Explicit action of Hecke operators on Siegel modular forms

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We develop an algorithm for determining an explicit set of coset representatives (indexed by lattices) for the action of the Hecke operators $T(p)$, $T_j(p^2)$ on Siegel modular forms of fixed degree and weight. This algorithm associates each coset representative with a particular lattice Ω, $p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda$ where Λ is a fixed reference lattice. We then evaluate the action of the Hecke operators on Fourier series. Since this evaluation yields incomplete character sums for $T_j(p^2)$, we complete these sums by replacing this operator with a linear combination of $T_\ell(p^2)$, $0 \leq \ell \leq j$. In all cases, this yields a clean and simple description of the action on Fourier coefficients.

1. INTRODUCTION

Given a weight k, degree n Siegel modular form F, it is easy to see that we can write F as a series supported on isometry classes of even integreal, positive semi-definite lattices:

$$F(\tau) = \sum_{\text{cls}\Lambda} c(\Lambda) e^*\{\Lambda\tau\}$$

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where, with T any matrix representing the quadratic form on Λ, $O(\Lambda)$ the orthogonal group of Λ and $e\{T\tau\} = \exp(\pi i Tr(T\tau))$,

$$e^*\{\Lambda\tau\} = \sum_{G\in O(\Lambda)\backslash \mathrm{GL}_n(\mathbb{Z})} e\{T[G]\tau\}.$$

(Actually, when k is odd, we equip Λ with an "orientation," and G varies over $O^+(\Lambda)\backslash \mathrm{SL}_n(\mathbb{Z})$; see the discussion at the beginning of the next section.)

In this paper, we evaluate the action of the Hecke operators $T(p)$ and $T_j(p^2)$ on the Fourier coefficients of F. To do this, we first develop an algorithm for determining the coset representatives giving the action of the Hecke operators. This algorithm simultaneously computes the coset representatives and associates each representative with a particular lattice Ω, $p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda$ (where Λ is a reference lattice, fixed throughout the algorithm). Having written F as a series supported on lattices, it is relatively simple to then evaluate the action of these coset representatives. However, this evaluation involves some incomplete character sums. To complete these and thereby have a clean description of the Fourier coefficients of the image of F, we replace $T_j(p^2)$ by $\widetilde{T}_j(p^2)$, a linear combination of $T_\ell(p^2)$, $0 \le \ell \le j$. (So the $\widetilde{T}_j(p^2)$ generate the same algebra as the $T_j(p^2)$.) Then we find that the Λ-th coefficient in the Fourier expansion of $F|\widetilde{T}_j(p^2)$ is given by

$$\sum_{p\Lambda\subseteq\Omega\subseteq\frac{1}{p}\Lambda} p^{E_j(\Lambda,\Omega)}\alpha_j(\Lambda,\Omega)c(\Omega),$$

where E_j and α_j are explicitly computed constants reflecting the geometry of Ω and Λ, and $c(\Omega)$ is the Ω-th coefficient of F (see Theorem 4.1). A similar expression for the action of $T(p)$ is given in Theorem 4.2 (this latter result was also obtained by Maass in [3]). We also describe the action of the Hecke operators when the Siegel modular form has non-trivial level and character (see Theorem 6.1).

We conclude the introduction with the formal definition of the Hecke operators and some basic notation.

Previously, descriptions of the coset representatives have been given by, for example, Freitag [3, Theorem 3.9] and Andrianov [1, Lemmas 3.3.32-3.3.33]. The coset representatives for $T(p)$ are given explicitly enough to enable easy analysis of the action of $T(p)$; this is not the case for the $T_j(p^2)$. While one could begin with these prior descriptions of coset representatives to obtain an explicit set of representatives, we find it simpler to start from the beginning.

Let $\Gamma = \mathrm{Sp}_n(\mathbb{Z})$. For each prime p, there are $n+1$ operators $T(p)$ and $T_j(p^2)$, $1 \le j \le n$, that generate the local Hecke algebra. We associate

$T(p)$ with the matrix $\underline{\delta} = \begin{pmatrix} pI_n & \\ & I_n \end{pmatrix}$, and each $T_j(p^2)$ with the matrix $\underline{\delta} = \underline{\delta}_j = \begin{pmatrix} \delta_j & \\ & \delta_j^{-1} \end{pmatrix}$ where $\delta_j = \begin{pmatrix} pI_j & \\ & I_{n-j} \end{pmatrix}$. Then the action of the operator is defined by mapping F to

$$(\det \underline{\delta})^{k/2} \sum_{\gamma \in (\Gamma' \cap \Gamma) \backslash \Gamma} F | \underline{\delta}^{-1} \gamma,$$

where $\Gamma' = \underline{\delta} \Gamma \underline{\delta}^{-1}$. Note that Andrianov [1, Section 3.3] defines the operators $T_j(p^2)$ via the double coset $\Gamma p \underline{\delta}_{n-j}^{-1} \Gamma$; this is equivalent to the definition given above with the caveat that we've interchanged the roles of j and $n-j$. (In truth, in our formal definition for $T(p)$ we will introduce an extra normalizing factor of $p^{-n(n+1)/2}$. Additional normalizing factors will be introduced when we define the operate $\widetilde{T}_j(p^2)$ as well.)

Also, for a $\mathbb{Z}$-lattice $\Delta = \mathbb{Z}\mathbf{a}_1 \oplus \cdots \oplus \mathbb{Z}\mathbf{a}_n$, $\mathbf{a}_i \in \mathbb{Z}^{j,1}$, we write $\operatorname{rank}_p(\Delta)$ and $\operatorname{span}_p(\Delta)$ to denote the rank and span of $\{\overline{\mathbf{a}}_1, \ldots, \overline{\mathbf{a}}_n\}$ where $\overline{\mathbf{a}}_i$ denotes the canonical image of $\mathbf{a}_i$ in $V = (\mathbb{Z}/p\mathbb{Z})^{j,1}$. We use the symbol $\{\Lambda : \Delta\}$ to denote the invariant factors of Δ in Λ (see [5, 81:11]) and the symbol $\operatorname{mult}_{\{\Lambda:\Delta\}}(r)$ for the multiplicity of r in the invariant factors.

Except in Section 6, we consider Siegel forms of "level 1", i.e. forms for the full modular group $\operatorname{Sp}_n(\mathbb{Z})$.

2. LATTICE INTERPRETATION OF $T_J(P^2)$

A weight k, degree n Siegel modular form F has a Fourier expansion supported on symmetric even integral $n \times n$ matrices T with $T \geq 0$. (This is classical; for example see Freitag [3, p.43] or Andrianov [1, 2.3.1,2.3.4].) We consider each T to be a quadratic form on a rank n $\mathbb{Z}$-lattice Λ relative to some basis for Λ. As T varies, the pair (Λ, T) varies over all isometry classes of rank n lattices with even integral positive semi-definite quadratic forms. Also, the isometry class of (Λ, T) is that of (Λ, T') if and only if $T' = T[G]$ for some $G \in \mathrm{GL}_n(\mathbb{Z})$. Since $F(\tau[G]) = (\det G)^k F(\tau)$ for all $G \in \mathrm{GL}_n(\mathbb{Z})$, it follows that $c(T[G]) = (\det G)^k c(T)$, where $c(T)$ denotes the T-th Fourier coefficient of F. Hence, using the language of lattices, when k is even we can rewrite the Fourier expansion of F in the form

$$F(\tau) = \sum_{\mathrm{cls}\Lambda} c(\Lambda) e^*\{\Lambda\tau\},$$

where $c(\Lambda) = c(T)$ for any matrix T representing the quadratic form on Λ.

When k is odd, we consider two "orientations" of Λ. So with $\Lambda \simeq T$ relative to a basis for Λ with orientation σ, we set $c(\Lambda,\sigma) = c(T)$ and

$$e^*\{(\Lambda,\sigma)\tau\} = \sum_{G\in O^+(\Lambda)\backslash \mathrm{SL}_n(\mathbb{Z})} e\{T[G]\tau\}.$$

Then we can rewrite the Fourier expansion of F in the form

$$F(\tau) = \sum_{\mathrm{cls}\Lambda} c(\Lambda,\sigma) e^*\{\Lambda\tau\}$$

where the sum runs over all oriented isometry classes.

Now fix $j \le n$ and let δ_j and Γ' be associated to the operator $T_j(p^2)$, as defined above. Our goal is to find coset representatives for the quotient $(\Gamma'\cap\Gamma)\backslash\Gamma$. To this end, let $M \in \Gamma$, and the top j rows of M be denoted by $M_j = (A|B) = (\mathbf{a}_1,\dots,\mathbf{a}_n|\mathbf{b}_1,\dots,\mathbf{b}_n)$. So the $\mathbf{a}_i, \mathbf{b}_i$ represent the columns of M_j. We will reduce this (by right multiplication by elements of Γ) to a special form which we will then show is in the desired intersection, $\Gamma'\cap\Gamma$.

