I}^2), \chi_{\mathcal{N}}\right)$, we may assume $\mathcal{I} \subseteq \mathcal{O}$. Choose $a \in \mathcal{P} - \mathcal{P}^2$ such that $a\mathcal{O}$ is relatively prime to $\mathcal{N}$ and $a \equiv 1 \pmod{\mathcal{N}}$. Let $\{b_k\}$ be a set of coset representatives for $(\mathcal{P}^{-2}\mathcal{I}^{-2}\partial^{-1}/\mathcal{P}^{-1}\mathcal{I}^{-2}\partial^{-1})^\times$ such that $b_k\mathcal{P}^2\mathcal{I}^2\partial$ is relatively prime to $a\mathcal{O}$; then for each k, use strong approximation to choose $c_k \in \mathcal{N}\mathcal{P}^2\mathcal{I}^2\partial$ and $d_k \in \mathcal{O}$ such that $ad_k - b_kc_k = 1$. Take $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma_1(\mathcal{N}, \mathcal{P}^2\mathcal{I}^2)$ such that $a' \in \mathcal{P}^2$, $\mathcal{P} \not| d'$, and $a'd' - b'c' = 1$, and take $\{b''_j\}$ to be a set of representatives for $\mathcal{P}^{-2}\mathcal{I}^{-2}\partial^{-1}/\mathcal{I}^{-2}\partial^{-1}$. Then one easily sees that

$$\left\{\widetilde{\begin{pmatrix} 1 & b_j \\ 0 & 1 \end{pmatrix}}\right\} \cup \left\{\widetilde{\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}}\right\} \cup \left\{\widetilde{\begin{pmatrix} a & b_k \\ c_k & d_k \end{pmatrix}}\right\}$$

is a complete set of coset representatives for $\left(\widetilde{\Gamma}_1(\mathcal{N}, \mathcal{P}^2\mathcal{I}^2) \cap \widetilde{\Gamma}_1(\mathcal{N}, \mathcal{I}^2)\right) \Big\backslash \widetilde{\Gamma}_1(\mathcal{N}, \mathcal{P}^2\mathcal{I}^2)$.

Take $f \in \mathcal{M}_{\frac{m}{2}}\left(\widetilde{\Gamma}_0(\mathcal{N}, \mathcal{I}^2), \chi_{\mathcal{N}}\right)$. Then

$$f|\widetilde{A}' = f|\left[A', \frac{\theta(\mathcal{P}\mathcal{I}, A'\tau)}{\theta(\mathcal{P}\mathcal{I}, \tau)}\right]$$

and the transformation formula (2) in §2 of [6] shows that

$$\frac{\theta(\mathcal{P}\mathcal{I}, A'\tau)}{\theta(\mathcal{P}\mathcal{I}, \tau)} = \left(c' + d'\tfrac{1}{\tau}\right)^{\frac{1}{2}}\ \tau^{\frac{1}{2}}\ (d')^{-\frac{1}{2}} \sum_{\alpha \in \mathcal{P}\mathcal{I}/d'\mathcal{P}\mathcal{I}} e\left(\frac{b'}{d'}2\alpha^2\right).$$

(Recall that, as remarked earlier, we need not restrict d as [6], but we need to then use the extended transformation formula as it appears in [7].) On the other hand,

$$f|S(\mathcal{P}^2) = f|\left[A', N(\mathcal{P})^{-1}\frac{\theta(\mathcal{I}, A'\tau)}{\theta(\mathcal{P}^2\mathcal{I}, \tau)}\right]$$

and following the derivation in the proof of Proposition 6.1 of [6] we find that

$$\frac{\theta(\mathcal{I}, A'\tau)}{\theta(\mathcal{P}^2\mathcal{I}, \tau)} = \left(c' + d'\tfrac{1}{\tau}\right)^{\frac{1}{2}}\ \tau^{\frac{1}{2}}\ (d')^{-\frac{1}{2}} \sum_{\alpha \in \mathcal{P}^2\mathcal{I}/d'\mathcal{P}^2\mathcal{I}} e\left(\frac{b'}{d'}2\alpha^2\right) \sum_{\alpha \in d'\mathcal{I}/\mathcal{P}^2 d'\mathcal{I}} e\left(\frac{b'}{d'}2\alpha^2\right).$$

By Proposition 3.2 of [6], $\displaystyle\sum_{\alpha \in d'\mathcal{I}/d'\mathcal{P}^2\mathcal{I}} e\left(\frac{b'}{d'}2\alpha^2\right) = N(\mathcal{P})$; also, since $\mathcal{P} \not| d'$,

$$\sum_{\alpha \in \mathcal{P}^2\mathcal{I}/d'\mathcal{P}^2\mathcal{I}} e\left(\frac{b'}{d'}2\alpha^2\right) = \sum_{\alpha \in \mathcal{P}\mathcal{I}/d'\mathcal{P}\mathcal{I}} e\left(\frac{b'}{d'}2\alpha^2\right).$$

Thus $f|\widetilde{A}' = f|S(\mathcal{P}^2)$.

Now choose $\nu \in \mathcal{P}^{-1}\mathcal{I}^{-1}\partial^{-1}$ such that $(\nu\mathcal{P}\mathcal{I}\partial, d_k\mathcal{P}) = 1$ for all k. Fix some k; for simplicity write $A_k = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Set $\beta = \beta'\nu^2$ where $\beta' \in \mathcal{P}^{-1}\partial$ is chosen such that $a\beta + b \in \mathcal{P}^{-1}\mathcal{I}^{-2}\partial^{-1}$; we will show that

$$f\big|\widetilde{A}_k\big|\left[\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, 1\right] = N(\mathcal{P})^{\frac{1}{2}} \left(\sum_{\alpha\in\mathcal{P}\mathcal{I}/\mathcal{P}^2\mathcal{I}} e(2\beta\alpha^2)\right)^{-1} f\big|S(\mathcal{P}),$$

and then the lemma will follow. Now,

$$f\big|S(\mathcal{P}) = f\big|\left[A_k\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, N(\mathcal{P})^{-\frac{1}{2}} \frac{\theta\left(\mathcal{I}, A_k\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\tau\right)}{\theta(\mathcal{P}\mathcal{I},\tau)}\right];$$

again following the proof of Proposition 6.1 of [6] we find that

$$\begin{aligned} N(\mathcal{P})^{-\frac{1}{2}} &\frac{\theta\left(\mathcal{I}, A_k\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\tau\right)}{\theta(\mathcal{P}\mathcal{I},\tau)} \\ &= \big(c + (c\beta+d)\tfrac{1}{\tau}\big)^{\frac{1}{2}}\ \tau^{\frac{1}{2}}\ (c\beta+d)^{-\frac{1}{2}}\ N(\mathcal{P})^{-\frac{1}{2}} \sum_{\alpha\in\mathcal{I}/(c\beta+d)\mathcal{P}\mathcal{I}} e\left(\frac{a\beta+b}{c\beta+d}2\alpha^2\right) \end{aligned}$$

and since $a(c\beta+d) - c(a\beta+b) = 1$ and $e\big(a(a\beta+b)2\alpha^2\big) = 1$,

$$\begin{aligned} &= \big(c + (c\beta+d)\tfrac{1}{\tau}\big)^{\frac{1}{2}}\,\tau^{\frac{1}{2}}\ (c\beta+d)^{-\frac{1}{2}}\ N(\mathcal{P})^{-\frac{1}{2}} \sum_{\alpha\in\mathcal{I}/(c\beta+d)\mathcal{P}\mathcal{I}} e\left(-\frac{c(a\beta+b)^2}{c\beta+d}2\alpha^2\right) \\ &= \big(c + (c\beta+d)\tfrac{1}{\tau}\big)^{\frac{1}{2}}\,\tau^{\frac{1}{2}}\ (c\beta+d)^{-\frac{1}{2}}\ N(\mathcal{P})^{-\frac{1}{2}} \sum_{\alpha\in\mathcal{O}/(c\beta+d)\mathcal{P}} e\left(-\frac{c\nu^2}{c\beta+d}2\alpha^2\right) \end{aligned}$$

