AUTOMORPHIC FORMS AND SUMS OF SQUARES OVER FUNCTION FIELDS

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Abstract. We develop some of the theory of automorphic forms in the function field setting. As an application, we find formulas for the number of ways a polynomial over a finite field can be written as a sum of k squares, $k \geq 2$. As a consequence, we show every polynomial can be written as a sum of 4 squares. We also show, as in the classical case, that these representation numbers are asymptotic to the Fourier coefficients of the basic Eisenstein series.

Given a finite field $\mathbb{F}_q$ with q odd, we want to determine how many ways a polynomial in $\mathbb{F}_q[T]$ can be written as a sum of k squares. For $k \geq 3$ (or $k = 2$, -1 not a square in $\mathbb{F}_q$), the sum of k squares is an indefinite quadratic form, so there are infinitely many ways to write any polynomial over $\mathbb{F}_q$ as a sum of k squares. Hence we refine our question; we seek a formula for the restricted representation numbers $r_k(\alpha, m)$ where $r_k(\alpha, m)$ denotes the number of ways a polynomial α of degree n can be written as a sum of squares of k polynomials whose degrees are strictly bounded by $m > \frac{n}{2}$.

In the 1940's Carlitz and Cohen studied this problem with $n = 2m - 2$ or $m = 2n - 3$. Using the circle method, Cohen obtained exact formulas ([C1], [C2]) but it is not clear whether these numbers are nonzero. More recently Serre showed these numbers are nonzero for $k = 3$ (see [E-H]). In [M-W] elementary methods were used to give formulas in terms of Kloosterman sums (with no restriction as to the relation between m and n). Then Car [Car] refined the use of the circle method to obtain asymptotic formulas, showing that virtually every diagonal quadratic form represents any given polynomial α provided m is sufficiently large (here $\deg\alpha = 2m - 2$ or $2m - 3$).

In this paper we develop the theory of automorphic forms of integral and half-integral weights on the "upper half-plane" $\mathfrak{H} = SL_2(\mathbb{F}_q((\frac{1}{T})))/SL_2(\mathbb{F}_q[[\frac{1}{T}]])$ under the action of the full modular group $\Gamma = SL_2(\mathbb{F}_q[T])$. (Note that this is not the upper half-plane used by Weil; instead of SL, Weil used GL to define $\mathfrak{H}$, and unlike the classical case, these two definitions of $\mathfrak{H}$ are not equivalent. Furthermore, the

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theta series we use does not transform uniformly on Weil's upper half-plane.) Using Poincare series and a power of the (normalized) theta series presented in Proposition 2.13 of [H-R], we very easily compute $r_k(\alpha, m)$, obtaining generalizations of Cohen's results. Again, it is not obvious from these formulas that these numbers are nonzero when $m < \deg\alpha \le 2m-2$. As an application of the spectral theory developed herein, we show that, as in the classical case, the main term of $r_k(\alpha, m)$ is given by the Fourier coefficient of the Eisenstein series at $s = k/4$ (where we have removed the singularity if $k = 3$ or 4). From this we conclude that $r_k(\alpha, m) \neq 0$ for m sufficiently large, subject only to the necessary condition that $m > (\deg\alpha)/2$. This gives confirmation to the belief that Eisenstein series closely approximate theta series, whatever the setting.

We begin by reviewing the definitions and Fourier expansions of Eisenstein series (see Definition 2.5 and Proposition 2.11 [H-R] for the case of half-integral weight) and by developing some spectral theory (cf. §3.7 of [T]) to obtain the Fourier expansions of a set of Poincare series that span the space of automorphic forms. The discrete nature of $\mathfrak{H}$ in the function field setting makes these computations significantly easier than in the classical setting. Indeed, we use the following two trivial observations to great advantage:

(1) $\mathcal{D} = \left\{ \begin{pmatrix} T^m & 0 \\ 0 & T^{-m} \end{pmatrix} : \ m \ge 0 \right\}$ is a fundamental domain for $\Gamma \backslash \mathfrak{H}$, and

(2) A function on $\mathcal{D}$ is supported only on its 0th Fourier coefficient.

It immediately follows that there are no nonzero cusp forms for Γ, and that the functional equation satisfied by the 0th coefficient of the Eisenstein series E_s implies the functional equation for E_s. Furthermore, for each $z_0 \in \mathcal{D}$, it is easy to construct Poincare series P_{z_0} whose support in $\mathcal{D}$ consists only of $\{z_0\}$. Thus by observation (2) one only needs the 0th coefficient of an automorphic form restricted to $\mathcal{D}$, $f|_{\mathcal{D}}$, to write the automorphic form f as a sum of the Poincare series, yielding formulas for all the coefficients of f (Theorem 2). Applying this to $\theta^k(z)$, the kth power of the (normalized) analogue of the classical theta function, we obtain precise formulas for $r_k(\alpha, m)$. In particular (Corollary 2), when $\alpha \neq 0$ and $m > \deg\alpha$,

$$r_k(\alpha, m) = \begin{cases} q^{m(k-2)+1}(1 - \epsilon q^{-k/2})\sigma_{1-k/2}(\alpha, \epsilon) & \text{if } k \text{ is even,} \\ q^{m(k-2)+1}(1 - q^{1-k})F_{(k-1)/2}(\alpha, \epsilon) & \text{if } k \text{ is odd} \end{cases}$$

where $\epsilon = \pm 1$, σ is the divisor function (twisted by ϵ), and F is (essentially) the Dirichlet series that appears in the Fourier coefficients of the metaplectic Eisenstein series (see Proposition 2.11 of [H-R]). From this we find that any polynomial α can be written as a sum of 4 squares with degrees bounded by 1/2 the degree of α. Finally, since it is not transparent that $r_k(\alpha, m) \neq 0$ for $m \leq \deg\alpha \leq 2m-2$, we estimate the L^2-function $\theta^k - E_{k/4}$ to show that, when $q \neq 3$ of $k \geq 5$, $r_k(\alpha, m) \neq 0$ for m sufficiently large and is asymptotic to the Fourier coefficient of the Eisenstein series, as in the classical case (Theorem 4).

We note here that together with techniques used in [W] and [H-W], we expect the methods used herein can be extended to yield precise formulas for the representation numbers of any quadratic form on a function field.