Set $\Lambda = \mathbb{Z}x_1\oplus\cdots\oplus\mathbb{Z}x_n$, a formal $\mathbb{Z}$-lattice, and let $\Lambda^\# = \mathbb{Z}y_1\oplus\cdots\oplus\mathbb{Z}y_n$. (Later x_i will be replaced by $\overline{\mathbf{a}}_i$, and y_i by $\overline{\mathbf{b}}_i$.) When we write ΛC, C a matrix, we mean the formal $\mathbb{Z}$-lattice generated by the basis $(x_1,\dots,x_n)C$.

Our algorithm for computing the coset representative for a given M is described in the following steps.

Step 1. We focus on the rank of $(\mathbf{a}_1,\dots,\mathbf{a}_n)$ modulo p. First, let r_0 be the rank of these columns modulo p. Necessarily, $r_0 \le j$. By multiplying M_j on the right by a matrix of the form $\begin{pmatrix} E & \\ & E\end{pmatrix}$, with E a permutation matrix, we can put our matrix into the form $(A_0, A_1|B)$, where A_0 is $j\times r_0$ and its columns are linearly independent modulo p.

Then, using a matrix of the form

$$\left(\begin{array}{cc|cc} I_{r_0} & X & & \\ & I_{n-r_0} & & \\ \hline & & I_{r_0} & \\ & & -{}^tX & I_{n-r_0}\end{array}\right),$$

with X of size $r_0\times(n-r_0)$ chosen modulo p, we can convert the A-part of the matrix to have the form (A_0, pA_1) where A_0 is of size $j\times r_0$. Effectively we solve the system

$$A_0X + A_1 \equiv 0 \pmod{p}.$$

The top j rows of our matrix now has the form $(A_0, pA_1|B)$.

Note that the matrix E is not uniquely determined by M. To remedy this, we provide an alternate description of this step as follows.

STEP 1'. Equivalently, let $\Omega_0 = \ker(\Lambda \to \Lambda(A) \bmod p)$ where $M_j = (\mathbf{a}_1, \ldots, \mathbf{a}_n | \mathbf{b}_1, \ldots, \mathbf{b}_n) = (A|B)$ and $\Lambda \to \Lambda(A) \bmod p$ denotes the map that takes x_i to $\overline{\mathbf{a}}_i$. Thus $\Lambda = \Lambda_0 \oplus \Delta_1$, $\Omega_0 = p\Lambda_0 \oplus \Delta_1$, where $\mathrm{rank}_{\mathbb{Z}}\Delta_1 = n - r_0$. Note that Δ_1 is uniquely determined modulo $p\Lambda$. Take any (change of basis) matrix $C_0 \in \mathrm{GL}_n(\mathbb{Z})$ such that $\Omega_0 = \Lambda C_0 \begin{pmatrix} pI_{r_0} & \\ & I_{n-r_0} \end{pmatrix}$. So $\Delta_1 = \Lambda C_0 \begin{pmatrix} 0 \cdot I_{r_0} & \\ & I_{n-r_0} \end{pmatrix}$. Thus, with renewed notation, $M_j \begin{pmatrix} C_0 & \\ & {}^tC_0^{-1} \end{pmatrix}$ has the form $(\mathbf{a}_1, \ldots, \mathbf{a}_n | \mathbf{b}_1, \ldots, \mathbf{b}_n) = (A_0, pA_1 | B)$.

Note that while C_1 is not uniquely determined by M, the lattice Ω_1 is.

STEP 2. By Lemma 7.2, $\mathrm{rank}_p(\mathbf{a}_1, \ldots, \mathbf{a}_{r_0}, \mathbf{b}_{r_0+1}, \ldots, \mathbf{b}_n) = j$ so that by multiplying by a matrix of the form

$$\left(\begin{array}{cc|cc} I_{r_0} & & & \\ & E_{n-r_0} & & \\ \hline & & I_{r_0} & \\ & & & E_{n-r_0} \end{array}\right),$$

where E_{n-r_0} is a permutation matrix, we will guarantee that the $\mathrm{rank}_p(\mathbf{a}_1, \ldots, \mathbf{a}_{r_0}, \mathbf{b}_{r_0+1}, \ldots, \mathbf{b}_j) = j$. This permutation does not affect the first r_0 columns of either the A or B part of our matrix but does permute the next $n - r_0$ columns. We now write the top j rows as

$$(A_0, pA_1, pA_3 | B_0, B_1, B_3)$$

where A_0 and B_0 are of size $j \times r_0$, A_1 and B_1 are of size $j \times (j - r_0)$,and A_3 and B_3 are of size $j \times (n - j)$ with $\mathrm{rank}_p(A_0, B_1) = j$. (The choice of the subscript notation will be clear in a moment.)

By Lemma 7.2, the columns of B_0 are in $\mathrm{span}_p A = \mathrm{span}_p A_0$. We now force all the columns of B_3 to be in $\mathrm{span}_p A$. We can do this by using a matrix of the form

$$\left(\begin{array}{ccc|ccc} I_{r_0} & & & & & \\ & I_{j-r_0} & & & & \\ & -{}^tX & I_{n-j} & & & \\ \hline & & & I_{r_0} & & \\ & & & & I_{j-r_0} & X \\ & & & & & I_{n-j} \end{array}\right),$$

where X is size $(j-r_0)\times(n-j)$. Effectively, we are replacing B_3 by B_1X+B_3, where $B_3 \equiv A_0X' - B_1X \pmod p$.

This matrix multiplication has no effect on A_0 or B_1 so that the property $\mathrm{rank}_p(A_0,B_1)=j$ remains in effect. The r_0+1 to nth columns of the A-part are combined, but still preserve the property that all the columns are divisible by p.

Now (with renewed notation) the top j rows of our matrix have the form

$$(A_0, pA_1, pA_3 | B_0, B_1, B_3),$$

with $B_0, B_3 \subseteq \mathrm{span}_p A_0$ and A_3, B_3 of size $j\times(n-j)$.

Step 2'. With C_0 as in Step 1', we have $M_j\begin{pmatrix} C_0 & \\ & {}^tC_0^{-1}\end{pmatrix} = (\mathbf{a}_1,\dots,\mathbf{a}_n|\mathbf{b}_1,\dots,\mathbf{b}_n) = (A_0,pA_1|B)$. Simultaneously, we obtained splittings $\Lambda=\Lambda_0\oplus\Delta_1$, $\Omega_0=p\Lambda_0\oplus\Delta_1$ with Δ_1 uniquely determined modulo $p\Lambda$. This splitting of Λ corresponds to a splitting $\Lambda^\#=\Lambda_0'\oplus\Lambda_1'$ with Λ_0' determined uniquely modulo $p\Lambda^\#$. (Here C_0 carries $(x_1,\dots,x_n)$ to a basis for the splitting $\Lambda_0\oplus\Delta_1$, and ${}^tC_0^{-1}$ carries $(y_1,\dots,y_n)$ to a basis for the splitting $\Lambda^\#=\Lambda_0'\oplus\Lambda_1'$; thinking of $(y_1,\dots,y_n)$ as the basis dual to $(x_1,\dots,x_n)$, it is clear that C_0, C_0^{-1} behave as claimed.) We let U be the subspace of $V=(\mathbb{Z}/pZ)^{j,1}$ spanned by $\overline{\mathbf{a}}_1,\dots,\overline{\mathbf{a}}_{r_0}$, and we take

$$\Omega_1' = \ker\left(\Lambda^\# \to \Lambda^\#(B) \bmod p \to V/U\right);$$

here $\Lambda^\#\to\Lambda^\#(B) \bmod p$ denotes the map taking y_i to $\overline{\mathbf{b}}_i$, and $\Lambda^\#(B) \bmod p\to V/U$ denotes the canonical projection map. By Lemma 7.2, $\overline{\mathbf{b}}_1,\dots,\overline{\mathbf{b}}_{r_0}$ are in the span of $\overline{\mathbf{a}}_1,\dots,\overline{\mathbf{a}}_{r_0}$, so $\Lambda_0'\subseteq\Omega_1'$. Also by Lemma 7.2, $\Omega_1'/p\Lambda^\#$ has dimension $n-j+r_0$. So we can find a change of basis matrix C_1 such that

$$\Omega_1' = \Lambda^{\#}\,{}^tC_1^{-1}\begin{pmatrix} I_{r_0} & & \\ & pI_{j-r_0} & \\ & & I_{n-j}\end{pmatrix}$$

with $\Lambda_0'=\Lambda^\#\,{}^tC_1^{-1}\begin{pmatrix} I_{r_0} & \\ & 0\end{pmatrix}$. So ${}^tC_1^{-1}$ carries $(y_1,\dots,y_n)$ to a basis for a splitting $\Lambda^\#=\Lambda_0'\oplus\Lambda_2'\oplus\Lambda_3'$; note that $\Lambda_0'\oplus\Lambda_3'$ is uniquely determined modulo $p\Lambda^\#$. Also, C_1 carries $(x_1,\dots,x_n)$ to a basis for the corresponding splitting

$$\Lambda=\Lambda_0\oplus\Delta_2\oplus\Lambda_3$$

with Δ_2 uniquely determined modulo $p\Lambda$. (Here Δ_1 has been split as $\Delta_2 \oplus \Lambda_3$.) Set

$$\Omega_1 = \Lambda C_1 \begin{pmatrix} pI_{r_0} & & \\ & I_{j-r_0} & \\ & & pI_{n-j} \end{pmatrix};$$

so $\Delta_1 = \Lambda C_1 \begin{pmatrix} 0 & \\ & I_{n-r_0} \end{pmatrix}$ and $\Omega_1 = p\Lambda_0 \oplus \Delta_2 \oplus p\Lambda_3$ (so C_1 is a refinement of C_0, and since Δ_2 is uniquely determined modulo $p\Lambda$, Ω_1 is well defined).