(note that $\nu\mathcal{P}\mathcal{I}\partial$ is relatively prime to $(c\beta+d)\mathcal{P}$). Now, d is relatively prime to 4 since $4|c$; thus by reciprocity of Gauss sums (Theorem 161 of [3]) we have

$$\begin{aligned} (c\beta+d)^{-\frac{1}{2}}\ N(\mathcal{P})^{-\frac{1}{2}} &\sum_{\alpha\in\mathcal{O}/(c\beta+d)\mathcal{P}} e\left(-\frac{c\nu^2}{c\beta+d}2\alpha^2\right) \\ &= i^{-\frac{n}{2}}\ N(c\nu^2\mathcal{P}\partial)^{-\frac{1}{2}} \sum_{\alpha\in\mathcal{O}/c\nu^2\mathcal{P}\partial} e\left(\frac{c\beta+d}{c\nu^2}2\alpha^2\right) \end{aligned}$$

and using the techniques of §3 of [6],

$$= i^{-\frac{n}{2}}\ N(c\nu^2\mathcal{P}\partial)^{-\frac{1}{2}} \sum_{\alpha\in\mathcal{P}/c\nu^2\mathcal{P}\partial} e\left(\frac{c\beta+d}{c\nu^2}2\alpha^2\right) \sum_{\alpha\in c\nu^2\partial/c\nu^2\mathcal{P}\partial} e\left(\frac{c\beta+d}{c\nu^2}2\alpha^2\right).$$

For $\alpha \in \mathcal{P}$, $\dfrac{c\beta+d}{c\nu^2}2\alpha^2 \equiv \dfrac{d}{c\nu^2}2\alpha^2 \pmod{2\partial^{-1}}$ (since $\beta = \nu^2\beta'$ with $\beta' \in \mathcal{P}^{-1}\partial$) so

$$\sum_{\alpha\in\mathcal{P}/c\nu^2\mathcal{P}\partial} \mathrm{e}\left(\frac{c\beta+d}{c\nu^2}2\alpha^2\right) = \sum_{\alpha\in\mathcal{P}/c\nu^2\mathcal{P}\partial} e\left(\frac{d}{c\nu^2}2\alpha^2\right)$$

$$= \sum_{\alpha\in\mathcal{O}/c\nu^2\partial} e\left(\frac{d}{c\nu^2}2\alpha^2\right)$$

(note that $\mathrm{ord}_{\mathcal{P}} c\nu^2\partial = 0$). Also, $\dfrac{c\beta+d}{c\nu^2}2\alpha^2 \equiv 2\beta\left(\dfrac{\alpha}{\nu}\right)^2 \pmod{2\partial^{-1}}$ for $\alpha \in c\nu^2\partial$, so

$$\sum_{\alpha\in c\nu^2\partial/c\nu^2\mathcal{P}\partial} e\left(\frac{c\beta+d}{c\nu^2}2\alpha^2\right) = \sum_{\alpha\in c\nu^2\partial/c\nu^2\mathcal{P}\partial} e\left(2\beta\left(\frac{\alpha}{\nu}\right)^2\right)$$

$$= \sum_{\alpha\in\mathcal{P}\mathcal{I}/\mathcal{P}^2\mathcal{I}} e(2\beta\alpha^2).$$

On the other hand, formula (1) of [6] and the techniques used above show that

$$\frac{\theta\left(\mathcal{P}\mathcal{I}, A_k\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}\tau\right)}{\theta\left(\mathcal{P}\mathcal{I}, \begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}\tau\right)}$$

$$= \left(c+(c\beta+d)\tfrac{1}{\tau}\right)^{\frac{1}{2}} \tau^{\frac{1}{2}}\ d^{-\frac{1}{2}} \sum_{\alpha\in\mathcal{P}\mathcal{I}/d\mathcal{P}\mathcal{I}} e\left(-\frac{cb^2}{d}2\alpha^2\right)$$

$$= \left(c+(c\beta+d)\tfrac{1}{\tau}\right)^{\frac{1}{2}} \tau^{\frac{1}{2}}\ d^{-\frac{1}{2}} \sum_{\alpha\in\mathcal{O}/d\mathcal{O}} e\left(-\frac{c\nu^2}{d}2\alpha^2\right)$$

and by reciprocity of Gauss sums,

$$= \left(c+(c\beta+d)\tfrac{1}{\tau}\right)^{\frac{1}{2}} \tau^{\frac{1}{2}}\ i^{-\frac{n}{2}}\ N(c\nu^2\partial)^{-\frac{1}{2}} \sum_{\alpha\in\mathcal{O}/c\nu^2\partial} e\left(\frac{d}{c\nu^2}2\alpha^2\right).$$

q.e.d.

Our goal in this section is to determine the effect of the Hecke operators on the Fourier coefficients of a half-integral weight form. When $\mathbf{K} = \mathbf{Q}$, we know that for

$$f(\tau) = \sum_{n \geq 0} a(n) e(2n\tau) \in \mathcal{M}_{\frac{m}{2}}\left(\widetilde{\Gamma}_0(N), \chi\right),$$

we have $f(\tau)\big|T(p^2) = \sum_{n \geq 0} b(n) e(2n\tau)$ where

$$b(n) = a(p^2 n) + \chi(p) p^{\frac{m-3}{2}} (-1|p)^{\frac{m-1}{2}} (n|p) a(n) + \chi(p^2) p^{m-2} a(n/p^2).$$

By defining "Fourier coefficients" attached to integral ideals, we expect to get a similar description of the effect of the Hecke operators on any half-integral weight Hilbert modular form. This, in fact, is one of the things Shimura does for integral weight forms in [5]; so mimicing Shimura, we decompose a space of half-integral weight Hilbert modular forms as described below.

Whenever $\mathcal{I}$ and $\mathcal{J}$ are fractional ideals in the same (nonstrict) ideal class, the mapping $f \to f\Big|\left[\begin{pmatrix} \alpha^{-2} & 0 \\ 0 & 1 \end{pmatrix}, N(\alpha^2)^{\frac{1}{4}}\right]$ is an isomorphism from the space $\mathcal{M}_{\frac{m}{2}}\left(\widetilde{\Gamma}_0(\mathcal{N}, \mathcal{I}^2), \chi_{\mathcal{N}}\right)$ onto $\mathcal{M}_{\frac{m}{2}}\left(\widetilde{\Gamma}_0(\mathcal{N}, \mathcal{J}^2), \chi_{\mathcal{N}}\right)$ where α is any element of $\mathbf{K}^\times$ such that $\alpha\mathcal{I} = \mathcal{J}$ (notice that this isomorphism is independent of the choice of α). Hence we can consider $T(\mathcal{P}^2)$ and $S(\mathcal{P})$ as operators on the space

$$\mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi_{\mathcal{N}}) = \prod_{\lambda=1}^{h'} \mathcal{M}_{\frac{m}{2}}\left(\widetilde{\Gamma}_0(\mathcal{N}, \mathcal{I}_\lambda^2), \chi_{\mathcal{N}}\right)$$

where $\mathcal{I}_1, \ldots, \mathcal{I}_{h'}$ represent all the distinct (nonstrict) ideal classes such that $I_1^2, \ldots, I_{h'}^2$ represent distinct strict ideal classes (see §61 of [3]). Just as in the case where m is even (see Lemma 1.1 and Proposition 1.2 of [7]), we have

$$\mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi_{\mathcal{N}}) = \underset{\chi}{\oplus} \mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi)$$

where the sum is over all Hecke characters χ extending $\chi_{\mathcal{N}}$ with $\chi_\infty = 1$,

$$\mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi) = \{F \in \mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi_{\mathcal{N}}) : F\big|S(\mathcal{J}) = \chi^*(\mathcal{J})F \ \text{ for all fractional ideals } \mathcal{J},\ (\mathcal{J}, \mathcal{N}) = 1\ \},$$

and χ^* is the ideal character induced by χ. (For $\mathcal{J}$ a fractional ideal relatively prime to $\mathcal{N}$, $\chi^*(\mathcal{J}) = \chi(\widetilde{a})$ where $\widetilde{a}$ is an idele of $\mathbf{K}$ such that $\widetilde{a}_{\mathcal{P}} = 1$ for all primes $\mathcal{P}|\mathcal{N}\infty$, and $\widetilde{a}\mathcal{O} = \mathcal{J}$. Also note that there are Hecke characters χ extending $\chi_{\mathcal{N}}$ with $\chi_\infty = 1$ since $\chi_{\mathcal{N}}(u) = 1$ for all $u \in \mathcal{U}$.)