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§**1. Preliminaries.** Let $\mathbb{F} = \mathbb{F}_q$ be a finite field of odd order $q = p^r$ and let T be an indeterminate. Set $\mathbb{A} = \mathbb{F}[T]$ and $\mathbb{K} = \mathbb{F}(T)$. Consider the valuation $|\cdot| = |\cdot|_\infty$ defined by

$$|\alpha/\beta| = q^{\deg\alpha - \deg\beta} \qquad \text{where} \qquad \alpha, \beta \in \mathbb{A};$$

we agree that $|0| = -\infty$. Now let $\mathbb{K}_\infty$ denote the completion of $\mathbb{K}$ with respect to the valuation $|\cdot|$; so $\mathbb{K}_\infty = \mathbb{F}((\frac{1}{T}))$. We extend the function deg to $\mathbb{K}_\infty$ in the obvious way; so $\deg y = \mathrm{ord}_\pi y$ where $\pi = \frac{1}{T}$. Let $\mathcal{O}_\infty = \{\alpha \in \mathbb{K}_\infty : \ |\alpha| \leq 1 \ \} = \mathbb{F}[[\frac{1}{T}]]$. Then $SL_2(\mathcal{O}_\infty)$ is the maximal compact subgroup of $SL_2(\mathbb{K}_\infty)$; we define an "upper half-plane" $\mathfrak{H}$ by

$$\mathfrak{H} = SL_2(\mathbb{K}_\infty)/SL_2(\mathcal{O}_\infty).$$

The group $\Gamma = SL_2(\mathbb{A})$ acts on $\mathfrak{H}$ by left multiplication. We have the following elementary lemma.

Lemma 1.

(a) (Iwasawa decomposition) The set

$$\left\{ \begin{pmatrix} T^m & xT^{-m} \\ 0 & T^{-m} \end{pmatrix} : \ m \in \mathbb{Z}, \ x \in T^{2m+1}\mathbb{A} \ \right\}$$

is a complete set of representatives for $\mathfrak{H}$.

(b) (Fundamental domain) The set

$$\left\{ \begin{pmatrix} T^m & 0 \\ 0 & T^{-m} \end{pmatrix} : \ m \geq 0 \ \right\}$$

is a complete set of representatives for $\Gamma \backslash \mathfrak{H}$.

Proof. First, to establish (a), take $z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{K}_\infty)$. Then

$$z \equiv \begin{cases} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & 1 \end{pmatrix} & \text{if } \deg c \leq \deg d, \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{d}{c} & 1 \\ -1 & 0 \end{pmatrix} & \text{if } \deg d < \deg c \end{cases}$$

where $z \equiv z'$ means z and z' represent the same coset of $\mathfrak{H}$. Thus z is equivalent to a matrix of the form

$$\begin{pmatrix} w & x' \\ 0 & w^{-1} \end{pmatrix}$$

with $w, x' \in \mathbb{K}_\infty$, $w \neq 0$. Now, $w = T^m u$ for some $m \in \mathbb{Z}$ and $u \in {\mathcal{O}_\infty}^\times$. So

$$z \equiv \begin{pmatrix} w & x' \\ 0 & w^{-1} \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} \equiv \begin{pmatrix} T^m & x'' \\ 0 & T^{-m} \end{pmatrix}$$

for some $x'' \in \mathbb{K}_\infty$. Writing x'' as $T^{-m}(x + T^{2m}v)$ where $x \in T^{2m+1}\mathbb{A}$, $v \in \mathcal{O}_\infty$, we see that

$$z \equiv \begin{pmatrix} T^m & x'' \\ 0 & T^{-m} \end{pmatrix} \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} T^m & xT^{-m} \\ 0 & T^{-m} \end{pmatrix}.$$

Now suppose $z \equiv \begin{pmatrix} T^m & xT^{-m} \\ 0 & T^{-m} \end{pmatrix} \equiv \begin{pmatrix} T^{m'} & x'T^{-m'} \\ 0 & T^{-m'} \end{pmatrix}$ where $m, m' \in \mathbb{Z}$, $x \in T^{2m+1}\mathbb{A}$, $x' \in T^{2m'+1}\mathbb{A}$. Then

$$\begin{pmatrix} T^m & xT^{-m} \\ 0 & T^{-m} \end{pmatrix}^{-1} \begin{pmatrix} T^{m'} & x'T^{-m'} \\ 0 & T^{-m'} \end{pmatrix} = \begin{pmatrix} T^{m'-m} & (x-x')T^{-m'-m} \\ 0 & T^{m-m'} \end{pmatrix} \in SL_2(\mathcal{O}_\infty).$$

Thus $T^{m'-m}, T^{m-m'} \in \mathcal{O}_\infty$, which implies $m = m'$. So $T^{-m'-m}(x'-x) = T^{-2m}(x'-x) \in \mathcal{O}_\infty$. We know $x, x' \in T^{2m+1}\mathbb{A}$, so

$$T^{-2m}(x'-x) \in \mathcal{O}_\infty \cap T\mathbb{A} = \{0\}.$$

Hence $x = x'$, and the representation is unique.

From (a), the proof of (b) follows easily from the observations that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\left\{ \begin{pmatrix} u & a \\ 0 & u^{-1} \end{pmatrix} : a \in \mathbb{A}, u \in \mathbb{F}^\times \right\}$ generate Γ, and for $z = \begin{pmatrix} T^{-m} & xT^m \\ 0 & T^m \end{pmatrix} \in \mathfrak{H}$, $x \in T^{1-2m}\mathbb{A}$,

$$-\frac{1}{z} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z \equiv \begin{cases} \begin{pmatrix} T^m & 0 \\ 0 & T^{-m} \end{pmatrix} & \text{if } x = 0, \\ \begin{pmatrix} T^{n-m} & -\frac{1}{x}T^{m-n} \\ 0 & T^{m-n} \end{pmatrix} & \text{if } x \neq 0, \deg x = -n. \end{cases} \qquad \square$$

§**2. Fourier Series.** Suppose $f : \mathfrak{H} \to \mathbb{C}$ is invariant under the action of Γ_∞ on $\mathfrak{H}$ where $\Gamma_\infty = \left\{ \begin{pmatrix} u & \alpha \\ 0 & u^{-1} \end{pmatrix} : \alpha \in \mathbb{A}, u \in \mathbb{F}^x \right\}$. Then for a fixed $y = T^m$, we can consider f_y, defined by $f_y(x) \equiv f\left(\begin{pmatrix} y & xy^{-1} \\ 0 & y^{-1} \end{pmatrix}\right)$, as a function on the finite abelian (additive) subgroup $T^{1+2m}\mathbb{A}/\mathbb{A}$ of $\mathbb{K}_\infty/\mathbb{A}$. Thus elementary Fourier analysis shows that f can be written as a Fourier series as follows: For $x = \sum_{j=-\infty}^{N} x_j T^j \in \mathbb{K}_\infty$, let $\mathrm{e}\{x\} = \exp(2\pi i \mathrm{Tr}(x_1)/p)$ where Tr denotes the trace from $\mathbb{F}$ to $\mathbb{Z}/p\mathbb{Z}$. Then we have

$$f\left(\begin{pmatrix} y & xy^{-1} \\ 0 & y^{-1} \end{pmatrix}\right) = \sum_{\beta \in T^2\mathbb{A}} c_\beta(f, y) e\{\beta x\},$$

where $c_\beta(f, T^m) = \chi_{\mathcal{O}_\infty}(\beta T^{2m}) p^{1+2m} \sum_{x \in T^{1+2m}\mathbb{A}/\mathbb{A}} f\left(\begin{pmatrix} T^m & xT^{-m} \\ 0 & T^{-m} \end{pmatrix}\right) e\{-\beta x)$.