Thus, with renewed notation, $M_j \begin{pmatrix} C_1 & \\ & {}^tC_1^{-1} \end{pmatrix}$ has the form

$$(\mathbf{a}_1, \ldots, \mathbf{a}_n | \mathbf{b}_1, \ldots, \mathbf{b}_n) = (A_0, pA_1, pA_3 | B_0, B_1, B_3),$$

with $B_0, B_3 \subseteq \mathrm{span}_p A_0$.

Step 3. We next look at the rank of the A-part of our matrix modulo p^2. More precisely, we consider $\mathrm{rank}_p(A_0, A_1)$, denoted by $r_0 + r_1$. Let $r_2 = j - r_0 - r_1$. We can find a permutation which permutes the columns of the A-part so that the first $r_0 + r_1$ columns are linearly independent modulo p^2. Such a permutation matrix will have the form

$$P = \begin{pmatrix} I_{r_0} & & \\ & E_{r_1+r_2} & \\ & & I_{n-j} \end{pmatrix}.$$

The effect of multiplying $(A_0, pA_1, pA_3 | B_0, B_1, B_3)$ by $\begin{pmatrix} P & \\ & P \end{pmatrix}$ is to simultaneously permute the $r_0 + 1, \ldots, j$ columns of A_1 and of B_1, but the relationship between A_0 and B_1 is not affected. This puts the top j rows of our matrix in the form

$$(A_0, pA_1, pA_2, pA_3, | B_0, B_1, B_2, B_3),$$

where $\mathrm{rank}_p(A_0, A_1) = r_0 + r_1$, and the old A_1 (resp. B_1) has been split into parts A_1 and A_2 (resp. B_1 and B_2). The new matrices A_1, B_1 are of size $j \times r_1$, and A_2, B_2 are of size $j \times r_2$.

We now solve the congruence $-A_2 \equiv A_0X_0 + A_1X_1 \pmod p$, then we multiply $(A_0, pA_1, pA_2, pA_3|B_0, B_1, B_2, B_3)$ by $\begin{pmatrix} C & \\ & {}^tC^{-1} \end{pmatrix}$ where

$$C = \begin{pmatrix} I_{r_0} & & pX_0 & \\ & I_{r_1} & X_1 & \\ & & I_{r_2} & \\ & & & I_{n-j} \end{pmatrix}.$$

(So X_0 is of size $r_0 \times r_2$, X_1 of size $r_1 \times r_2$.) Thus pA_2 gets replaced by something completely divisible by p^2. This will combine columns of our new B_2 with the B_1, but we still have that $\mathrm{rank}_p(A_0, B_1, B_2) = j$, and (with renewed notation) our matrix is now of the form

$$(A_0, pA_1, p^2A_2, pA_3, |B_0, B_1, B_2, B_3),$$

where $\mathrm{rank}_pA_0 = r_0$, $\mathrm{rank}_p(A_0, A_1) = \mathrm{rank}_p(A_0, B_1) = r_0 + r_1$, and $B_0, B_3 \subseteq \mathrm{span}_pA_0$.

Step 3'. Recall that $\Omega_1 = p\Lambda_0 \oplus \Delta_2 \oplus p\Lambda_3$ where $\Lambda = \Lambda_0 \oplus \Delta_2 \oplus \Lambda_3$; say (renewing our notation) the $(x_1, \dots, x_n)$ is a basis corresponding to this splitting of Λ. Also, we have

$$M_j \begin{pmatrix} C_1 & \\ & {}^tC_1^{-1} \end{pmatrix} = (\mathbf{a}_1, \dots, \mathbf{a}_n|\mathbf{b}_1, \dots, \mathbf{b}_n) = (A_0, pA_1, pA_3|B_0, B_1, B_3).$$

Let $p\Omega_2 = \ker\left(\Omega_1 \to \frac{1}{p}\Omega_1(A) \bmod p\right)$ where $\Omega_1 \to \frac{1}{p}\Omega_1(A) \bmod p$ denotes the map taking x_i to $\overline{\mathbf{a}}_i$. (So $\frac{1}{p}\Omega_1(A) \bmod p$ is spanned by the images in V of the columns of A_0, A_1 and pA_3.) With our earlier notation, $\Lambda_0(A) \bmod p$ has dimension r_0, and $\frac{1}{p}\left(p\Lambda_0 \oplus \Delta_2\right)(A) \bmod p$ has dimension $r_0 + r_1$. Thus

$$p\Omega_2 = p^2\Lambda_0 \oplus p\Lambda_1 \oplus \Lambda_2 \oplus p\Lambda_3$$

where Λ_2 is determined uniquely modulo $p\Omega_1$, and $\mathrm{rank}_{\mathbb{Z}}\Lambda_2 = r_2 = j - r_0 - r_1$. Take a (change of basis) matrix $C_2 \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$\begin{aligned} \Omega_0 &= \Lambda C_2 \begin{pmatrix} pI_{r_0} & \\ & I_{n-r_0} \end{pmatrix} \\ \Omega_1 &= \Lambda C_2 \begin{pmatrix} pI_{r_0} & & \\ & I_{n-j-r_0} & \\ & & pI_{n-j} \end{pmatrix} \end{aligned}$$

$$p\Omega_2 \;=\; \Lambda C_2 \begin{pmatrix} p^2 I_{r_0} & & & \\ & pI_{r_1} & & \\ & & I_{r_2} & \\ & & & pI_{n-j} \end{pmatrix}.$$

(So C_2 refines our choice of C_1.)

Thus, $M_j \begin{pmatrix} C_2 & \\ & {}^tC_2^{-1} \end{pmatrix}$ has the form

$$(A_0, pA_1, p^2A_2, pA_3, |B_0, B_1, B_2, B_3)$$

where $\text{rank}_p A_0 = r_0$, $\text{rank}_p(A_0, A_1) = \text{rank}_p(A_0, B_1) = r_0 + r_1$, and $B_0, B_3 \subseteq \text{span}_p A_0$.

Remark 2. 1. Knowing Ω_2 and Λ_1, we can reconstruct Ω_0 and Ω_1, for $\Omega_0 = \Omega_2 \cap \Lambda$, and $\Omega_1 = \Lambda_1 + p(\Lambda + \Omega_2)$. Note that $\Lambda_1 + p(\Lambda + \Omega_2)$ is an r_1 dimensional subspace of $(\Omega_2 \cap \Lambda)/p(\Omega_2 + \Lambda)$, and that distinct pairs (Ω_2, Ω_1) correspond to distinct pairs (Ω_2, Λ_1).

While C_2 is not uniquely determined by M, the lattice Ω_2 is.

STEP 4. With C_2 as above, we have

$$M_j \begin{pmatrix} C_2 & \\ & {}^tC_2^{-1} \end{pmatrix} = (A_0, pA_1, p^2A_2, pA_3, |B_0, B_1, B_2, B_3).$$

We solve $A_0Y_3 \equiv -B_3 \pmod{p}$, with Y_3 unique modulo p since A_0 has maximal p-rank.

Next, we solve $A_0Y_0' \equiv -B_0 \pmod{p}$; so Y_0' is uniquely determined modulo p. Also, since $M \begin{pmatrix} C_2 & \\ & {}^tC_2^{-1} \end{pmatrix} \in \Gamma$, we know $B_0{}^tA_0 \equiv Y_0'[{}^tA_0]$ $\pmod{p}$ is symmetric modulo p, and hence we can choose Y_0' symmetric over $\mathbb{Z}$.