When defining "Fourier coefficients" attached to integral ideals for an integral weight form F, Shimura uses the fact that for $u \in \mathcal{U}^+$,

$$F\Big| \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} = F.$$

In the case of half-integral weight forms, we have no analogous equation. However, we can decompose $\mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi_{\mathcal{N}})$ as follows.

Let $\mathbf{K}^+ = \{a \in \mathbf{K} : \ a \gg 0 \ \}$ and $\dot{K}^2 = \{a^2 : \ a \in \mathbf{K}, \ a \neq 0 \ \}$; set $G = \mathbf{K}^+/\dot{\mathbf{K}}^2$ and $H = \mathcal{U}^+\dot{\mathbf{K}}^2/\dot{\mathbf{K}}^2 \left(\approx \mathcal{U}^+/\mathcal{U}^2\right)$. For each character $\phi \in \widehat{G}$ =the character group of G, define

$$\mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi_{\mathcal{N}}, \phi) = \left\{ F \in \mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi_{\mathcal{N}}) : \ F\Big| \left[\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, 1 \right] = \phi(u) F \text{ for all } u \in \mathcal{U}^+ \ \right\}.$$

Then we have

Lemma 2.3. *With the above definitions,*

$$\mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi_{\mathcal{N}}) = \bigoplus_{\phi} \mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi_{\mathcal{N}}, \phi)$$

where the sum runs over a complete set of representatives ϕ for $\widehat{G}/H^\perp$ with $H^\perp = \{\phi \in \widehat{G} : \ \phi|_H = 1 \ \}$. Each space $\mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi_{\mathcal{N}}, \phi)$ is invariant under all the Hecke operators $T(\mathcal{P}^2)$ where $\mathcal{P}$ is a prime ideal not dividing $\mathcal{N}$.

Remark. The restriction map defines an isomorphism from $\widehat{G}/H^\perp$ onto $\widehat{H} \approx \widehat{\mathcal{U}^+/\mathcal{U}^2}$, but there is no canonical way to extend an element of $\widehat{\mathcal{U}^+/\mathcal{U}^2}$ to an element of $\widehat{G}/H^\perp$.

Proof: Given $F \in \mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi)$, set

$$F_\phi = \frac{1}{[\mathcal{U}^+ : \mathcal{U}^2]} \sum_{u \in \mathcal{U}^+/\mathcal{U}^2} \overline{\phi}(u) F\Big| \left[\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, 1 \right].$$

One easily verifies that $F \in \mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi_{\mathcal{N}}, \phi)$. Also,

$$\sum_{\phi \in \widehat{G}/H^\perp} F_\phi = \frac{1}{[\mathcal{U}^+ : \mathcal{U}^2]} \sum_{u \in \mathcal{U}^+/\mathcal{U}^2} \left(\sum_{\phi} \overline{\phi}(u) \right) F\Big| \left[\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, 1 \right] = F$$

since duality shows that $\sum_\phi \overline{\phi}(u)$ is only nonzero when $u=1$. Furthermore, for $\phi_1, \phi_2 \in \widehat{G}$, $\mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi_{_\mathcal{N}}, \phi_1)$ and $\mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi_{_\mathcal{N}}, \phi_2)$ either are equal or have trivial intersection, depending on whether $\phi_1\overline{\phi}_2 \in H^\perp$. Thus $\mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi_{_\mathcal{N}}) = \oplus_\phi \mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi_{_\mathcal{N}}, \phi)$ as claimed.

Now, given $u \in \mathcal{U}^+$, $\mathcal{P}$ a prime ideal not dividing $\mathcal{N}$, and $\{\widetilde{A}_j\}$ a set of coset representatives for

$$\left(\widetilde{\Gamma}_1(\mathcal{N}, \mathcal{I}^2) \cap \widetilde{\Gamma}_1(\mathcal{N}, \mathcal{P}^2\mathcal{I}^2)\right) \Big\backslash \widetilde{\Gamma}_1(\mathcal{N}, \mathcal{P}^2\mathcal{I}^2),$$

we see that $\left\{ \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} A_j \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a set of coset representatives for

$$\left(\Gamma_1(\mathcal{N}, \mathcal{I}^2) \cap \Gamma_1(\mathcal{N}, \mathcal{P}^2\mathcal{I}^2)\right) \Big\backslash \Gamma_1(\mathcal{N}, \mathcal{P}^2\mathcal{I}^2).$$

Standard techniques for evaluating Gauss sums show that

$$\frac{\theta(\mathcal{I}, A_j u\tau)}{\theta(\mathcal{I}, u\tau)} = (u|d_j)\ \frac{\theta(\mathcal{I}, A_j^u \tau)}{\theta(\mathcal{I}, \tau)}$$

where $A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$ and $A_j^u = \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} A_j \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$. Since $d_j \equiv a_j d_j \equiv v^2 \pmod{\mathcal{N}}$ for some $v \in \mathcal{U}$, the Law of Quadratic Reciprocity (Theorem 165 of [3]) shows that $(u|d_j) = 1$; hence

$$\left[\begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}, 1 \right] \widetilde{A}_j \left[\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, 1 \right] = \widetilde{A_j^u}$$

and thus $T(\mathcal{P}^2)$ acts invariantly on the space $\mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi_{_\mathcal{N}}, \phi)$. **q.e.d.**

Unfortunately, we also have

Lemma 2.4. *Given $\phi \in \widehat{G}$ and $\mathcal{P}$ a prime ideal not dividing $\mathcal{N}$, we have*

$$S(\mathcal{P}) : \mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi_{_\mathcal{N}}, \phi) \to \mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi_{_\mathcal{N}}, \phi\psi_{_\mathcal{P}})$$

where $\psi_{_\mathcal{P}}$ is an element of $\widehat{G}$ such that $\psi_{_\mathcal{P}}(u) = (u|\mathcal{P})$ for all $u \in \mathcal{U}^+$. Consequently, given any Hecke character χ extending $\chi_{_\mathcal{N}}$ (with $\chi_\infty = 1$),

$$\mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi) \cap \mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi_{_\mathcal{N}}, \phi) = \{0\}$$

unless $\mathcal{U}^+ = \mathcal{U}^2$.

Proof: Let $C = \begin{pmatrix} * & * \\ * & d \end{pmatrix}$ be a matrix as in the definition of $S(\mathcal{P})$; so $\det C = 1$, and

$$F\big|S(\mathcal{P}) = f\big| \left[C, N(\mathcal{P})^{-\frac{1}{2}} \frac{\theta(\mathcal{I}, C\tau)}{\theta(\mathcal{P}\mathcal{I}, \tau)} \right]$$

for $f \in \mathcal{M}_{\frac{m}{2}}(\widetilde{\Gamma}_0(\mathcal{N},\mathcal{I}^2),\chi_{_\mathcal{N}})$. Then for $u \in \mathcal{U}^+$, the techniques used to prove Proposition 6.1 of [6] show that

$$\left[\begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}, 1\right] \left[C, N(\mathcal{P})^{-\frac{1}{2}} \frac{\theta(\mathcal{I}, C\tau)}{\theta(\mathcal{P}\mathcal{I}, \tau)}\right] \left[\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, 1\right]$$
$$= \left[C^u, (u|d)(u|\mathcal{P})N(\mathcal{P})^{-\frac{1}{2}} \frac{\theta(\mathcal{I}, C^u\tau)}{\theta(\mathcal{P}\mathcal{I}, \tau)}\right]$$

where $C^u = \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} C \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$. Since $d \equiv 1 \pmod{\mathcal{N}}$ (recall the definition of $S(\mathcal{P})$) we see again by the Law of Quadratic Reciprocity that $(u|d) = 1$. Hence for $F \in \mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi_{_\mathcal{N}}, \phi)$,

$$F\big|S(\mathcal{P})\big|\left[\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, 1\right] = (u|\mathcal{P})\ F\big|\left[\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, 1\right]\big|S(\mathcal{P}) = \phi(u)(u|\mathcal{P})\ F\big|S(\mathcal{P}),$$

showing that $F\big|S(\mathcal{P}) \in \mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi_{_\mathcal{N}}, \phi\psi_{_\mathcal{P}})$.