Remarks. Note that since $c_\beta(f, y) = 0$ whenever $\beta y^2 \notin \mathcal{O}_\infty$, we have that the Fourier series for f is finite for any fixed y, and in particular, $f(z) = c_0(y)$ for $\deg y \geq 0$. Note also that for any $u \in \mathcal{O}_\infty$, $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in SL_2(\mathcal{O}_\infty)$ so

$$\begin{pmatrix} y & xy^{-1} \\ 0 & y^{-1} \end{pmatrix} \equiv \begin{pmatrix} y & (x + uy^2)y^{-1} \\ 0 & y^{-1} \end{pmatrix}.$$

Thus for f to be well-defined, we must have $c_{f,\beta}(y) = 0$ whenever $\beta y^2 \notin \mathcal{O}_\infty$. In particular, this means that $f(z) = c_0(f, y)$ for $z = \begin{pmatrix} y & xy^{-1} \\ 0 & y^{-1} \end{pmatrix}$ with $\deg y \geq 0$.

§**3. Automorphic Forms.** The analogue of the classical (unnormalized) theta series is

$$\widetilde{\theta}(z) = \sum_{\beta \in T\mathbb{A}} \chi_{\mathcal{O}_\infty}(\beta^2 y^2)\ \mathrm{e}\{\beta^2 x\}$$

where $\chi_{\mathcal{O}_\infty}$ is the characteristic function of $\mathcal{O}_\infty$. In [M-W] the authors studied restricted representation numbers by studying the kth power of this function (for k even and q prime), for

$$\widetilde{\theta}^k(z) = \sum_{\alpha \in \mathbb{A}} r_k(\alpha, m) \mathrm{e}\{T^2\alpha^2 x\}$$

(where $y = \mathrm{Im} z = T^{-m}$). In Proposition 2.13 of [H-R], the authors realized a normalization $\theta(z)$ of the above theta series as a residue of an Eisenstein series:

$$\theta(z) = \gamma(y)|y|^{1/2}\widetilde{\theta}(z)$$

and (as in Definition 2.2 of [H-R])

$$\gamma(T^m) = q^{m/2}\int \mathrm{e}\{T^m x^2\}\mathrm{d}x / \int \mathrm{e}\{x^2\}\mathrm{d}x$$

where the integrals are over "sufficiently large" (fractional) ideals of $\mathcal{O}_\infty$. (Here $\mathrm{d}x$ denotes additive Haar measure normalized so the measure of $\mathcal{O}_\infty$ is 1, and hence the measure of $\mathfrak{P}_\infty^m$ is q^{-m} since $1 = \sum_{x\in\mathcal{O}_\infty/\mathfrak{P}_\infty^m}\int_{\mathfrak{P}_\infty^m}\mathrm{d}x$.) We will evaluate $\gamma(y)$ in Lemma 2 below.

Let $f : \mathfrak{H} \to \mathbb{C}$. We say f is an automorphic form of weight k if f transforms under $\Gamma = SL_2(\mathbb{A})$ as does the kth power of $\widetilde{\theta}$. That is, f is an automorphic form of weight k if for all $\delta \in SL_2(\mathbb{A})$ we have

$$f(\delta z) = \frac{\widetilde{\theta}^k(\delta z)}{\widetilde{\theta}^k(z)} f(z).$$

We say an automorphic form f is a cusp form if $f(z) = \sum_{\substack{\beta\in T^2\mathbb{A}\\ \beta\neq 0}} c_\beta(f,y)\,\mathrm{e}\{\beta x\}$, i.e. $c_0(f,y) = 0$ for all y.

We thank both Wolfgang Schmidt and Eric Rains for showing us two different ways of evaluating a Gauss sum over a finite field with p^r elements. (Here we simply quote the result; we are happy to provide these proofs to any interested reader.)

Lemma 2. *For all $\delta \in \Gamma$, $\theta(\delta z)/\theta(z) = \pm 1$. Also, with $q = p^r$,*

$$\gamma(T^{-m}) = \begin{cases} 1 & \text{if } 2|m, \\ (-1)^{r+1} & \text{if } 2\not| m,\ p \equiv 1 \pmod 4, \\ (-1)^{r+1}\sqrt{-1}^{\,r} & \text{if } 2\not| m,\ p \equiv 3 \pmod 4). \end{cases}$$

Proof. By Theorem 2.3 of [M-W] (which extends naturally to include the current setting),

$$\widetilde{\theta}(\delta z)/\widetilde{\theta}(z) = \sqrt{z} = \begin{cases} q^m & \text{if } x = 0, \\ q^{n/2}(x_n/\mathbb{F})^n/\gamma(T^n) & \text{if } x \neq 0,\ \deg x = n \end{cases}$$

where $(x_n/\mathbb{F})$ indicates whether x_n is a square in $\mathbb{F}$. So

$$\frac{\theta(\delta z)}{\theta(z)} = \frac{\sqrt{z}\gamma(\mathrm{Im}\delta z)|\mathrm{Im}\delta z|^{1/2}}{\gamma(\mathrm{Im}z)|\mathrm{Im}z|^{1/2}}.$$

We first evaluate $\gamma(\mathrm{Im}z)$.