Similarly,

$$M \begin{pmatrix} C_2 & \\ & {}^tC_2^{-1} \end{pmatrix} \left(\begin{array}{c|c} I_n & \begin{matrix} Y_0' & 0 & Y_3 \\ 0 & & \\ {}^tY_3 & & \end{matrix} \\ \hline 0 & I_n \end{array}\right) \in \Gamma,$$

so we can solve $(A_0, A_1)Y \equiv -(\frac{1}{p}(B_0 + A_0Y_0') + A_3{}^tY_3, B_1) \pmod{p}$ uniquely modulo p with a symmetric $r_0 + r_1$ dimensional matrix Y.

Decompose $Y = \begin{pmatrix} Y_0'' & Y_2 \\ {}^tY_2 & Y_1 \end{pmatrix}$ with Y_0'' symmetric of dimension r_0 and set $Y_0 = Y_0' + pY_0''$. Note that since the columns of A_0, B_1 are linearly independent over $\mathbb{Z}/p\mathbb{Z}$, rank_p of the columns of $B_1 - A_0Y_2$ must equal rank_p of the columns of B_1, which is r_1. Then since $A_1Y_1 \equiv B_1 - A_0Y_2 \pmod{p}$, rank_p of the columns of A_1Y_1 must be r_1, and so the $r_1 \times r_1$ matrix Y_1 is invertible over $\mathbb{Z}/p\mathbb{Z}$ (i.e. $p \not| \det Y_1$). Notice that in this construction, the column structure of M_j uniquely determines Y_1, Y_2, Y_3 modulo p and Y_0 modulo p^2.

In this way we construct a matrix X of whose inverse has the form

$$X^{-1} = \left(\begin{array}{cccc|cccc} I_{r_0} & & & & Y_0 & pY_2 & 0 & Y_3 \\ & \frac{1}{p}I_{r_1} & & & {}^tY_2 & Y_1 & 0 & \\ & & \frac{1}{p^2}I_{r_2} & & 0 & 0 & I_{r_2} & \\ & & & I_{n-j} & {}^tY_3 & & & \\ \hline & & & & I_{r_0} & & & \\ & & & & & pI_{r_1} & & \\ & & & & & & p^2I_{r_2} & \\ & & & & & & & I_{n-j} \end{array}\right)$$

so that

$$M_j \begin{pmatrix} C_2 & \\ & {}^tC_2^{-1} \end{pmatrix} X^{-1}$$

has the form

$$(A_0, A_1, A_2, pA_3, |p^2B_0, p^2B_1, p^2B_2, pB_3). \tag{1}$$

However, $X \notin \Gamma$. Write

$$X = \begin{pmatrix} D & -B' \\ 0 & D^{-1} \end{pmatrix} = \left(\begin{array}{cc|cc} D_j & & -Y' & -Y_3' \\ & I_{n-j} & -{}^tY_3' & \\ \hline & & D_j^{-1} & \\ & & & I_{n-j} \end{array}\right).$$

Since $(D_j, -Y')$ is a coprime symmetric pair, there exist matrices U, V such that $\begin{pmatrix} D_j & -Y' \\ U & V \end{pmatrix} \in \mathrm{Sp}_j(\mathbb{Z})$ and hence

$$X' = \left(\begin{array}{cc|cc} D_j & & -Y' & -Y_3' \\ & I_{n-j} & -{}^tY_3' & \\ \hline U & & V & -UY_3' \\ & 0 & & I_{n-j} \end{array}\right) \in \Gamma. \tag{2}$$

Certainly,

$$M \begin{pmatrix} C_2 & \\ & {}^tC_2^{-1} \end{pmatrix} X'^{-1} \in \Gamma.$$

Also, the top j rows of the product have the form

$$(A_0, pA_1, p^2A_2, pA_3, |B_0, B_1, B_2, B_3),$$

so by Lemma 7.1, this product is in Γ'.

Furthermore, it is easy to see that

$$X' = X''X, \text{ where } X'' \in \Gamma'. \tag{3}$$

Thus for each $M \in \Gamma$, we have used the column structure of M_j to produce a coset representative in a particular form. Since left multiplication by an element of $\Gamma' \cap \Gamma$ does not change the column structure of M_j, the algorithm produces a complete (nonredundant) set of coset representatives. This argument proves the following.

PROPOSITION 2.1. *Each coset of* $(\Gamma' \cap \Gamma)\backslash\Gamma$ *is identified with*

1.a lattice Ω *(*Ω_2 *in the above notation) such that* $p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda$ *and* $r_0 + r_2 \le j$ *where* $r_0 = \mathrm{mult}_{\{\Lambda:\Omega\}}(p)$ *and* $r_2 = \mathrm{mult}_{\{\Lambda:\Omega\}}(1/p)$*;*

2.a subspace $\Lambda_1 + p(\Omega+\Lambda)$ *of* $(\Omega\cap\Lambda)/p(\Omega+\Lambda)$, $\mathrm{rank}\Lambda_1 = r_1 = j - r_0 - r_2$*;*

3.a matrix $B' = \begin{pmatrix} Y_0 & Y_2 & 0 & Y_3 \\ p\,{}^tY_2 & Y_1 & 0 & \\ 0 & 0 & I & \\ {}^tY_3 & & & \end{pmatrix}$ *where* Y_0 *is unique modulo* p^2*, symmetric,* $r_0 \times r_0$*;* Y_1 *is unique and invertible modulo* p*, symmetric,* $r_1 \times r_1$*;* Y_2 *is unique modulo* p*,* $r_0 \times r_1$*; and* Y_3 *is unique modulo* p*,* $r_0 \times n - j$*.*

By (3), the action of a representative for this coset is given by the product $\begin{pmatrix} D & B' \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} C^{-1} & \\ & {}^tC \end{pmatrix}$ where $D = D(\Omega)$ is the diagonal matrix given in Step 4, determined by Ω, and $C = C(\Omega, \Lambda_1)$ is a change of basis matrix C_2 as in Step 3'.

Notice also that given any triple (Ω, Λ_1, B') meeting these conditions, (D, B') is a coprime symmetric pair; hence $\begin{pmatrix} D & B' \\ U & V \end{pmatrix} \begin{pmatrix} C^{-1} & \\ & {}^tC \end{pmatrix} \in \Gamma$ for some U, V.

The following is now immediate.

Corollary 2.1. *For F a Siegel modular form of degree n, we have*

$$F|T_j(p^2) = \sum_{(\Omega,\Lambda_1,B')} F \left| \begin{pmatrix} \delta_j^{-1} & \\ & \delta_j \end{pmatrix} \begin{pmatrix} D & B' \\ & D^{-1} \end{pmatrix} \begin{pmatrix} C^{-1} & \\ & {}^tC \end{pmatrix} \right.$$

where (Ω, Λ_1, B') varies over all triples meeting the three conditions of Proposition 2.1 and $D = D(\Omega)$, $C = C(\Omega, \Lambda_1)$.

3. LATTICE INTERPRETATION OF $T(P)$

Recall that for $T(p)$, we seek the coset representatives for $(\Gamma' \cap \Gamma)\backslash\Gamma$, where

$$\Gamma' = \begin{pmatrix} pI_n & \\ & I_n \end{pmatrix} \Gamma \begin{pmatrix} \frac{1}{p}I_n & \\ & I_n \end{pmatrix}.$$

A similar, but simpler, algorithm can be used in this case. Namely, we use Step 1 to put the top n rows into the form $(A_0, pA_1|B)$, where A_0 has maximal rank modulo p. Next we use a relaxation of Step 4 to put the top n rows into the form $(A_0, A_1|pB)$. The rest of the argument is essentially the same.

In this way we prove the following (cf. Maass [4]).

Proposition 3.1. *For F a Siegel modular form of degree n, we have*

$$F|T(p) = p^{n(k-n-1)/2} \sum_{(\Omega,B')} F \left| \begin{pmatrix} \frac{1}{p}I & \\ & I \end{pmatrix} \begin{pmatrix} D & B' \\ & D^{-1} \end{pmatrix} \begin{pmatrix} C^{-1} & \\ & {}^tC \end{pmatrix} \right.$$

where (Ω, B') varies over all pairs meeting the following conditions:

1.Ω is a lattice such that $p\Lambda \subseteq \Omega \subseteq \Lambda$; set $r = \mathrm{mult}_{\{\Lambda:\Omega\}}(p)$.

2.B' is a matrix of the form $\begin{pmatrix} Y & \\ & I \end{pmatrix}$ where Y is symmetric, $r \times r$ and determined modulo p.

Here $C = C(\Omega)$ is a change of basis matrix so that $\Omega = \Lambda C \begin{pmatrix} pI_r & \\ & I_{n-r} \end{pmatrix}$ and $D = D(\Omega) = \begin{pmatrix} I_r & \\ & pI_{n-r} \end{pmatrix}$.