Now, to finish proving the lemma, we simply observe that there are an infinite number of primes $\mathcal{P}$ such that $(u|\mathcal{P}) = -1$ if $u \in \mathcal{U}^+ - \mathcal{U}^2$ (see 65:19 of [4]). **q.e.d.**

The preceding two lemmas compel us to define "Fourier coefficients" attached to integral ideals as follows.

Given $F = (\ldots, f_\lambda, \ldots) \in \mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi_{_\mathcal{N}})$ where $f_\lambda(\tau) = \sum_\zeta a_\lambda(\zeta)e(2\zeta\tau)$, $\phi \in \widehat{G}$ and $\mathcal{M} \neq 0$ an integral ideal, we define the $\mathcal{M}, \phi$-Fourier coefficient of F by:

(i) $\mathbf{a}(\mathcal{M}, \phi) = \frac{1}{[\mathcal{U}^+:\mathcal{U}^2]} \sum_{u \in \mathcal{U}^+/\mathcal{U}^2} \overline{\phi}(\xi u)a_\lambda(\xi u)N(\mathcal{I}_\lambda)^{-\frac{m}{2}}$ if $\mathcal{M} = \xi\mathcal{I}_\lambda^{-2}$ for some λ and some $\xi \gg 0$;

(ii) $\mathbf{a}(\mathcal{M}, \phi) = 0$ if $\mathcal{M}$ cannot be written as $\xi\mathcal{I}_\lambda^{-2}$ (with $\xi \gg 0$);

(iii) $\mathbf{a}(0, \phi) = a_\lambda(0)N(\mathcal{I}_\lambda)^{-\frac{m}{2}}$ if $a_\lambda(0)N(\mathcal{I}_\lambda)^{-\frac{m}{2}} = a_\mu(0)N(\mathcal{I}_\mu)^{-\frac{m}{2}}$ for all λ, μ.

Thus for $\mathcal{M} = \xi\mathcal{I}_\lambda^{-2}$, $\xi \gg 0$, $\mathbf{a}(\mathcal{M}, \phi)$ is $N(\mathcal{I}_\lambda)^{-\frac{m}{2}}$ times the ξ-Fourier coefficient of the λ-component of F_ϕ. Since $F = \sum_\phi F_\phi$, the collection of all the $\mathcal{M}, \phi$-Fourier coefficients ($\phi \in \widehat{G}/H^\perp$) characterize any form F whose $0, \phi$-Fourier coefficients can be defined.

We now describe the effect of the Hecke operators on these Fourier coefficiets.

Theorem 2.5. *Let $F = (\ldots, f_\lambda, \ldots) \in \mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi)$ where χ is a Hecke character extending $\chi_{_\mathcal{N}}$ with $\chi_\infty = 1$. Take $\mathcal{P}$ to be a prime ideal not dividing $\mathcal{N}$, and take $\psi_{_\mathcal{P}} \in (\widehat{\mathbf{K}^+/\dot{\mathbf{K}}^2})$ such that $\psi_{_\mathcal{P}}(\xi) = (\xi|\mathcal{P})$ for all $\xi \in \mathbf{K}^+$ with $\mathrm{ord}_\mathcal{P}\xi = 0$. Let $\mathbf{a}(\mathcal{M}, *)$ and $\mathbf{b}(\mathcal{M}, *)$ denote the*

$\mathcal{M}, *$-*Fourier coefficients of* F *and of* $F\big|T(\mathcal{P}^2)$ *(respectively). Then for any* $\phi \in (\widehat{\mathbf{K}^+/\dot{\mathbf{K}}^2})$, *we have*

$$\mathbf{b}(\mathcal{M},\phi) = \begin{cases} \mathbf{a}(\mathcal{P}^2\mathcal{M},\phi) + \chi^*(\mathcal{P})N(\mathcal{P})^{\frac{m-3}{2}}(-1|\mathcal{P})^{\frac{m-1}{2}}\mathbf{a}(\mathcal{M},\phi\psi_{\mathcal{P}}) & \\ \quad +\chi^*(\mathcal{P}^2)N(\mathcal{P})^{m-2}\mathbf{a}(\mathcal{M}\mathcal{P}^{-2},\phi) & \text{if } \mathcal{P}\not|\,\mathcal{M}, \\ \mathbf{a}(\mathcal{P}^2\mathcal{M},\phi) + \chi^*(\mathcal{P}^2)N(\mathcal{P})^{m-2}\mathbf{a}(\mathcal{M}\mathcal{P}^{-2},\phi) & \text{if } \mathcal{P}|\mathcal{M}. \end{cases}$$

Proof: Take $\rho,\gamma \in \mathbf{K}^\times$ such that $\mathcal{I}_\lambda^2\mathcal{P}^2 = \rho^2\mathcal{I}_\mu^2$ and $\mathcal{I}_\lambda^2\mathcal{P}^4 = \gamma^2\mathcal{I}_\eta^2$. Then by Lemma 2.2 the μ-component of $F\big|T(\mathcal{P}^2)$ is

$$N(\mathcal{P})^{\frac{m}{2}-2}\Bigg(f_\lambda|\sum_b\left[\begin{pmatrix}1&b\\0&1\end{pmatrix},1\right]$$
$$+\chi^*(\mathcal{P})f_\mu|\left[\begin{pmatrix}\rho^2&0\\0&1\end{pmatrix},N(\rho^2)^{-\frac{1}{4}}\right]|\sum_\beta\left[\begin{pmatrix}1&\beta\\0&1\end{pmatrix},\frac{N(\mathcal{P})^{\frac{1}{2}}}{\sum_\alpha e(-2\beta\alpha^2)}\right]$$
$$+\chi^*(\mathcal{P}^2)f_\eta|\left[\begin{pmatrix}\gamma^2&0\\0&1\end{pmatrix},N(\gamma^2)^{-\frac{1}{4}}\right]\Bigg)|\left[\begin{pmatrix}\rho^{-2}&0\\0&1\end{pmatrix},N(\rho^2)^{\frac{1}{4}}\right]$$

where b runs over $\mathcal{P}^{-2}\mathcal{I}_\lambda^{-2}\partial^{-1}/\mathcal{I}_\lambda^{-2}\partial^{-1}$, β runs over $(\mathcal{P}^{-3}\mathcal{I}_\lambda^{-2}\partial^{-1}/\mathcal{P}^{-2}\mathcal{I}_\lambda^{-2}\partial^{-1})^\times$, and α runs over $\mathcal{I}_\lambda\mathcal{P}/\mathcal{I}_\lambda\mathcal{P}^2$. (Recall that $F \in \mathcal{M}_{\frac{m}{2}}(\mathcal{N},\chi)$ so $f_\lambda\big|S(\mathcal{I})|\left[\begin{pmatrix}\omega^2&0\\0&1\end{pmatrix},N(\omega^2)^{-\frac{1}{4}}\right] = \chi^*(\mathcal{I})f_\sigma$ where $\omega\mathcal{I}^2\mathcal{I}_\lambda^2 = \mathcal{I}_\sigma^2$.) It is easily seen that

$$\begin{aligned} &f_\lambda|\sum_b\left[\begin{pmatrix}1&b\\0&1\end{pmatrix},1\right]|\left[\begin{pmatrix}\rho^{-2}&0\\0&1\end{pmatrix},N(\rho^2)^{-\frac{1}{4}}\right](\tau)\\ &\quad= N(\mathcal{I}_\lambda\mathcal{P}\mathcal{I}_\mu^{-1})^{-\frac{m}{2}}N(\mathcal{P}^2)\sum_{\xi\in\mathcal{P}^2\mathcal{I}_\lambda^2}a_\lambda(\xi)e(2\xi\rho^{-2}\tau)\\ &\quad= N(\mathcal{I}_\lambda\mathcal{P}\mathcal{I}_\mu^{-1})^{-\frac{m}{2}}N(\mathcal{P}^2)\sum_{\xi\in\mathcal{I}_\mu^2}a_\lambda(\rho^2\xi)e(2\xi\tau), \end{aligned}$$

and that

$$\begin{aligned} &f_\eta|\left[\begin{pmatrix}\gamma^2&0\\0&1\end{pmatrix},N(\gamma^2)^{-\frac{1}{4}}\right]|\left[\begin{pmatrix}\rho^{-2}&0\\0&1\end{pmatrix},N(\rho^2)^{\frac{1}{4}}\right](\tau)\\ &\quad= N(\mathcal{P}\mathcal{I}_\mu\mathcal{I}_\eta^{-1})^{\frac{m}{2}}\sum_{\xi\in\mathcal{P}^2\mathcal{I}_\eta^2}a_\eta(\xi\rho^2\gamma^{-2})e(2\xi\tau). \end{aligned}$$