For $m \geq 0$ we have

$$\begin{aligned}
\int_{T^m\mathcal{O}_\infty} \mathrm{e}\{T^{-m}x^2\}\mathrm{d}x &= \sum_{x\in T^m\mathcal{O}_\infty/\mathfrak{P}_\infty^m} \int_{\mathcal{O}_\infty} \mathrm{e}\{T^{-m}(x+x_0)^2\}\mathrm{d}x_0 \\
&= \sum_{x\in T^m\mathcal{O}_\infty/\mathfrak{P}_\infty^m} \mathrm{e}\{T^{-m}x^2\} \int_{\mathcal{O}_\infty} \mathrm{d}x_0 \\
&= \sum_{u_i\in\mathbb{F}} \mathrm{e}\left\{T^{-m}\left(\sum_{i=1}^m u_iT^i\right)^2\right\} \\
&= \sum_{u_i\in\mathbb{F}} \mathrm{e}\left\{\sum_{i=1}^m u_iu_{m+1-i}T\right\} \\
&= \begin{cases} \prod_{i=1}^{m/2} \sum_{u_i\in\mathbb{F}} \left(\sum_{v\in\mathbb{F}} \mathrm{e}\{2u_ivT\}\right) & \text{if } 2|m, \\ \prod_{i=1}^{(m-1)/2} \sum_{u_i\in\mathbb{F}} \left(\sum_{v\in\mathbb{F}} \mathrm{e}\{2u_ivT\}\right)\left(\sum_{u\in\mathbb{F}} \mathrm{e}\{u^2T\}\right) & \text{if } 2 \not| m. \end{cases}
\end{aligned}$$

(Here the sum on u corresponds to $i = (m+1)/2$.) Since $v \mapsto \mathrm{e}\{2u_ivT\}$ is a nontrivial character when $u_i \neq 0$, we get

$$\int_{T^m\mathcal{O}_\infty} \mathrm{e}\{T^{-m}x^2\}\mathrm{d}x = \begin{cases} q^{m/2} & \text{if } 2|m, \\ q^{(m-1)/2}\sum_{u\in\mathbb{F}} \mathrm{e}\{u^2T\} & \text{if } 2 \not| m. \end{cases}$$

A similar argument shows

$$\int_{T^m\mathcal{O}_\infty} \mathrm{e}\{x^2\}\mathrm{d}x = \sum_{x\in T^m\mathcal{O}_\infty/\mathfrak{P}_\infty^m} \mathrm{e}\{x^2\} \int_{\mathfrak{P}_\infty^m} \mathrm{d}x_0 = q^mq^{-m} = 1.$$

So for $m \geq 0$,

$$\gamma(T^{-m}) = \begin{cases} 1 & \text{if } 2|m, \\ q^{-1/2}\sum_{u\in\mathbb{F}} \mathrm{e}\{u^2T\} & \text{if } 2 \not| m. \end{cases}$$

(Thus when $2 \not| m$, $\gamma(T^{-m})$ is a Gauss sum.) Evaluating the Gauss sum yields the result on γ.

For $m \geq 0$ we also have

$$\int_{\mathcal{O}_\infty} \mathrm{e}\{T^m x^2\} \mathrm{d}x = \sum_{x \in \mathcal{O}_\infty/\mathfrak{P}_\infty^m} \mathrm{e}\{T^m x^2\} \int_{\mathfrak{P}_\infty^m} \mathrm{d}x_0$$
$$= q^{-m} \sum_{x \in T^m \mathcal{O}_\infty/\mathcal{O}_\infty} \mathrm{e}\{T^{-m} x^2\},$$

and this last sum we just evaluated. Hence for all $m \in \mathbb{Z}$ we have

$$\gamma(T^m) = \begin{cases} 1 & \text{if } 2|m, \\ q^{-1/2} \sum_{u \in \mathbb{F}} \mathrm{e}\{u^2 T\} & \text{if } 2 \not| m. \end{cases}$$

Again, evaluating the Gauss sum yields the result on γ.

Now take $\delta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and write $z = \begin{pmatrix} T^m & xT^{-m} \\ 0 & T^{-m} \end{pmatrix}$ with $x \in T^{2m-1}\mathbb{A}$, or $x = T^{2m}$ (which, under the right action of $SL_2(\mathcal{O}_\infty)$, is equivalent to $x = 0$); let $n = \deg x$. Then $T^{2m}/x \in SL_2(\mathcal{O}_\infty)$ so

$$\delta z \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z \begin{pmatrix} 1 & 0 \\ -T^{2m}/x & 1 \end{pmatrix} = \begin{pmatrix} T^m/x & * \\ 0 & T^{-m}x \end{pmatrix}.$$

Hence

$$\theta(\delta z)/\theta(z) = \frac{\gamma(T^{m-n})(x_n/\mathbb{F})^n}{\gamma(T^m)\gamma(T^n)} = \pm 1.$$

Since $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generate Γ, it follows that $\theta(\delta z)/\theta(z) = \pm 1$ for all $\delta \in \Gamma$. □

Remark. Given an automorphic form f of weight k, we can normalize f to a weight 0 or weight 1/2 automorphic form g by setting

$$g(z) = \begin{cases} |\mathrm{Im} z|^{k/2} f(z) & \text{if } k \text{ is even}, \\ \gamma(\mathrm{Im} z)^k |\mathrm{Im} z|^{k/2} f(z) & \text{if } k \text{ is odd}. \end{cases}$$

For this reason, we now only consider these two weights.

Theorem 1. *There are no nonzero cusp forms.*

Proof. Say f is a cusp form. We know that for any $z = \begin{pmatrix} T^m & 0 \\ 0 & T^{-m} \end{pmatrix}$, $m \geq 0$, we have $f(z) = c_0(f, T^m) = 0$. Hence f is 0 on a fundamental domain for $\Gamma \backslash \mathfrak{H}$ and so f is identically 0. □

§**4. Weight 0 Eisenstein Series.** For $s \in \mathbb{C}$ with $\Re s > 1$, we define the weight 0 Eisenstein series to be

$$E_s(z) = \sum_{\delta \in \Gamma_\infty \backslash \Gamma} |\mathrm{Im} \delta z|^{2s}$$

where $s \in \mathbb{C}$ with $\Re s > 1$, $z \in \mathfrak{H}$, and $\mathrm{Im} \begin{pmatrix} y & xy^{-1} \\ 0 & y^{-1} \end{pmatrix} = y$. To show E_s is absolutely convergent for $\Re s > 1$, we consider $s \in \mathbb{R}$, $s > 1$. Since each element of $\mathfrak{H}$ is in the Γ-orbit of some $z = \begin{pmatrix} T^m & 0 \\ 0 & T^{-m} \end{pmatrix}$ with $m \geq 0$, it suffices to consider only such z, which is what we do in the following computation.