4. THE ACTION OF THE HECKE OPERATORS

Let F be a Siegel modular form of weight k and degree n, and write

$$F(\tau) = \sum_T c(T)e\{T\tau\}$$

where T varies over all symmetric, even integral, $n \times n$ matrices with $T \geq 0$.

With $\delta = \delta_j$, Corollary 2.1 gives us

$$F|T_j(p^2)(\tau) = \sum_{\substack{T \\ (\Omega,\Lambda_1,B')}} \det\left(\delta^{-1}D\right)^k c(T)e\{T[\delta^{-1}DC^{-1}]\tau\}e\{T\delta^{-1}B'D\delta^{-1}\},$$

where $D = D(\Omega)$, $C = C(\Omega, \Lambda_1)$. Write B' as in Condition 3. of Proposition 2.1; then

$$\delta^{-1}(B'D)\delta^{-1} = \begin{pmatrix} Y_0/p^2 & Y_2/p & 0 & Y_3/p \\ {}^tY_2/p & Y_1/p & 0 & \\ 0 & 0 & I & \\ {}^tY_3/p & & & \end{pmatrix}.$$

For any Ω and Λ_1, let $P = P(\Omega)$ be the permutation matrix such that

$$\widetilde{B} = {}^tP\left(\delta^{-1}B'D\delta^{-1}\right)P = \begin{pmatrix} Y_0/p^2 & Y_2/p & Y_3/p & 0 \\ {}^tY_2/p & Y_1/p & & \\ {}^tY_3/p & & & \\ 0 & & & I \end{pmatrix}. \qquad (4)$$

With $\widetilde{D} = \widetilde{D}(\Omega) = {}^tP\left(\delta^{-1}D\right)P = \begin{pmatrix} \frac{1}{p}I_{r_0} & & \\ & I & \\ & & pI_{r_2} \end{pmatrix}$, and $\widetilde{C} = \widetilde{C}(\Omega, \Lambda_1) = CP$, we have

$$\Omega = \Lambda\widetilde{C}\widetilde{D}^{-1}$$

(although the basis of Ω is ordered differently) and

$$\Lambda_1 = \Lambda\widetilde{C}\begin{pmatrix} 0 \cdot I_{r_0} & & \\ & I_{r_1} & \\ & & 0 \cdot I \end{pmatrix}.$$

Thus by interchanging the sums on T and Ω and replacing T by $T[{}^tP(\Omega)]$, we get $c(T[{}^tP(\Omega)]) = c(T)$, and

$$F|T_j(p^2)(\tau) = \sum_{\substack{(\Omega,\Lambda_1) \\ T}} p^{k(r_2-r_0)}c(T)e\{T\left[\widetilde{D}\widetilde{C}^{-1}\right]\tau\}\sum_{\widetilde{B}} e\{T\widetilde{B}\}, \qquad (5)$$

where $\widetilde{B}$ varies over the matrices of the form in (4), with $\mathrm{rank}_p Y_1 = r_1 = j - r_0 - r_2$. Note that $\det\left(\delta^{-1}D\right) = p^{r_2 - r_0}$ and recall that $r_0 = \mathrm{mult}_{\{\Lambda:\Omega\}}(p)$ and $r_2 = \mathrm{mult}_{\{\Lambda:\Omega\}}(1/p)$.

The invertibility condition on Y_1 modulo p means that the exponential sum on $\widetilde{B}$ is an incomplete character sum. To complete this sum, we will replace T_j by a new operator which is a linear combination of the T_t for $0 \le t \le j$, where T_0 is the identity operator. More specifically, we make the following definition.

Definition 4.1. With the notation above, define the operator $\widetilde{T}_j$ by

$$\widetilde{T}_j(p^2) = p^{j(k-n-1)} \sum_{0 \le t \le j} \beta_p(n-t, j-t) T_t(p^2)$$

where

$$\beta_p(m,\ell) = \prod_{i=1}^{\ell} \frac{p^{m-\ell+i} - 1}{p^i - 1}$$

is the number of ℓ-dimensional subspaces of an m-dimensional space over $\mathbb{Z}/p\mathbb{Z}$.

We need a bit more notation. For $p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda$, let $\alpha_j(\Lambda,\Omega)$ be the number of totally isotropic subspaces of $(\Omega \cap \Lambda)/p(\Omega + \Lambda)$ with dimension $d - n + j$ where $d = \dim(\Omega \cap \Lambda)/p(\Omega + \Lambda)$, and the quadratic form on this quotient is $\frac{1}{2}Q$ modulo p. (So $\alpha_j(*) = 0$ if $d - n + j < 0$.) In Proposition 4.1 and the remarks following we give explicit formulas for these α_j. Also, set

$$\begin{aligned} E_j(\Lambda,\Omega) &= k\left(m(1/p) - m(p) + j\right) + m(p)\left(m(p) + m(1) + 1\right) \\ &\quad + \frac{1}{2} m_j(1)\left(m_j(1) + 1\right) - j(n+1). \end{aligned}$$

where $m(a) = \mathrm{mult}_{\{\Lambda:\Omega\}}(a)$ and $m_j(a) = m(a) - n + j$.

We can now prove our main theorem.

Theorem 4.1. *With F as above,*

$$F|\widetilde{T}_j(p^2)(\tau) = \sum_{\mathrm{cls}\Lambda} \left(\sum_{p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda} p^{E_j(\Lambda,\Omega)} \alpha_j(\Lambda,\Omega) c(\Omega) \right) e^*\{\Lambda\tau\}.$$

Here $\mathrm{cls}\Lambda$ *varies over all classes of (oriented) integral lattices. (Note that* $c(\Omega) = 0$ *unless* Ω *is even integral. Also, when* k *is odd, by* $c(\Omega)$ *we really mean* $c(\Omega,\sigma)$ *where* σ *is the orientation on* Ω *induced by that on* Λ*. Since* *an orientation σ on Λ induces a compatible orientation, which we also call σ, on any lattice Ω in the space $\mathbb{Q}\Lambda$, we can suppress the reference to the orientation without causing confusion.)*

Proof. First suppose k is even. With Definition 4.1 and equation (5), we have

$$F|\widetilde{T}_j(p^2)(\tau) = \sum_{0\le t\le j} \beta_p(n-t,j-t) \sum_{\substack{(\Omega,\Delta_1)\\ T}} p^{k(r_2-r_0)}c(T)e\{T\left[\widetilde{D}\widetilde{C}^{-1}\right]\tau\}\sum_{\widetilde{B}} e\{T\widetilde{B}\},$$

where the sums on $\Omega, \Delta_1, \widetilde{B}$ are as in equation (5) but with j replaced by t and Λ_1 replaced by Δ_1. Since there are no Δ_1 meeting these conditions when $r_0 + r_2 > t$, we can interchange the sums on Ω and t to get

$$F|\widetilde{T}_j(p^2)(\tau) = \sum_{\Omega,T} \sum_{\substack{0\le t\le j\\ \Delta_1}} p^{k(r_2-r_0)}c(T)e\{T\left[\widetilde{D}\widetilde{C}_t^{-1}\right]\tau\}\beta_p(n-t,j-t)\sum_{\widetilde{B}} e\{T\widetilde{B}\}, \tag{6}$$

where Ω is limited by the condition $r_0 + r_2 \le j$, and Δ_1 satisfies the condition $\mathrm{rank}_{\mathbb{Z}}\Delta_1 = t - r_0 - r_2$, $\widetilde{C}_t = \widetilde{C}(\Omega, \Delta_1)$.

Now take $t < r_1$; by Lemma 7.3, as we vary G over incongruent modulo p matrices in $\mathrm{GL}_{r_1}(\mathbb{Z})$ and Y_1 over incongruent modulo p symmetric matrices in $\mathrm{GL}_t(\mathbb{Z})$, the product

$${}^tG \begin{pmatrix} Y_1 & \\ & 0 \end{pmatrix} G = Y_1'$$

varies over all modulo p symmetric $r_1 \times r_1$ matrices with $\mathrm{rank}_p Y_1' = t$. Let

$$\widetilde{G} = \begin{pmatrix} I_{r_0} & & \\ & {}^tG & \\ & & I \end{pmatrix}.$$

Replacing T by $T\left[{}^t\widetilde{G}\right]$, we have $c\left(T\left[{}^t\widetilde{G}\right]\right) = c(T)$, $\widetilde{G}\widetilde{D} = \widetilde{D}\widetilde{G}$, and $\widetilde{C}_t\widetilde{G}^{-1}$ simply represents another choice for the matrix $\widetilde{C}_t = \widetilde{C}(\Omega, \Delta_1)$. Each matrix $\widetilde{C}(\Omega, \Delta_1)$ can be viewed as a choice of $\widetilde{C}(\Omega, \Lambda_1)$ for any lattice Λ_1 containing Δ_1 with $\mathrm{rank}_{\mathbb{Z}}\Lambda_1 = j - r_0 - r_2$, via

$$\Delta_1 = \Lambda_1 G \begin{pmatrix} I_t & \\ & 0 \end{pmatrix}.$$

As remarked in Lemma 7.3, the number of such Λ_1 (equivalently, G or $\widetilde{G}$) is $\beta_p(t,r)$. Hence the effect of replacing the operator $T_j(p^2)$ by $\widetilde{T}_j(p^2)$ is to remove the condition that Y_1 have full rank modulo p.