Now we work a little:

$$f_\mu | \left[\begin{pmatrix} \rho^2 & 0 \\ 0 & 1 \end{pmatrix}, N(\rho^2)^{-\frac{1}{4}} \right] | \sum_\beta \left[\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \frac{N(\mathcal{P})^{\frac{1}{2}}}{\sum_\alpha e(-2\beta\alpha^2)} \right] | \left[\begin{pmatrix} \rho^{-2} & 0 \\ 0 & 1 \end{pmatrix}, N(\rho^2)^{\frac{1}{4}} \right]$$
$$= N(\mathcal{P})^{-\frac{m}{2}} \sum_\beta \left(\sum_\alpha e(-2\beta\alpha^2) \right)^m \sum_{\xi \in \mathcal{I}_\mu^2} a_\mu(\xi) e(2\xi\beta\rho^2) e(2\xi\tau).$$

Taking $\beta_0 \in \mathcal{P}^{-3}\mathcal{I}_\lambda^{-2}\partial^{-1} \ - \ \mathcal{P}^{-2}\mathcal{I}_\lambda^{-2}\partial^{-1}$, standard techniques for evaluating Gauss sums show us that

$$\sum_\beta \left(\sum_\alpha e(-2\beta\alpha^2) \right)^m e(2\xi\beta\rho^2) = \sum_{\beta' \in \mathcal{O}/\mathcal{P}} (-\beta'|\mathcal{P})^m \left(\sum_\alpha e(2\beta_0\alpha^2) \right)^m e(2\xi\beta_0\beta'\rho^2)$$

and $\left(\sum_\alpha e(2\beta_0\alpha^2)\right)^2 = N(\mathcal{P})(-1|\mathcal{P})$. So

$$\sum_\beta \left(\sum_\alpha e(-2\beta\alpha^2) \right)^m e(2\xi\beta\rho^2)$$
$$= N(\mathcal{P})^{\frac{m-1}{2}} (-1|\mathcal{P})^{\frac{m+1}{2}} \left(\sum_{\beta' \in \mathcal{O}/\mathcal{P}} (\beta'|\mathcal{P}) e(2\beta'\beta_0\xi\rho^2) \right) \left(\sum_\alpha e(2\beta_0\alpha^2) \right)$$

which is equal to 0 when $\xi \in \mathcal{P}\mathcal{I}_\mu^2$. When $\xi \neq \mathcal{P}\mathcal{I}_\mu^2$ and $\nu \in \mathcal{I}_\mu^{-1} \ - \ \mathcal{P}\mathcal{I}_\mu^{-1}$, $\beta'\xi\nu^2$ runs over $\mathcal{O}/\mathcal{P}$ as β' does; in this case

$$\sum_{\beta' \in \mathcal{O}/\mathcal{P}} (\beta'|\mathcal{P}) e(2\beta'\beta_0\xi\rho^2) = \sum_{\beta'} (\beta'\xi\nu^2|\mathcal{P}) e(2\beta'\beta_0\xi^2\nu^2\rho^2) = (\xi\nu^2|\mathcal{P}) \sum_{\alpha \in \mathcal{P}\mathcal{I}_\lambda/\mathcal{P}^2\mathcal{I}_\lambda} e(2\beta_0\alpha^2).$$

Thus

$$f_\mu | \sum_\beta \left[\begin{pmatrix} 1 & \rho^2\beta \\ 0 & 1 \end{pmatrix}, N(\mathcal{P})^{\frac{1}{2}} \left(\sum_\alpha e(-2\beta\alpha^2) \right)^{-1} \right] (\tau)$$
$$= N(\mathcal{P})^{\frac{1}{2}} (-1|\mathcal{P})^{\frac{m-1}{2}} \sum_{\xi \in \mathcal{I}_\mu^2} (\xi\nu^2|\mathcal{P}) a_\mu(\xi) e(2\xi\tau).$$

This means that for $\mathcal{M} = \xi\mathcal{I}_\mu^{-2}, \ \xi \gg 0$,

$$\mathbf{b}(\mathcal{M},\phi) = \frac{N(\mathcal{I}_\mu)^{-\frac{m}{2}}}{[\mathcal{U}^+ : \mathcal{U}^2]} N(\mathcal{P})^{\frac{m}{2}-2}$$

$$\cdot \Big(N(\mathcal{P})^{2-\frac{m}{2}}\ N(\mathcal{I}_\mu)^{\frac{m}{2}}\ N(\mathcal{I}_\lambda)^{-\frac{m}{2}} \sum_{u\in\mathcal{U}^+/\mathcal{U}^2} \overline{\phi}(\xi u) a_\lambda(u\xi\rho^2)$$

$$+ \chi^*(\mathcal{P}) N(\mathcal{P})^{\frac{1}{2}} (-1|\mathcal{P})^{\frac{m-1}{2}} \sum_{u\in\mathcal{U}^+/\mathcal{U}^2} \overline{\phi}(\xi u)(u\xi\nu^2|\mathcal{P}) a_\mu(u\xi)$$

$$+ \chi^*(\mathcal{P}^2) N(\mathcal{P})^{\frac{m}{2}} N(\mathcal{I}_\mu)^{\frac{m}{2}} N(\mathcal{I}_\eta)^{-\frac{m}{2}} \sum_{u\in\mathcal{U}^+/\mathcal{U}^2} \overline{\phi}(\xi u) a_\eta(u\xi\rho^2\gamma^{-2}) \Big).$$

Noting that $(u\xi\nu^2|\mathcal{P}) = 0$ when $\mathcal{P}|\mathcal{M}$, the theorem now follows from the definition of the $\mathcal{M},\phi$-Fourier coefficients of F. **q.e.d.**

Corollary 2.6. *If $F \in \mathcal{M}_{\frac{m}{2}}(\mathcal{N},\chi)$ is an eigenform for all $T(\mathcal{P}^2)$ $(\mathcal{P} \nmid \mathcal{N})$ whose $0,*$-Fourier coefficients can be defined and are nonzero, then*

$$F\big|T(\mathcal{P}^2) = (1 + \chi^*(\mathcal{P}^2) N(\mathcal{P})^{m-2})F.$$

§**3. Relations on representation numbers of odd rank lattices.** Let L be a lattice of rank m over $\mathcal{O}$ where m is odd; since lattices of rank 1 are already well understood, we restrict our attention here to the case where $m \geq 3$. Then, as shown in Theorem 3.7 of [6], $\theta(L,\tau) = \sum_{x\in L} e(Q(x)\tau)$ is a Hilbert modular form of weight $\frac{m}{2}$, level $\mathcal{N}$ and character $\chi_{_L}$ for the group $\{\widetilde{A} \in \widetilde{\Gamma}_0(\mathcal{N},\mathcal{I}^2) :\ \det A = 1\ \}$ where $\mathcal{I}$ is the smallest fractional ideal such that $\mathbf{n}L \subseteq \mathcal{I}^2$ (so for every prime $\mathcal{P}$, $\text{ord}_{\mathcal{P}}\mathbf{n}L \cdot \mathcal{I}^{-2}$ is minimal), $\mathcal{N} = (\mathbf{n}L^{\#})^{-1}\mathcal{I}^{-2}$, and $\chi_{_L}$ is a quadratic character modulo $\mathcal{N}$. (Here $L^{\#}$ denotes the dual lattice of L, and $\mathbf{n}L$ is the fractional ideal generated by $\{\frac{1}{2}Q(x) :\ x \in L\ \}$; note that Proposition 3.4 of [6] shows that $4\mathcal{O}|\mathcal{N}$.) Since $\theta(L,u^2\tau) = \theta(L,\tau)$ for any $u \in \mathcal{U}$, we have $\theta(L,\tau) \in \mathcal{M}_{\frac{m}{2}}(\widetilde{\Gamma}_0(\mathcal{N},\mathcal{I}^2),\chi_{_L})$.