First, notice that the cosets of $\Gamma_\infty \backslash \Gamma$ are represented by matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where (c, d) varies over relatively prime pairs of elements of $\mathbb{A}$ with c monic or $(c, d) = (0, 1)$. Also, $c = 0$ only when we consider the coset of I, and $d = 0$ only when we consider the coset of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. When $c, d \neq 0$, $m \geq 0$ and $\deg d \geq 2m + \deg c$, we have $(cT^{2m})/d \in \mathcal{O}_\infty$ so (recalling that $\mathfrak{H} = SL_2(\mathbb{K}_\infty)/SL_2(\mathcal{O}_\infty)$),

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} z &= \begin{pmatrix} 1 & b/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/d & 0 \\ c & d \end{pmatrix} z \\ &\equiv \begin{pmatrix} 1 & b/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} T^m/d & 0 \\ cT^m & d/T^m \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -(cT^{2m})/d & 1 \end{pmatrix} \\ &= \begin{pmatrix} T^m/d & * \\ 0 & d/T^m \end{pmatrix}. \end{aligned}$$

Similarly, when $c, d \neq 0$, $m \geq 0$ and $\deg d < 2m + \deg c$,

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} z &\equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} z \begin{pmatrix} d/(cT^{2m}) & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1/(cT^m) & * \\ 0 & cT^m \end{pmatrix}. \end{aligned}$$

The above equivalence also holds when $d = 0$, for then $c = 1$ and

$$\begin{aligned} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z &\equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} T^{-m} & 0 \\ 0 & T^m \end{pmatrix} \\ &= \begin{pmatrix} 1/(cT^m) & * \\ 0 & cT^m \end{pmatrix}. \end{aligned}$$

Hence for z, s as above,

$$E_s(z) = q^{2ms} + \sum_{c\ monic} \sum_{\substack{(c,d)=1 \\ \deg d \geq 2m+\deg c}} |T^m/d|^{2s} + \sum_{c\ monic} \sum_{\substack{(c,d)=1 \\ \deg d < 2m+\deg c}} |1/(cT^m)|^{2s}.$$

Now,

$$\sum_{\substack{n\in\mathbb{A} \\ n \text{ monic}}} |n|^{-2s} = (1-q^{1-2s})^{-1}$$

so

$$\begin{aligned}\sum_{c\ monic} \sum_{\substack{(c,d)=1 \\ \deg d \geq 2m+\deg c}} |T^m/d|^{2s} &= q^{2ms}(1-q^{1-2s}) \sum_{\substack{c\ monic \\ \deg d \geq 2m+\deg c}} |d|^{-2s} \\ &= q^{2ms}(1-q^{1-2s}) \sum_{\substack{\ell\geq 0 \\ r\geq 0}} q^{\ell}(q-1)q^{(r+\ell+2m)(1-2s)} \\ &= \frac{q^{2m-2ms}(q-1)}{(1-q^{2-2s})}.\end{aligned}$$

Also,

$$\begin{aligned}\sum_{c \text{ monic}} \sum_{\substack{c \text{ monic} \\ \deg d < 2m+\deg c}} |1/(cT^m)|^{2s} &= q^{-2ms}(1-q^{1-2s}) \sum_{\ell\geq 0} q^{\ell(1-2s)} q^{\ell+2m} \\ &= \frac{q^{2m-2ms}(1-q^{1-2s})}{(1-q^{2-2s})}.\end{aligned}$$

Notice that this shows E_s is absolutely convergent for $s \in \mathbb{C}$, $\Re s > 1$, and that for such s and $m \geq 0$,

$$E_s\left(\begin{pmatrix} T^m & 0 \\ 0 & T^{-m} \end{pmatrix}\right) = q^{2ms} + \frac{q(1-q^{-2s})}{(1-q^{2-2s})} q^{2m-2ms}.$$

One easily sees that E_s is invariant under the action of Γ.

Lemma 3. *$E_s(z) = \sum_{\beta'\in\mathbb{A}} b_\beta(y,s)e\{T^2\beta x\}$ where*

$$b_0(y,s) = |y|^{2s} + \frac{q(1-q^{-2s})}{(1-q^{2-2s})}\ |y|^{2-2s}$$

and for $\beta \neq 0$ with $T^2\beta' y = \beta y^2 \in \mathcal{O}_\infty$, $n = deg\beta'$, $m = -degy \geq n+2$,

$$b_{\beta'}(y,s) = q^{1-m}(1-q^{-2s})(q^{m(2s-1)} - q^{(n+1-m)(2s-1)})\ \sigma_{1-2s}(\beta').$$

Here

$$\sigma_s(\beta') = \sum_{\substack{\alpha|\beta' \\ \alpha \text{ monic}}} |\alpha|^s = \sum_{\ell=0}^{n} q^{\ell s}\tau_\ell(\beta')$$

where $\beta' = \beta T^{-2}$ *and* $\tau_\ell(\beta') = \#\{monic\ \alpha|\beta' :\ deg\alpha = \ell\ \}$.

Proof. The preceding computation proves the lemma for $b_0(T^m, s)$ with $m \geq 0$. So now take $m > 0$; we compute $b_\beta(T^{-m}, s)$ for $\beta = T^2\beta'$, $\beta' \in \mathbb{A}$ and $\deg\beta \leq 2m$. (Recall that for $\deg\beta > 2m$ we have $b_\beta(T^{-m}, s) = 0$ since then $\chi_{\mathcal{O}_\infty}(\beta T^{-2m}) = 0$.) Set $z = \begin{pmatrix} T^{-m} & xT^m \\ 0 & T^m \end{pmatrix}$ with $x \in \mathfrak{P}_\infty/\mathfrak{P}_\infty^{2m}$; we take $x = T^{-2m}$ as the representative of $\mathfrak{P}_\infty^{2m}$. For $c \neq 0$ and $\deg d \geq \deg c$, $\begin{pmatrix} 1 & 0 \\ -c/(T^{2m}(cx+d)) & 1 \end{pmatrix} \in SL_2(\mathcal{O}_\infty)$ so

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z \equiv \begin{pmatrix} 1/(T^m(cx+d)) & * \\ 0 & T^m(cx+d) \end{pmatrix}.$$

When $\deg d < \deg c$, $x - d/c$ runs over $\mathfrak{P}_\infty/\mathfrak{P}_\infty^{2m}$ as x does, and $\begin{pmatrix} 1 & 0 \\ -1/(T^{2m}x) & 1 \end{pmatrix} \in SL_2(\mathcal{O}_\infty)$, so

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z \equiv \begin{pmatrix} 1/(T^m cx) & * \\ 0 & T^m cx \end{pmatrix}.$$