Thus

$$\begin{aligned} F|\widetilde{T}_j(p^2)(\tau) &= \sum_{\Omega,\Lambda_1,T} p^{k(r_2-r_0)} c(T) e\{T[\widetilde{D}\widetilde{C}^{-1}]\tau\} \times \\ &\qquad \sum_{\substack{Y_0 \pmod{p^2} \\ Y_1,Y_2,Y_3 \pmod{p}}} e\left\{ T \begin{pmatrix} Y_0/p^2 & Y_2/p & Y_3/p & 0 \\ {}^tY_2/p & Y_1/p & & \\ {}^tY_3/p & & & \\ 0 & & & I \end{pmatrix} \right\}. \end{aligned} \tag{7}$$

Evaluating the character sum gives us

$$F|\widetilde{T}_j(p^2)(\tau) = \sum_{\Omega,\Lambda_1,T} p^{E_j(\Lambda,\Omega)} c(T) e\{T[\widetilde{D}\widetilde{C}^{-1}]\tau\},$$

where Ω and Λ_1 vary as above, and $T \in \mathbf{U}_0 = \mathbf{U}_0(r_0,r_2)$, i.e. $T = T_0 + {}^tT_0$ where T_0 is upper triangular and satisfies

$$T_0 \equiv \begin{pmatrix} 0 & p* & p* & * \\ & p* & & \\ & & * & \\ & & & * \end{pmatrix} \pmod{p^2}.$$

Fix r_0, r_2 (i.e., fix $\widetilde{D}$). For $G, G' \in \mathrm{GL}_n(\mathbb{Z})$, $\Lambda G\widetilde{D}^{-1} = \Lambda G'\widetilde{D}^{-1}$ if and only if $G' \in G\mathbf{U}$ where

$$\mathbf{U} = \mathrm{GL}_n(\mathbb{Z}) \cap \begin{pmatrix} \mathbb{Z}^{r_0,r_0} & \mathbb{Z}^{r_0,n-r_0-r_2} & \mathbb{Z}^{r_0,r_2} \\ p\mathbb{Z}^{n-r_0-r_2,r_0} & \mathbb{Z}^{n-r_0-r_2,n-r_0-r_2} & \mathbb{Z}^{n-r_0-r_2,r_2} \\ p^2\mathbb{Z}^{r_2,r_0} & p\mathbb{Z}^{r_2,n-r_0-r_2} & \mathbb{Z}^{r_2,r_2} \end{pmatrix}.$$

Thus the sublattices $\Omega = p\Lambda_0 \oplus \Delta_1 \oplus \frac{1}{p}\Lambda_2$ with $\{\Lambda : \Omega\} = (r_2, n-r_0-r_2, r_0)$ are in one-to-one correspondence with coset representatives $\{G\}$ for $\mathrm{GL}_n(\mathbb{Z})/\mathbf{U}$. Similarly, for fixed such Ω, the dimension $r_1 = j - r_0 - r_2$ subspaces $\overline{\Lambda}_1$ of $(\Omega \cap \Lambda)/p(\Omega + \Lambda)$ are in one-to-one correspondence with coset representatives $\{H\}$ for $\mathbf{W}/\mathbf{W}'$, where

$$\mathbf{W} = \mathrm{GL}_n(\mathbb{Z}) \cap \left\{ \begin{pmatrix} I_{r_0} & & \\ & * & \\ & & I_{r_2} \end{pmatrix} \right\},$$

and

$$\mathbf{W}' = \mathrm{GL}_n(\mathbb{Z}) \cap \left\{ \begin{pmatrix} I_{r_0} & & \\ & G_1 & \\ & & I_{r_2} \end{pmatrix},\ G_1 \in \begin{pmatrix} \mathbb{Z}^{r_1,r_1} & \mathbb{Z}^{r_1,n-j} \\ p\mathbb{Z}^{n-j,r_1} & \mathbb{Z}^{n-j,n-j} \end{pmatrix} \right\}.$$

(So for these choices for r_0, r_2, the matrices $\widetilde{C}$ are of the form GH.) Thus for fixed Ω and G such that $\Omega = \Lambda G \widetilde{D}^{-1}$ we have

$$\sum_{\substack{\Lambda_1 \\ T \in \mathbf{U}_0}} c(T) e\{T[\widetilde{D}\widetilde{C}^{-1}(\Omega, \Lambda_1)]\tau\} = \sum_{\substack{H \\ T \in \mathbf{U}_0}} c(T) e\{T[\widetilde{D} H^{-1} G^{-1}]\tau\}.$$

Note that $\widetilde{D} H^{-1} = H^{-1} \widetilde{D}$ and recall that $c(T) = c(T[H])$; thus replacing T by $T[H]$ we have

$$\sum_{\substack{\Lambda_1 \\ T \in \mathbf{U}_0}} c(T) e\{T[\widetilde{D}\widetilde{C}^{-1}(\Omega, \Lambda_1)]\tau\} = \sum_{T} \sum_{\substack{H \\ T[H^{-1}] \in \mathbf{U}_0}} c(T) e\{T[\widetilde{D} G^{-1}]\tau\}.$$

Identify T with Ω; so by our choice of G, $T[\widetilde{D}G^{-1}]$ is identified with Λ. Then, presuming T and $T[\widetilde{D}]$ are both even integral, the number of H with $T[H^{-1}] \in \mathbf{U}_0$ is $\alpha_j(\Lambda, \Omega)$.

Thus for k even,

$$F|\widetilde{T}_j(p^2)(\tau) = \sum_{\substack{\mathrm{cls}\Lambda \\ p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda}} c(\Omega) \alpha_j(\Lambda, \Omega) e^*\{\Lambda\tau\}$$

(recall that $c(\Omega) = 0$ unless Ω is even integral).

When k is odd, the argument is the same. Notice that we can always choose our lattices $C = C(\Lambda, \Omega)$ to have determinant 1. Then the orientation σ on Λ induces a compatible orientation, which we again call σ, on Ω (recall that $\Omega = \Lambda C D$). Were we to take C with $\det C = -1$ then C would induce on Ω the orientation $\overline{\sigma}$, but we would also pick up a factor of $-1 = \det C$. Since $c(\Omega, \sigma) = -c(\Omega, \overline{\sigma})$, the result is independent of our choice of $\det C$. ▌

Similarly, Proposition 3.1 provides an alternate proof of Maass' result [4].

THEOREM 4.2. *With F as above,*

$$F|T(p)(\tau) = \sum_{\mathrm{cls}(\Lambda)} \sum_{p\Lambda \subseteq \Omega \subseteq \Lambda} p^{E(\Lambda,\Omega)} c(\Omega^{1/p}) e^*\{\Lambda\tau\},$$

where where $E(\Lambda, \Omega) = m(1)k + m(p)(m(p)+1)/2 - n(n+1)/2$, $m(a) = \mathrm{mult}_{\{\Lambda:\Omega\}}(a)$, and $\Omega^{1/p}$ denotes that lattice Ω scaled by $\frac{1}{p}$ (so when Q is *the quadratic form associated to* Ω, $\frac{1}{p}Q$ *is the quadratic form associated to* $\Omega^{1/p}$*).*

The next proposition gives us a tool to explicitly evaluate the terms $\alpha_j(\Lambda, \Omega)$ in the statement of Theorem 4.1.

Proposition 4.1. *Let* V *be a* d*-dimensional space over* $\mathbb{Z}/p\mathbb{Z}$*. Set* $r = \dim R$ *where* $R = \mathrm{rad}V$*. Then the number of dimension* l *totally isotropic subspaces of* V *is*

$$\sum_{t=0}^{r} \beta_p(r,t) \prod_{i=0}^{l-t-1} p^{r-t} \frac{\phi_p(d-r-2i, \mu)}{p^{l-t-i}-1},$$

where $\phi_p(d, \mu)$ *is the number of isotropic vectors in a regular space* W *of dimension* d *and* $\mu = 1$ *if* W *is hyperbolic and equals* -1 *otherwise. In particular,*

$$\phi_p(d, \mu) = \begin{cases} p^{d-1} - 1 & \text{if } 2 \not| d \\ (p^{d/2} - \mu)(p^{d/2-1} + \mu) & \text{if } 2|d, \end{cases}$$

Proof. The proof is simple combinatorics. For a fixed t, $0 \le t \le r$, the β_p term counts the number of t-dimensional subspaces R_t of R. The second product counts the number of $l-t$ dimensional totally isotropic subspaces of V/R_t which are independent of R/R_t. (Andrianov [1, Prop A.2.14] does a similar computation in the case V is regular. See also Artin [2, pp143–146].) ▮

Remark 4. 1. To evaluate $\alpha_j(\Lambda, \Omega)$, we first observe that

$$\begin{aligned} \Lambda &= \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2 \\ \Omega &= p\Lambda_0 \oplus \Lambda_1 \oplus \frac{1}{p}\Lambda_2 \end{aligned}$$

and

$$\Lambda \cap \Omega / p(\Lambda + \Omega) \simeq \Lambda_1 / p\Lambda_1.$$

So, in the proposition, take $V = \Lambda_1/p\Lambda_1$ and $\ell = m(1) - n + j$.