Lemma 3.1. *Let $\mathcal{P}$ be a prime ideal not dividing $\mathcal{N}$. Then setting $L_{\mathcal{P}} = \mathcal{O}_{\mathcal{P}}L$, we have*

$$L_{\mathcal{P}} \simeq \pi^2\big\langle 1,\ldots,1,\varepsilon_{_\mathcal{P}}\big\rangle$$

for some $\pi \in \mathbf{K}_{\mathcal{P}}$ and $\varepsilon_{_\mathcal{P}} \in \mathcal{O}_{\mathcal{P}}^{\times}$.

Proof: Since $4\mathcal{O}|\mathcal{N}$, $\mathcal{P}$ must be nondyadic. Then from the remarks immediately preceding 92:1 of [4], we see that $L_{\mathcal{P}} \simeq \big\langle \alpha_1,\ldots,\alpha_m\big\rangle$ where $\alpha_1,\ldots,\alpha_m \in \mathbf{K}_{\mathcal{P}}$. Since $\mathcal{P} \nmid \mathcal{N}$ and $(\mathbf{n}L^{\#})^{-1}(\mathbf{n}L)^{-1}|\mathcal{N}$, we know that $\mathcal{P} \nmid (\mathbf{n}L^{\#})^{-1}(\mathbf{n}L)^{-1}$ and hence $L_{\mathcal{P}}$ is modular; thus by

92:1 of [4], $L_{\mathcal{P}} \simeq \rho\langle 1,\ldots,1,\varepsilon_{\mathcal{P}}\rangle$ for some $\varepsilon_{\mathcal{P}} \in \mathcal{O}_{\mathcal{P}}^{\times}$ and $\rho \in \mathbf{K}_{\mathcal{P}}$ such that $\rho\mathcal{O}_{\mathcal{P}} = \mathbf{n}L_{\mathcal{P}}$. Furthermore, since $\mathcal{N} = (\mathbf{n}L^{\#})^{-1}\mathcal{I}^{-2}$ and $\mathcal{P}\nmid\mathcal{N}$, the fractional ideal $\mathbf{n}L^{\#}$ and hence $\mathbf{n}L$ must have even order at $\mathcal{P}$, so we may choose $\rho = \pi^2$ with $\pi \in \mathbf{K}_{\mathcal{P}}$. **q.e.d.**

Notice that in the preceding lemma the square class of $\varepsilon_{\mathcal{P}}$ is independent of the choice of π; thus we can make the following

Definition. With $\mathcal{P}$ a prime, $\mathcal{P}\nmid\mathcal{N}$, let $\varepsilon_{\mathcal{P}} \in \mathcal{O}_{\mathcal{P}}^{\times}$ be as in Lemma 3.1; set $\varepsilon_L(\mathcal{P}) = (2\varepsilon_{\mathcal{P}}|\mathcal{P})$ where $(*|*)$ is the quadratic residue symbol. For an integral ideal $\mathcal{A}$ relatively prime to $\mathcal{N}$, set

$$\varepsilon_L(\mathcal{A}) = \prod_{\mathcal{P}|\mathcal{A}} \varepsilon_L(\mathcal{P})^{\mathrm{ord}_{\mathcal{P}}(\mathcal{A})}.$$

A straightforward computation analogous to that used to prove Lemma 3.8 of [8] proves

Lemma 3.2. *For $a \in \mathbf{K}^{\times}$ with a relatively prime to $\mathcal{N}$, $\chi_L(a) = \varepsilon_L(a\mathcal{O})$.*

Next we have

Proposition 3.3. *Let $\mathcal{P}$ be a prime, $\mathcal{P}\nmid\mathcal{N}$. Then*

$$\theta(L,\tau)\big|S(\mathcal{P}) = N(\mathcal{P})^{\frac{m}{2}}\varepsilon_L(\mathcal{P})\theta(\mathcal{P}L,\tau) \text{ and so } \theta(L,\tau)\big|S(\mathcal{P}^2) = N(\mathcal{P})^m\theta(\mathcal{P}^2L,\tau).$$

Proof: Following the proof of Proposition 6.1 of [6] and using the extended transformation formula from §4 of [7], we find that for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathcal{P} & \mathcal{P}^{-1}\mathcal{I}^{-2}\partial^{-1} \\ \mathcal{N}\mathcal{P}\mathcal{I}^2\partial & \mathcal{O} \end{pmatrix}$ with $\det A = 1$ and $d \equiv 1 \pmod{\mathcal{N}}$,

$$\theta(L,A\tau) = \left(c + d\tfrac{1}{\tau}\right)^{\frac{m}{2}}\tau^{\frac{m}{2}}d^{-\frac{m}{2}} \sum_{x\in\mathcal{P}L/d\mathcal{P}L} e\left(\frac{b}{d}Q(x)\right) \sum_{x\in dL/d\mathcal{P}L} e\left(\frac{b}{d}Q(x)\right)\cdot\theta(\mathcal{P}L,\tau),$$

and

$$\theta(\mathcal{I},A\tau) = \left(c + d\tfrac{1}{\tau}\right)^{\frac{1}{2}}\tau^{\frac{1}{2}}d^{-\frac{1}{2}} \sum_{\alpha\in\mathcal{P}\mathcal{I}/d\mathcal{P}\mathcal{I}} e\left(\frac{b}{d}2\alpha^2\right) \sum_{\alpha\in d\mathcal{I}/d\mathcal{P}\mathcal{I}} e\left(\frac{b}{d}2\alpha^2\right)\cdot\theta(\mathcal{P}\mathcal{I},\tau).$$

Thus

$$\theta(L,\tau)\big|S(\mathcal{P}) = N(\mathcal{P})^{\frac{m}{2}} \sum_{x\in\mathcal{P}L/d\mathcal{P}L} e\left(\frac{b}{d}Q(x)\right)\left(\sum_{\alpha\in\mathcal{P}\mathcal{I}/d\mathcal{P}\mathcal{I}} e\left(\frac{b}{d}2\alpha^2\right)\right)^{-m}$$

$$\cdot \sum_{x\in dL/d\mathcal{P}L} e\left(\frac{b}{d}Q(x)\right)\left(\sum_{\alpha\in d\mathcal{I}/d\mathcal{P}\mathcal{I}} e\left(\frac{b}{d}2\alpha^2\right)\right)^{-m}\theta(\mathcal{P}L,\tau).$$

We know from Lemma 3.1 that $L_\mathcal{P} \simeq \pi^2\langle 1, \ldots, 1, \epsilon_\mathcal{P}\rangle$ where $\epsilon_\mathcal{P} \in \mathcal{O}_\mathcal{P}^\times$; thus Propositions 3.1-3.3 and the arguments used to prove Theorem 3.7 of [6] show that

$$\sum_{x \in dL/d\mathcal{P}L} e\left(\frac{b}{d}Q(x)\right)\left(\sum_{\alpha \in d\mathcal{I}/d\mathcal{P}\mathcal{I}} e\left(\frac{b}{d}2\alpha^2\right)\right)^{-m} = (2\epsilon_\mathcal{P}|\mathcal{P}) = \varepsilon_L(\mathcal{P})$$

and that

$$\sum_{x \in \mathcal{P}L/d\mathcal{P}L} e\left(\frac{b}{d}Q(x)\right)\left(\sum_{\alpha \in \mathcal{P}\mathcal{I}/d\mathcal{P}\mathcal{I}} e\left(\frac{b}{d}2\alpha^2\right)\right)^{-m} = \chi_L(d) = 1$$