Thus, using Fourier transform,

$$\begin{aligned}
b_\beta(T^{-m}, x) &= q^{1-2m} \sum_{x \in \mathfrak{P}_\infty/\mathfrak{P}_\infty^{2m}} E_s\left(\begin{pmatrix} T^{-m} & xT^m \\ 0 & T^m \end{pmatrix}\right) \mathrm{e}\{-\beta x\} \\
&= q^{1-2m} \sum_{\delta \in \Gamma_\infty \backslash \Gamma} \sum_{x \in \mathfrak{P}_\infty/\mathfrak{P}_\infty^{2m}} \left| \mathrm{Im}\delta \begin{pmatrix} T^{-m} & xT^m \\ 0 & T^m \end{pmatrix} \right|^{2s} \mathrm{e}\{-\beta x\} \\
&= q^{1-2m-2ms} \sum_{x \in \mathfrak{P}_\infty/\mathfrak{P}_\infty^{2m}} \mathrm{e}\{-\beta x\} \\
&\quad + q^{1-2m} \sum_{c \text{ monic}} \sum_{\substack{(c,d)=1 \\ \deg d \geq \deg c}} \sum_{x \in \mathfrak{P}_\infty/\mathfrak{P}_\infty^{2m}} |T^m d|^{-2s} \mathrm{e}\{-\beta x\} \\
&\quad + q^{1-2m} \sum_{c \text{ monic}} \sum_{\substack{(c,d)=1 \\ \deg d < \deg c}} \sum_{x \in \mathfrak{P}_\infty/\mathfrak{P}_\infty^{2m}} |T^m cx|^{-2s} \mathrm{e}\{-\beta x\}.
\end{aligned}$$

First say $\beta = 0$. Then

$$\sum_{c \text{ monic}} \sum_{\substack{(c,d)=1 \\ \deg d \geq \deg c}} |T^m d|^{-2s} = q^{-2ms}(1-q^{1-2s}) \sum_{\substack{c \text{ monic} \\ \deg d \geq \deg c}} |d|^{-2s}$$

$$= q^{-2ms}(1-q^{1-2s}) \sum_{\substack{\ell \geq 0 \\ r \geq 0}} q^{\ell}(q-1) q^{(\ell+r)(1-2s)}$$

$$= \frac{q^{-2ms}(q-1)}{(1-q^{2-2s})}.$$

Also, by separating the sums on x and c we find

$$q^{1-2m} \sum_{c \text{ monic}} \sum_{\substack{(c,d)=1 \\ \deg d < \deg c}} \sum_{x \in \mathfrak{P}_\infty / \mathfrak{P}_\infty^{2m}} |T^m c x|^{-2s}$$

$$= q^{1-2m}(1-q^{1-2s}) q^{-2ms} \sum_{\ell \geq 0} q^{\ell(2-2s)} \left(q^{4ms} + \sum_{1 \leq r \leq 2m} (q-1) q^{2m-r-1} q^{2rs} \right)$$

$$= \frac{q^{2ms-2m+1}(1-q^{1-2s}) + (q-1)q^{-2ms}(q^{(2m-1)(2s-1)}-1)}{(1-q^{2-2s})}.$$

Consequently,

$$b_0(T^{-m}, s) = q^{-2ms} + \frac{q(1-q^{-2s})}{(1-q^{2-2s})} q^{m(2s-2)}.$$

Now say $\beta = \beta' T^2 \neq 0$ with $n = \deg \beta' \leq 2m-2$. Then, writing $\beta' = \sum_{j=0}^{n} \beta_j T^j$,

$$\sum_{x \in \mathfrak{P}_\infty / \mathfrak{P}_\infty^{2m}} \mathrm{e}\{-\beta x\} = \sum_{\alpha_i \in \mathbb{F}_q} \mathrm{e} \left\{ - \sum_{j=0}^{n} \beta_j T^{j+2} \sum_{i=1}^{2m-1} \alpha_i T^{-i} \right\}$$

$$= \sum_{\alpha_i \in \mathbb{F}_q} \mathrm{e} \left\{ - \sum_{i=0}^{n} \beta_i \alpha_{i-1} T \right\}$$

$$= \prod_{i=0}^{n} \sum_{u \in \mathbb{F}_q} \mathrm{e}\{-\beta_i u T\}$$

$$= 0$$

(since for some i, $\beta_i \neq 0$). Also, since $\displaystyle\sum_{\substack{d \in \mathbb{A} \\ \deg d < \deg c}} \mathrm{e}\{\beta d / c\} = 0$ unless c divides β',

$$\sum_{c \text{ monic}} \sum_{\substack{(c,d)=1 \\ \deg d < \deg c}} |c|^{-2s} e\{\beta d / c\} = (1-q^{1-2s}) \sum_{\ell \geq 0} q^{-2\ell s} \tau_\ell(\beta')$$

$$= (1-q^{1-2s}) \sigma_{1-2s}(\beta').$$

Next,

$$\sum_{x\in\mathfrak{P}_\infty/\mathfrak{P}_\infty^{2m}} |x|^{-2s}\mathrm{e}\{-\beta x\} = q^{4ms} + \sum_{r=1}^{2m-1} q^{2rs} \sum_{\substack{u\in\mathbb{F}_q^\times \\ x_0\in\mathfrak{P}_\infty^{r+1}/\mathfrak{P}_\infty^{2m}}} \mathrm{e}\{-\beta(uT^{-r}+x_0)\}$$

$$= q^{4ms} + \sum_{r=n+1}^{2m-1} q^{2rs} q^{2m-1-r} \sum_{u\in\mathbb{F}_q^\times} \mathrm{e}\{-\beta uT^{-r}\}$$

since $x_0 \mapsto \mathrm{e}\{-\beta x_0\}$ is the trivial character on $\mathfrak{P}_\infty^{r+1}/\mathfrak{P}_\infty^{2m}$ if and only if $r \geq n+1$. When $r = n+1$, the sum on u yields -1, and when $r \geq n+2$ the sum on u yields $q-1$. Hence by substituting and simplifying, we find that

$$b_\beta(T^{-m}, s) = q^{1-m}(1-q^{-2s})(q^{m(2s-1)} - q^{(n+1-m)(2s-1)})\sigma_{1-2s}(\beta').\square$$

Corollary 1. *E_s can be analytically continued to $\mathbb{C}$ with a pole at $s = 1$. Then we have the functional equation*

$$g(s)E_s = g(1-s)E_{1-s}$$

where $g(s) = q^{-2s}\Gamma(2s)\zeta(2s)$, $\Gamma(2s) = (1-q^{-2s})^{-1}$, *and* $\zeta(2s) = (1-q^{1-2s})^{-1}$.