5. A SIMPLE RELATION BETWEEN OPERATORS

PROPOSITION 5.1. *With $\widetilde{T}_j(p^2)$ as defined in Definition 4.1 (so $\widetilde{T}_n(p^2)$ is the so-called average Hecke operator), we have*

$$\widetilde{T}_n(p^2) = T(p)^2 - \sum_{0\le j<n} p^{k(n-j)+j(j+1)/2-n(n+1)/2}\widetilde{T}_j(p^2).$$

Proof. With

$$F(\tau) = \sum_{\mathrm{cls}(\Lambda)} c(\Lambda)e^*\{\Lambda\tau\},$$

and T any Hecke operator, we write

$$F|T(\tau) = \sum_{\mathrm{cls}(\Lambda)} c|T(\Lambda)e^*\{\Lambda\tau\}.$$

By Theorem 4.2,

$$c|T(p)^2(\Lambda) = \sum_{p\Lambda\subseteq\Lambda'\subseteq\frac{1}{p}\Lambda} \left(\sum_{\substack{p(\Lambda+\Lambda')\subseteq\Omega\subseteq\Lambda\cap\Lambda' \\ \Omega^{1/p}\ \text{integral}}} p^{E(\Lambda,\Omega)+E(\Omega,p\Lambda')} \right) .c(\Lambda')$$

Fix Λ' and set $m_0 = \mathrm{mult}_{\{\Lambda:\Lambda'\}}(1/p)$, $m_1 = \mathrm{mult}_{\{\Lambda:\Lambda'\}}(1)$, $m_2 = \mathrm{mult}_{\{\Lambda:\Lambda'\}}(p)$. $\Omega^{1/p}$ being integral is equivalent to $\Omega/p(\Lambda+\Lambda')$ being a totally isotropic subspace of $(\Lambda\cap\Lambda')/p(\Lambda+\Lambda')$. We have $\dim(\Lambda\cap\Lambda')/p(\Lambda+\Lambda') = m_1$; then for $j \le n$, the number of Ω in the second summation above with $\dim\Omega/p(\Lambda+\Lambda') = m_1-(n-j)$ is $\alpha_j(\Lambda,\Lambda')$. For such Ω,

$$\begin{aligned} E(\Lambda,\Omega) &= (m_0+m_1-n+j)k + (m_2+n-j)(m_2+n-j+1)/2, \\ E(\Omega,p\Lambda') &= (m_0+n-j)k + (m_2+m_1-n+j)(m_2+m_1-n+j+1)/2. \end{aligned}$$

By simple algebra, the sum of these two expressions is

$$k(n-j) + j(j+1)/2 - n(n+1)/2 + E_j(\Lambda,\Lambda').$$

This and Theorem 4.1 complete this proof. ▮

6. SIEGEL MODULAR FORMS WITH LEVEL

Say that F is a Siegel moudlar form with level N and character χ modulo N; that is, F satisfies the transformation law

$$F|M = \chi(\det D_M)F,$$

for any $M = \begin{pmatrix} A_M & B_M \\ C_M & D_M \end{pmatrix} \in \Gamma_0(N)$, where

$$\Gamma_0(N) = \left\{ M \in \mathrm{Sp}_n(\mathbb{Z}) : \ M \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}$$

(and $\lim_{\tau \to i\infty} F(\tau) < \infty$). Then in the definition of the Hecke operators we take $\Gamma = \Gamma_0(N)$, $\Gamma' = \underline{\delta}\Gamma\underline{\delta}^{-1}$ and for the operator T associated to $\underline{\delta}$ we define

$$F|T = (\det \underline{\delta})^{k/2} \sum_{\gamma} \overline{\chi}(\det D_\gamma) F|\underline{\delta}^{-1}\gamma$$

where γ runs over a set of representatives for $(\Gamma' \cap \Gamma)\backslash\Gamma$. Since $F|\underline{\delta}^{-1}\gamma' = \chi(\det D_{\gamma'})F|\underline{\delta}^{-1}$ for $\gamma' \in \Gamma'$, $F|T$ is well-defined. (As before, we include an additional normalizing factor of $p^{-n(n+1)/2}$ in the definition of $T(p)$.)

For $1 \le j \le n$, define

$$\widetilde{T}_j(p^2) = p^{j(k-n-1)} \sum_{0 \le t \le j} \chi(p^{j-t}) \beta_p(n-t, j-t) T_t(p^2).$$

(So when $p|N$, $\widetilde{T}_j(p^2) = p^{j(k-n-1)} T_j(p^2)$.)

Theorem 6.1. *Let F be a Siegel modular form of level N, character χ with coefficients $c(\Lambda)$.*

(1)The Λth coefficient of $F|T(p)$ is

$$\sum_{p\Lambda \subseteq \Omega \subseteq \Lambda} \chi([\Omega : p\Lambda]) p^{E(\Lambda,\Omega)} c(\Lambda^{1/p}).$$

(2)For each $1 \le j \le n$, the Λth coefficient of $F|\widetilde{T}_j(p^2)$ is

$$\sum_{p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda} \chi(p^{j-n}[\Omega : p\Lambda]) p^{E_j(\Lambda,\Omega)} \alpha_j(\Lambda, \Omega) c(\Omega).$$

Proof. We first consider $F|\widetilde{T}_j(p^2)$. To find the coset representatives for $T_j(p^2)$, we essentially proceed as for level 1 but we point out where there are differences in the construction of these representatives.

Proceed as before through Step 4, constructing the matrices C_2 and X such that $M_j \begin{pmatrix} C_2 & \\ & {}^tC_2^{-1} \end{pmatrix} X^{-1}$ has the same form as in (1).

First suppose $p|N$. Then X takes the form (because $r_0 = j$) given by

$$X = \left(\begin{array}{cc|cc} I_j & & Y_0 & Y_3 \\ & I_{n-j} & {}^tY_3 & 0 \\ \hline & & I_j & \\ & & & I_{n-j} \end{array}\right) \in \Gamma_0(N)$$

and

$$F|T_j(p^2) = \sum_{\Omega,Y_0,Y_3} F\left|\begin{pmatrix} \delta_j^{-1} & \\ & \delta_j \end{pmatrix} X \begin{pmatrix} C^{-1} & \\ & C \end{pmatrix}\right.,$$

where Ω varies over lattices such that $p\Lambda \subseteq \Omega \subseteq \Lambda$ with $[\Lambda : \Omega] = p^j$, $\Omega = \Lambda C \begin{pmatrix} pI_j & \\ & I \end{pmatrix}$, Y_3 varies over all $j \times n-j$ matrices modulo p, and Y_0 varies over all $j \times j$ symmetric matrices modulo p^2. Thus

$$F|T_j(p^2) = p^{j(-k+n+1)} \sum_{\Lambda,\Omega} c(\Omega) e\{\Lambda\tau\},$$

where Λ runs over all isometry classes of integral rank n lattices with positive semi-definite quadratic form, and Ω runs over all lattices such that $p\Lambda \subseteq \Omega \subseteq \Lambda$, $[\Lambda : \Omega] = p^j$. Hence, the Λth coefficient of $F|\widetilde{T}_j(p^2)$ is as claimed.

Now suppose $p \not| N$. Then constructing X' as in (2) is a bit more delicate, because we need $X' \in \Gamma_0(N)$, i.e., $U \equiv 0 \pmod{N}$. We know $(D_j, -Y')$ is a coprime symmetric pair, so $(D_j, -NY')$ is as well. Thus there are matrices U', V such that $\begin{pmatrix} D_j & -NY' \\ U' & V \end{pmatrix} \in \mathrm{Sp}_j(\mathbb{Z})$, and so with $U = NU'$,

$$X' = \left(\begin{array}{cc|cc} D_j & & -Y' & -Y_3' \\ & I_{n-j} & -{}^tY_3' & \\ \hline U & & V & -UY_3' \\ & 0 & & I_{n-j} \end{array}\right) \in \Gamma_0(N).$$

Write $X' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$; then since X' is sympletic, $A'{}^tD' \equiv I \pmod{N}$. Thus $\overline{\chi}(\det D') = \chi(\det A') = \chi(p^{r_1+2r_2})$. We know $r_0 + r_1 + r_2 = j$, so $\overline{\chi}(\det D') = \chi(p^{j-r_0+r_2})$. So

$$F|T_j(p^2) = \sum_{C_2,X'} \chi(p^{j-r_0+r_2}) F\left|\underline{\delta}_j^{-1} X' \begin{pmatrix} C_2^{-1} & \\ & {}^tC_2 \end{pmatrix}\right.,$$

where C_2, X' vary as before.