(since $d \equiv 1 \pmod{\mathcal{N}}$ and χ_L is a character modulo $\mathcal{N}$). **q.e.d.**

With this we prove

Proposition 3.4. *Let the notation be as above. Then*

$$\theta(L,\tau)\big|T(\mathcal{P}^2) = \varepsilon_L(\mathcal{P})N(\mathcal{P})^{\frac{m}{2}}\kappa^{-1}\sum_K \theta(K,\tau) + \varepsilon_L(\mathcal{P})N(\mathcal{P})^{\frac{m}{2}}(1 - N(\mathcal{P})^{\frac{m-3}{2}})\theta(\mathcal{P}L,\tau)$$

where

$$\kappa = \begin{cases} 1 & \text{if } m = 3 \\ N(\mathcal{P})^{\frac{m-5}{2}} \cdots N(\mathcal{P})^0 (N(\mathcal{P})^{\frac{m-3}{2}} + 1) \cdots (N(\mathcal{P}) + 1) & \text{if } m > 3 \end{cases}$$

and the sum runs over all $\mathcal{P}^2$-sublattices K of L (i.e. over all sublattices K of L such that $\mathbf{n}K = \mathcal{P}^2 \cdot \mathbf{n}L$ and the invariant factors $\{L : K\} = (\mathcal{O}, \ldots, \mathcal{O}, \mathcal{P}, \mathcal{P}^2, \ldots, \mathcal{P}^2)$ with $\mathcal{O}$ and $\mathcal{P}^2$ each appearing $\frac{m-1}{2}$ times). Furthermore, each $\mathcal{P}^2$-sublattice K of L lies in the genus of $\mathcal{P}L$, and hence $\theta(\mathcal{P}L,\tau), \theta(K,\tau) \in \mathcal{M}_{\frac{m}{2}}\left(\widetilde{\Gamma}_0(\mathcal{N}, \mathcal{P}^2\mathcal{I}^2), \chi_L\right)$.

Proof: An easy check shows that the Hecke operator $T(\mathcal{P}^2)$ defined in [6] is, in the notation of this paper, $T(\mathcal{P}^2)S(\mathcal{P}^{-2})$. Thus Theorem 7.4 of [6] together with the preceding proposition shows that $\theta(L,\tau)\big|T(\mathcal{P}^2)$ is as claimed. (**N.B.**: Part 2 of Theorem 7.4 has the wrong constants; for $m = 2k+1$ with m odd the theorem should read

$$\theta(L,\tau)\big|T(\mathcal{P}^2) = N(\mathcal{P})^{-\frac{m}{2}}\kappa^{-1}\sum_K \theta(\mathcal{P}^{-2}K,\tau) + N(\mathcal{P})^{-\frac{m}{2}}(1 - N(\mathcal{P})^{\frac{m-3}{2}})\theta(\mathcal{P}^{-1}L,\tau)$$

where the sum runs over all $\mathcal{P}^2$-sublattices K of L and κ is as above.)

Now let K be a $\mathcal{P}^2$-sublattice of L. Since $\mathbf{n}K = \mathbf{n}\mathcal{P}L$, disc$K$ = disc$\mathcal{P}L$ and $\mathcal{P}L_\mathcal{P}$ is modular, it follows that $\mathbf{K}_\mathcal{P}$ is modular as well, and that $\mathbf{K}_\mathcal{P} \simeq \mathcal{P}L_\mathcal{P}$. Clearly we have $K_{\mathcal{Q}} = L_{\mathcal{Q}} = \mathcal{P}L_{\mathcal{Q}}$ where $\mathcal{Q}$ is any prime other than $\mathcal{P}$; thus $K \in \mathrm{gen}\mathcal{P}L$, the genus of $\mathcal{P}L$. Finally, Theorem 7.4 of [6] shows that $\theta(\mathcal{P}^{-2}K,\tau)$ and $\theta(\mathcal{P}^{-1}L,\tau)$ lie in $\mathcal{M}_{\frac{m}{2}}\left(\widetilde{\Gamma}_0(\mathcal{N},\mathcal{P}^{-2}\mathcal{I}^2),\chi_{_L}\right)$, so $\theta(K,\tau) = N(\mathcal{P})^{-m}\theta(\mathcal{P}^{-2}K,\tau)\big|S(\mathcal{P}^2)$ and $\theta(\mathcal{P}L,\tau) = N(\mathcal{P})^{-m}\theta(\mathcal{P}^{-1}L,\tau)\big|S(\mathcal{P}^2)$ lie in $\mathcal{M}_{\frac{m}{2}}\left(\widetilde{\Gamma}_0(\mathcal{N},\mathcal{P}^2\mathcal{I}^2),\chi_{_L}\right)$ as claimed. **q.e.d.**

Completely analogous to Lemma 3.2 of [8], we have

Lemma 3.5. *Let $o(L')$ denote the order of $O(L')$, the orthogonal group of the lattice L', and define*

$$\theta(\mathrm{gen}L,\tau) = \sum_{L'} \frac{1}{o(L')}\theta(L',\tau)$$

where the sum runs over a complete set of representatives L' for the distinct isometry classes in $\mathrm{gen}L$, the genus of L. Then for a prime $\mathcal{P}\nmid\mathcal{N}$,

$$\theta(\mathrm{gen}L,\tau)\big|T(\mathcal{P}^2) = N(\mathcal{P})^{\frac{m}{2}}\varepsilon_{_L}(\mathcal{P})(1+N(\mathcal{P})^{m-2})\theta(\mathrm{gen}\mathcal{P}L,\tau).$$

As in §2, choose fractional ideals $\mathcal{I}_1,\ldots,I_{h'}$ representing the distinct (nonstrict) ideal classes such that $\mathcal{I}_1^2,\ldots,\mathcal{I}_{h'}^2$ are in distinct strict ideal classes; for convenience, we assume that $\mathcal{I}_1 = \mathcal{O}$ and that each $\mathcal{I}_\lambda$ is relatively prime to $\mathcal{N}$. Define the extended genus of L, $\mathrm{xgen}L$, to be the union of all genera $\mathrm{gen}\mathcal{I}L$ where $\mathcal{I}$ is a fractional ideal; set

$$\Theta(\mathrm{xgen}L,\tau) = (\ldots,N(\mathcal{I}_\lambda\mathcal{I})^{\frac{m}{2}}\theta(\mathrm{gen}\mathcal{I}_\lambda L,\tau),\ldots).$$

Then we have

Theorem 3.6. *Let χ be the Hecke character extending $\chi_{_L}$ such that $\chi_\infty = 1$ and $\chi^*(\mathcal{A}) = \varepsilon_{_L}(\mathcal{A})$ for any fractional ideal $\mathcal{A}$ which is relatively prime to $\mathcal{N}$. Then*

$$\Theta(\mathrm{xgen}L,\tau) \in \mathcal{M}_{\frac{m}{2}}(\mathcal{N},\chi) \subseteq \prod_\lambda \mathcal{M}_{\frac{m}{2}}\left(\widetilde{\Gamma}_0(\mathcal{N},\mathcal{I}_\lambda^2\mathcal{I}^2),\chi_{_L}\right)$$

and for every prime $\mathcal{P}\nmid\mathcal{N}$,

$$\Theta(\mathrm{xgen}L,\tau)\big|T(\mathcal{P}^2) = \varepsilon_{_L}(\mathcal{P})(1+N(\mathcal{P})^{m-2})\Theta(\mathrm{xgen}L,\tau).$$

Proof: Take $\mathcal{J}$ to be a fractional ideal relatively prime to $\mathcal{N}$. Then for each λ we have $\mathcal{J}\mathcal{I}_\lambda = \alpha\mathcal{I}_\mu$ for some μ and some $\alpha \in \mathbf{K}^\times$. By Proposition 3.1 we have

$$\begin{aligned} N(\mathcal{I}_\lambda)^{\frac{m}{2}}\theta(\mathrm{gen}\mathcal{I}_\lambda L,\tau)\big|S(\mathcal{J})\big| &\left[\begin{pmatrix}\alpha^{-2} & 0\\ 0 & 1\end{pmatrix}, N(\alpha^2)^{\frac{1}{4}}\right] \\ &= \varepsilon_{_L}(\mathcal{J})N(\alpha^{-1}\mathcal{J}\mathcal{I}_\lambda)^{\frac{m}{2}}\theta(\mathrm{gen}(\alpha^{-1}\mathcal{J}\mathcal{I}_\lambda L),\tau) \\ &= \varepsilon_{_L}(\mathcal{J})N(\mathcal{I}_\mu)^{\frac{m}{2}}\theta(\mathrm{gen}\mathcal{I}_\mu L,\tau); \end{aligned}$$

since we have chosen χ such that $\chi^*(\mathcal{J}) = \varepsilon_L(\mathcal{J})$, we have $\Theta(\text{xgen}L, \tau) \in \mathcal{M}_{\frac{m}{2}}(\mathcal{N}, \chi)$.