Proof. One sees from the preceding lemma that E_s can be analytically continued with a simple pole at $s = 1$. Then one sees that for any $s \neq 1$ and any $z = \begin{pmatrix} T^m & 0 \\ 0 & T^{-m} \end{pmatrix}$, $m \geq 0$, we have

$$E_s(z) = b_0(T^m, s) = \frac{g(1-s)}{g(s)} b_0(T^m, 1-s) = \frac{g(1-s)}{g(s)} E_{1-s}(z).$$

Since E_s and $\frac{g(1-s)}{g(s)}E_{1-s}$ are equal on a fundamental domain, they are equal on all of $\mathfrak{H}$. $\square$

§**5. Weight 1/2 Eisenstein Series.** For $\Re s > 1$, set

$$\widetilde{E}_s(z) = \sum_{\delta\in\Gamma_\infty\backslash\Gamma} \gamma(\mathrm{Im}\delta z)\frac{\theta(z)}{\theta(\delta z)}\ |\mathrm{Im}\delta z|^{2s}.$$

(Recall that $\frac{\theta(z)}{\theta(\delta z)} = \pm 1$.) Then from Proposition 2.11 of [H-R] we have

Lemma 4. *$\widetilde{E}_s(z) = \sum_{\beta' \in \mathbb{A}} \widetilde{b}_{\beta'}(y,s) e\{T^2 \beta x\}$ where*

$$\widetilde{b}_0(y,s) = \gamma(y)|y|^{2s} + \frac{q(1-q^{1-4s})}{(1-q^{3-4s})}\gamma(y)|y|^{2-2s}$$

and for $\beta' \neq 0$ with $T^2\beta' y = \beta y^2 \in \mathcal{O}_\infty$, we define $\widetilde{b}'_\beta(y,s)$ as follows. Write $\beta' = \beta_0 \beta_1^2$ where β_0 is square-free. For $\delta \in \mathbb{A}$ monic, let $\chi_{\beta_0}(\delta) = \left(\frac{\beta_0}{\delta}\right) \epsilon^{\deg \delta}$ where

$$\epsilon = \begin{cases} \gamma(T)^k & \text{if } k \text{ is even,} \\ \gamma(T)^{k-1} & \text{if } k \text{ is odd;} \end{cases}$$

set

$$a(s,\beta') = \sum_{\substack{\delta_1 \delta_2 | \beta_1 \\ \delta_1, \delta_2 \ \text{monic}}} \chi_{\beta_0}(\delta_1)\mu(\delta_1)|\delta_1|^{\frac{1}{2}-2s}|\delta_2|^{2-4s}.$$

Let $W = W^{(0)}$ be the Whittaker function (defined in Proposition 2.11 of [H-R]); so with $n' = -\deg \beta y^2$ and $\eta = \pm 1$ according to whether the lead coefficient of β is a square,

$$W(\beta y^2) = \begin{cases} \frac{(1+\eta q^{\frac{1}{2}-2s})}{(1-q^{2-4s})}\left[(1-q^{2(n'+1)(1-2s)}) - \eta q^{\frac{1}{2}-2s}(1-q^{2n'(1-2s)})\right] \\ \qquad \text{if } n' \text{ is even,} \\ \frac{(1-q^{1-4s})}{(1-q^{2-4s})}(1-q^{2(n'+1)(1-2s)}) \\ \qquad \text{if } n' \text{ is odd.} \end{cases}$$

Then $\widetilde{b}_\beta(y,s) = \gamma(y)|y|^{2-2s}q^{2-4s}L(2s-\frac{1}{2},\chi_{\beta_0})a(s,\beta')W(\beta y^2)(1-q^{1-4s})^{-1}$. Furthermore, $\widetilde{E}_s$ can be analytically continued with a pole at $s = 3/4$ and satisfies the functional equation

$$\widetilde{g}(s)\widetilde{E}_s = \widetilde{g}(1-s)\widetilde{E}_{1-s}$$

where $\widetilde{g}(s) = q^{1-4s}\Gamma(4s-1)\zeta(4s-1)$, $\Gamma(s) = (1-q^{-2s})^{-1}$, and $\zeta(s) = (1-q^{1-2s})^{-1}$.

§**6. Mellin Transform and Poincare Series.** Let $\psi : \mathbb{K}_\infty^\times / \mathcal{O}_\infty{}^\times \to \mathbb{C}$ have finite support; extend ψ to be a function on $\mathbb{K}_\infty^\times$. We define the Mellin transform of ψ, $M\psi : \mathbb{C} \to \mathbb{C}$, by

$$M\psi(s) = \int_{\mathbb{K}_\infty^\times} \psi(y)\ |y|^{2s} \mathrm{d}^\times y$$

where $\mathrm{d}^\times y$ denotes multiplicative Haar measure normalized so that $\int_{\mathcal{O}_\infty^\times} \mathrm{d}^\times y = 1$.

Lemma 5. *Let ψ be as above.*

(a) $M\psi(s) = \sum_{m\in\mathbb{Z}} \psi(T^m) q^{2ms}$.

(b) (Inverse Mellin transform) Let C denote the complex plane contour given by $s = \sigma + it$, $0 \le t \le \frac{\pi}{\ln q}$ with $\sigma > 1$ fixed. With $u = q^{-2s}$ and $y \in \mathbb{K}_\infty^\times$,

$$\psi(y) = \frac{1}{2\pi i}\int_C M\psi(-s)\ |y|^{2s}\frac{du}{u}.$$

Proof. The equality stated in (a) follows immediately from the equality

$$\int_{\mathbb{K}_\infty^\times} \psi(y)\ |y|^{2s} \mathrm{d}^\times y = \sum_{m\in\mathbb{Z}} \psi(T^m) q^{2ms} \int_{T^m\mathcal{O}_\infty^\times} \mathrm{d}^\times y.$$

To establish the equality stated in (b), we let $n = \deg y$. Then, recalling that ψ is finitely supported, we have

$$\begin{aligned}\frac{1}{2\pi i}\int_C M\psi(-s)\ |y|^{2s}\frac{\mathrm{d}u}{u} &= \sum_{m\in\mathbb{Z}} \psi(T^m)\cdot\frac{1}{2\pi i}\int_C u^{m-n}\frac{\mathrm{d}u}{u}\\ &= \psi(T^n) = \psi(y). \qquad \square\end{aligned}$$

With ψ as above, we define the weight 0 Poincare series $P\psi : \mathfrak{H} \to \mathbb{C}$ by

$$P\psi(z) = \sum_{\delta\in\Gamma_\infty\backslash\Gamma} \psi(\mathrm{Im}\delta z),$$

and we define the weight 1/2 Poincare series $\widetilde{P\psi} : \mathfrak{H} \to \mathbb{C}$ by

$$\widetilde{P\psi}(z) = \sum_{\delta\in\Gamma_\infty\backslash\Gamma} \gamma(\mathrm{Im}\delta z)\frac{\theta(z)}{\theta(\delta z)}\cdot\psi(\mathrm{Im}\delta z).$$

Thus for $\delta \in \Gamma$, $P\psi(\delta z) = P\psi(z)$ and $\widetilde{P\psi}(\delta z) = \frac{\theta(\delta z)}{\theta(z)}\widetilde{P\psi}(z)$. (Again, recall that $\frac{\theta(\delta z)}{\theta(z)} = \pm 1$.)