Let $X'' = X'X^{-1}$. Then writing $\underline{\delta}_j^{-1} X'' \underline{\delta}_j$ as $\begin{pmatrix} A'' & B'' \\ C'' & D'' \end{pmatrix}$, we see that $C'' \equiv 0 \pmod N$, $A'' = I$ and so $D'' \equiv I \pmod N$. Hence, $F|\underline{\delta}_j^{-1} X' = F|\underline{\delta}_j^{-1} X'' X = F|\underline{\delta}_j^{-1} X$. Therefore,

$$F|T_j(p^2) = \sum_{\Omega,\Lambda_1,B'} \chi(p^{j-r_0+r_2}) F \left| \begin{pmatrix} \delta_j^{-1} & \\ & \delta_j \end{pmatrix} \begin{pmatrix} D & B' \\ & D^{-1} \end{pmatrix} \begin{pmatrix} C^{-1} & \\ & {}^tC \end{pmatrix} \right.,$$

where Ω, Λ_1, B' vary as in Corollary 2.1, $D = D(\Omega)$ and $C = C(\Omega, \Lambda_1)$.

Next we proceed as before to evaluate the action of $T_j(p^2)$ on the Fourier coefficients of F. The argument is virtually identical except for the introduction of the character in the above formula and in the definition of $\widetilde{T}_j(p^2)$. Consequently,

$$\begin{aligned} F|\widetilde{T}_j(p^2)(\tau) \;=\; & p^{j(k-n+1)} \sum_{\Omega,\Lambda_1,T} \chi(p^{j-r_0+r_2}) p^{k(r_2-r_0)} c(T) e\{T[\widetilde{D}\widetilde{C}^{-1}]\tau\} \times \\ & \sum_{\substack{Y_0 \pmod{p^2} \\ Y_1,Y_2,Y_3 \pmod p}} e\left\{ T \begin{pmatrix} Y_0/p^2 & Y_2/p & Y_3/p & 0 \\ {}^tY_2/p & Y_1/p & & \\ {}^tY_3/p & & & \\ 0 & & & I \end{pmatrix} \right\}. \end{aligned}$$

(Here the notation is as in (7).) Hence, after evaluating the character sum and identifying T with Ω, we find the Λth coefficient of $F|\widetilde{T}_j(p^2)$ is as claimed.

Similarly,

$$F|T(p) = p^{n(k-n-1)/2} \sum_{\Omega,B'} \chi(\det D) F \left| \begin{pmatrix} \frac{1}{p} I & \\ & I \end{pmatrix} \begin{pmatrix} D & B' \\ & D^{-1} \end{pmatrix} \begin{pmatrix} C^{-1} & \\ & {}^tC \end{pmatrix} \right.,$$

where Ω, B', D, C are as in Proposition 3.1. Here $\det D = p^{n-r}$ when $[\Lambda : \Omega] = p^r$. Consequently, the Λth coefficient of $F|T(p)$ is

$$\sum_{p\Lambda \subseteq \Omega \subseteq \Lambda} \chi([\Omega : p\Lambda]) p^{E(\Lambda,\Omega)} c(\Omega^{1/p}),$$

as claimed. ∎

7. LEMMAS

Lemma 7.1. *Let $M \in \Gamma$ and let M_j denote the top j rows of M ($j \le n$). Suppose $M_j = (A_0, pA_1 | p^2B_0, pB_1)$ with A_0, B_0 integral $j \times j$ matrices, A_1, B_1 integral $j \times n-j$ matrices. Then $M \in \Gamma'$.*

Proof. Write $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_0 & pA_1 & p^2B_0 & pB_1 \\ A_1' & A_2 & B_1' & B_2 \\ C_0 & C_1 & D_0 & D_1 \\ C_1' & C_2 & D_1' & D_2 \end{pmatrix}$. Since (A, B) is a coprime pair, we must have $\mathrm{rank}_p A_0 = j$. Also, $A^tB \equiv \begin{pmatrix} 0 & A_0{}^tB_1' \\ 0 & * \end{pmatrix}$ mod p and A^tB is symmetric; hence $B_1' \equiv 0 \bmod p$. Finally, $I = A^tD - B^tC \equiv \begin{pmatrix} * & A_0{}^tD_1' \\ & * \end{pmatrix}$ mod p, so $D_1' \equiv 0 \bmod p$. ▌

Lemma 7.2. *Let $M \in \Gamma$. Suppose $M_j = (A_0, pA_1 | B_0, B_1)$ with A_0, B_0 integral $j \times r$ matrices, $\mathrm{rank}_p A_0 = r$, and A_1, B_1 integral $j \times n-r$ matrices. Then $B_0 \subseteq \mathrm{span}_p A_0$ and $\mathrm{rank}_p(A_0, B_1) = j$.*

Proof. Since $\mathrm{rank}_p A_0 = r$, we can find $E \in GL_j(\mathbb{Z})$ so that $EA_0 = \begin{pmatrix} I_r \\ 0 \end{pmatrix}$. Then

$$M' = \left(\begin{array}{cc|cc} E & & & \\ & I_{n-j} & & \\ \hline & & {}^tE^{-1} & \\ & & & I_{n-j} \end{array}\right) M \in \Gamma$$

with $M_j' = (EA_0, pEA_1, EB_0, EB_1)$. We know $EB_0{}^tA_0{}^tE$ is symmetric, and $EB_0{}^tA_0{}^tE \equiv (EB_0, 0) \bmod p$. Hence the bottom $j - r$ rows of EB_0 must be 0 (modulo p), so $(EA_0, EB_0) \equiv \begin{pmatrix} I_r & * \\ 0 & 0 \end{pmatrix}$ mod p. Thus $\mathrm{rank}_p(A_0, B_0) = \mathrm{rank}_p(EA_0, EB_0) = r = \mathrm{rank}_p A_0$, i.e. $B_0 \subseteq \mathrm{span}_p A_0$.

Also, $M \in GL_{2n}(\mathbb{Z})$ hence $\mathrm{rank}_p M = 2n$. Thus $\mathrm{rank}_p(A_0, pA_1, B_0, B_1) = \mathrm{rank}_p M_j = j$, and so $\mathrm{rank}_p(A_0, B_1)$ must be j. ▌

Lemma 7.3. *Let V be an r dimensional space over $\mathbb{Z}/p\mathbb{Z}$ (p prime) with an ordered basis $\{v_1, \ldots, v_r\}$. Fix a nonnegative integer $s < r$. For each dimension s subspace U of V, fix a matrix $G_U \in GL_r(\mathbb{Z}/p\mathbb{Z})$ so that $VG_U^{-1} \begin{pmatrix} 0 & \\ & I_s \end{pmatrix} = U$. Then for symmetric $t = r - s$ dimensional matrices*

$M, N \in GL_t(\mathbb{Z}/p\mathbb{Z})$ and U, W dimension s subspaces of V,

$$ {}^tG_U \begin{pmatrix} M & \\ & 0 \end{pmatrix} G_U = {}^tG_W \begin{pmatrix} N & \\ & 0 \end{pmatrix} G_W $$

if and only if $U = W$ and $M = N$.

Proof. Suppose $A = {}^tG_U \begin{pmatrix} M & \\ & 0 \end{pmatrix} G_U = {}^tG_W \begin{pmatrix} N & \\ & 0 \end{pmatrix} G_W$ for some (dimension s) U,W and (nonsingular) M, N. Thus A defines a quadratic form Q on V relative to the basis $\{v_1, \ldots, v_r\}$. We have $VG_U^{-1} = U' \oplus U$, and relative to this decomposition of V, ${}^tG_U^{-1} A G_U^{-1} = \begin{pmatrix} M & \\ & 0 \end{pmatrix}$ represents Q. Hence U is the radical of V relative to Q. Similarly, $VG_W^{-1} = W' \oplus W$, and ${}^tG_W^{-1} A G_W^{-1} = \begin{pmatrix} N & \\ & 0 \end{pmatrix}$ represents Q relative to this decomposition. Hence W is the radical of V relative to Q. This means that $U = W$ and so $G_U = G_W$; hence $M = N$.

The converse is trivial. ∎

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