Now take $\mathcal{P}$ to be a prime, $\mathcal{P} \nmid \mathcal{N}$, and take $\alpha \in \mathbf{K}^\times$ such that $\mathcal{P}\mathcal{I}_\lambda = \alpha\mathcal{I}_\mu$. Then by Lemma 3.5,

$$\begin{aligned} &N(\mathcal{I}_\lambda)^{\frac{m}{2}} \theta(\text{gen}\mathcal{I}_\lambda L, \tau) \big| T(\mathcal{P}^2) \big| \left[\begin{pmatrix} \alpha^{-2} & 0 \\ 0 & 1 \end{pmatrix}, N(\alpha^2)^{\frac{1}{4}} \right] \\ &\quad = \varepsilon_L(\mathcal{P})(1 + N(\mathcal{P})^{m-2}) N(\alpha^{-1}\mathcal{I}_\lambda \mathcal{P})^{\frac{m}{2}} \theta(\text{gen}(\alpha^{-1}\mathcal{P}\mathcal{I}_\lambda L), \tau) \\ &\quad = \varepsilon_L(\mathcal{P})(1 + N(\mathcal{P})^{m-2}) N(\mathcal{I}_\mu)^{\frac{m}{2}} \theta(\text{gen}\mathcal{I}_\mu L, \tau). \end{aligned}$$

q.e.d.

This theorem allows us to infer relations on averaged representation numbers which we define as follows.

Set

$$\mathbf{r}(L', \xi) = \#\{x \in L' : \ Q(x) = \xi\ \}, \text{ and } \mathbf{r}(\text{gen}L, \xi) = \sum_{L'} \frac{1}{o(L')} \mathbf{r}(L', \xi)$$

where the sum runs over a complete set of representatives L' for the isometry classes within gen$L$. For $\phi \in (\widehat{\mathbf{K}^+/\dot{\mathbf{K}}^2})$, set

$$\mathbf{r}(\text{gen}L, \xi, \phi) = \frac{1}{[\mathcal{U}^+ : \mathcal{U}^2]} \sum_{u \in \mathcal{U}^+/\mathcal{U}^2} \overline{\phi}(u\xi)\ \mathbf{r}(\text{gen}L, u\xi).$$

Then with the notation of §2, the $\mathcal{M}, \phi$-Fourier coefficient of $\Theta(\text{xgen}L, \tau)$ is $\mathbf{r}(\text{gen}\mathcal{I}_\lambda L, 2\xi, \phi)$ where $\mathcal{M} = \xi\mathcal{I}_\lambda^{-2}$, $\xi \gg 0$. Note that for any fractional ideal $\mathcal{J}$, we can find some $\alpha \in \mathbf{K}$ and some λ such that $\mathcal{J} = \alpha\mathcal{I}_\lambda$; then for $\xi \in \mathbf{n}L$, $\xi \gg 0$, and $\mathcal{M} = \xi\mathcal{I}_\lambda^{-2}\mathcal{I}^{-2}$, the $\mathcal{J}, \phi$-Fourier coefficient of $\Theta(\text{xgen}L, \tau)$ is

$$\mathbf{r}(\text{gen}\mathcal{I}_\lambda L, 2\alpha^{-2}\xi, \phi) = \mathbf{r}(\text{gen}\alpha\mathcal{I}_\lambda L, 2\xi, \phi) = \mathbf{r}(\text{gen}\mathcal{J}L, 2\xi, \phi).$$

Also, $\mathbf{r}(\text{gen}L, 0) = \mathbf{r}(\text{gen}\mathcal{J}L, 0)$, so the $0, \phi$-Fourier coefficients of $\Theta(\text{xgen}L, \tau)$ are defined to be $\mathbf{r}(\text{gen}L, 0)$. Now Theorems 2.5 and 3.6 together with Corollary 3.7 give us

Corollary 3.7. *Let $\xi \in \mathbf{n}L$, $\xi \gg 0$. Set $\mathcal{M} = \xi\mathcal{I}^{-2}$ (where $\mathcal{I}$ is the smallest fractional ideal such that $\mathbf{n}L \subseteq \mathcal{I}^2$). Let $\mathcal{P}$ be a prime ideal not dividing $\mathcal{N}$, and let ϕ be any element of $(\widehat{\mathbf{K}^+/\dot{\mathbf{K}}^2})$. If $\mathcal{P} \nmid \mathcal{M}$, then*

$$\begin{aligned} &(1 + N(\mathcal{P})^{m-2})\ \mathbf{r}(\text{gen}L, 2\xi, \phi) \\ &\quad = \mathbf{r}(\text{gen}\mathcal{P}^{-1}L, 2\xi, \phi) + \varepsilon_L(\mathcal{P}) N(\mathcal{P})^{\frac{m-3}{2}} (-1|\mathcal{P})^{\frac{m-1}{2}}\ \mathbf{r}(\text{gen}L, 2\xi, \phi\psi_{\mathcal{P}}) \\ &\qquad + N(\mathcal{P})^{m-2}\ \mathbf{r}(\text{gen}\mathcal{P}L, 2\xi, \phi) \end{aligned}$$

where $\psi_{\mathcal{P}}$ *is an element of* $(\widehat{\mathbf{K}^+/\dot{\mathbf{K}}^2})$ *such that* $\psi_{\mathcal{P}}(\zeta) = (\zeta|\mathcal{P})$ *for any* $\zeta \in \mathbf{K}^+$ *with* $\mathrm{ord}_{\mathcal{P}}\zeta = 0$. *If* $\mathcal{P}|\mathcal{M}$, *then*

$$(1+N(\mathcal{P})^{m-2})\mathbf{r}(\mathrm{gen}L, 2\xi, \phi) = \mathbf{r}(\mathrm{gen}\mathcal{P}^{-1}L, 2\xi, \phi) + N(\mathcal{P})^{m-2}\ \mathbf{r}(\mathrm{gen}\mathcal{P}L, 2\xi, \phi).$$

In the case that $\mathbf{K} = \mathbf{Q}$, *we have*

$$\mathbf{r}(\mathrm{gen}L, 2p^2a) = \left(1 - p^{\frac{m-3}{2}}\chi_{_L}(p)(-1|p)^{\frac{m-1}{2}}(2a|p) + p^{m-2}\right)\mathbf{r}(\mathrm{gen}L, 2a) - p^{m-2}\mathbf{r}\left(\mathrm{gen}L, \frac{2a}{p^2}\right)$$

for any $a \in \mathbf{Z}_+$; *note that* $\chi_{_L}(p) = (2\mathrm{disc}L|p)$.

Remark. If $\mathcal{P}\not| \,(\mathbf{n}L^{\#})^{-1}(\mathbf{n}L)^{-1}$ but $\mathcal{P}|\mathcal{N}$, then the preceding corollary can be used to give us relations on the averaged representation numbers of $\mathrm{xfam}L^{\alpha}$ where $\alpha \gg 0$ with $\mathrm{ord}_{\mathcal{P}}\alpha$ odd. Since $\mathbf{r}(\mathrm{fam}^+\mathcal{I}_\mu L^\alpha, \alpha\xi) = \mathbf{r}(\mathrm{fam}^+\mathcal{I}_\mu L, \xi)$, the above corollary can be extended to include all primes $\mathcal{P}\not| \,(\mathbf{n}L^{\#})^{-1}(\mathbf{n}L)^{-1}$.

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