Theorem 2. *For $j \ge 0$, define $\psi_j : \mathbb{K}_\infty^\times/\mathcal{O}_\infty{}^\times \to \mathbb{C}$ by $\psi_j(T^m) = \begin{cases} 1 & \text{if } j = m,\\ 0 & \text{otherwise.}\end{cases}$*

(a) (Weight 0 case) $P\psi_j(z) = \sum_{\substack{\beta'\in\mathbb{A}\\ \deg\beta'\le 2m-2}} a_{j,\beta'}(y)e\{T^2\beta' x\}$ where

$$a_{j,0}(T^{-m}) = \begin{cases} 1 & \text{if } -m = j > 0,\\ q+1 & \text{if } m = j = 0,\\ q^{1-2m} & \text{if } m = j > 0,\\ q^{-2j-1}(q^2-1) & \text{if } m > j,\\ 0 & \text{otherwise,}\end{cases}$$

and for $\beta' \in \mathbb{A}$ with $\beta' \neq 0$ and deg$\beta' \leq 2m-2$,

$$a_{j,\beta'}(T^{-m}) = q^{-m-j}\Big(q\tau_{m-j}(\beta') - \tau_{m-1-j}(\beta') - q\tau_{m-1+j}(\beta') + \tau_{m+j}(\beta')\Big).$$

Here $\tau_\ell(\beta') = \#\{monic\ \alpha|\beta' :\ \ deg\alpha = \ell\ \}$. (Note that $a_{j,\beta'}(T^{-m}) = 0$ if $j > m$.) Furthermore, if f is any weight 0 automorphic form then

$$f(z) = \sum_{j\geq 0} \frac{c_0(f, T^j)}{a_{j,0}(T^j)}\ P\psi_j(z)$$

where $c_0(f,*)$ denotes the 0th Fourier coefficient of f.

(b) (Weight 1/2 case) $\widetilde{P\psi}_j(z) = \sum_{\beta'\in\mathbb{A}} \widetilde{a}_{j,\beta'}(y)e\{T^2\beta' x\}$ where

$$\widetilde{a}_{j,0}(T^{-m}) = \begin{cases} \gamma(T^m) & \text{if } -m = j > 0, \\ q+1 & \text{if } m = j = 0, \\ \gamma(T^m)q^{-(m+3j)/2-1}(q^2-1) & \text{if } m > j \text{ and } m-j \text{ is even}, \\ \gamma(T^m)q^{1-2m} & \text{if } m = j, \\ 0 & \text{otherwise}, \end{cases}$$

and for $\beta' \in \mathbb{A}$ with $\beta' \neq 0$ and deg$\beta \leq 2m-2$,

$$\widetilde{a}_{j,\beta'}(T^{-m}) = \gamma(T^m)q^{1-(3m+j)/2}$$
$$\cdot\Big(R_{m-j}(\beta') - R_{m-2-j}(\beta') - q^{1-j}R_{m-1+j}(\beta') + q^{-1-j}R_{m+j}(\beta')\Big).$$

Here

$$S_\ell(\beta') = \sum_{\deg\alpha=\ell} \left(\frac{\beta_0}{\alpha}\right)\left(\frac{-1}{q}\right)^\ell,$$

$$T_\ell(\beta') = \sum_{\substack{\delta'\delta^2|\beta_1^2 \\ \deg\delta'\delta^2=\ell}} q^{\deg\delta}\left(\frac{\beta_0}{\delta'}\right)\mu(\delta'),$$

$$R_\ell(\beta') = \sum_{r=0}^{\ell} S_{\ell-r}(\beta')T_r(\beta')$$

where α, δ, and δ' are monic polynomials in $\mathbb{A}$ and μ denotes the Möbius function. (Note that $a_{j,\beta'}(T^{-m}) = 0$ if $j > m$.) Furthermore, if f is any weight 1/2 automorphic form then

$$f(z) = \sum_{j\geq 0} \frac{c_0(f, T^j)}{\widetilde{a}_{j,0}(T^j)}\ \widetilde{P\psi}_j(z)$$

where $c_0(f,*)$ *denotes the 0th Fourier coefficient of* f.

Proof. Say $\psi : \mathbb{K}_\infty^\times/\mathcal{O}_\infty{}^\times \to \mathbb{C}$ has finite support. Using inverse Mellin transform (with $u = q^{-2s}$), we see

$$P\psi(z) = \sum_{\delta\in\Gamma_\infty\backslash\Gamma} \frac{1}{2\pi i}\int_C M\psi(-s)|\mathrm{Im}\delta z|^{2s}\frac{\mathrm{d}u}{u}$$

and since for any $z \in \mathfrak{H}$ there are finitely many $\delta \in \Gamma_\infty\backslash\Gamma$ with $\mathrm{Im}\delta z \in$supportψ,

$$\begin{aligned} P\psi(z) &= \frac{1}{2\pi i}\int_C M\psi(-s)\sum_{\delta\in\Gamma_\infty\backslash\Gamma}|\mathrm{Im}\delta z|^{2s}\frac{\mathrm{d}u}{u} \\ &= \frac{1}{2\pi i}\int_C M\psi(-s)E_s(z)\frac{\mathrm{d}u}{u}. \end{aligned}$$

For fixed $z = \begin{pmatrix} y & xy^{-1} \\ 0 & y^{-1}\end{pmatrix}$ we know $P\psi(z)$ and $E_s(z)$ are given by finite Fourier series (recall that $c_{\beta'}(f,y) = 0$ whenever $T^2\beta'y^2 \notin \mathcal{O}_\infty$) so

$$P\psi(z) = \sum_{\beta'\in\mathbb{A}}\left(\frac{1}{2\pi i}\int_C M\psi(-s)b_{\beta'}(y,s)\frac{\mathrm{d}u}{u}\right)\mathrm{e}\{T^2\beta'x\}.$$

Now, $M\psi_j(-s) = u^j$ so with the preceding lemma it is trivial to compute $a_{j,\beta'}$.

Let $f : \mathfrak{H} \to \mathbb{C}$ be a weight 0 automorphic form; write $f(z) = \sum c_{\beta'}(y)\mathrm{e}\{T^2\beta'x\}$. Then for each $z = \begin{pmatrix} T^m & 0 \\ 0 & T^{-m}\end{pmatrix}$, $m \geq 0$,

$$f(z) = \sum_{j\geq 0}\frac{c_0(f,T^j)}{a_{j,0}(T^j)}\ P\psi_j(z).$$

For each $z \in \mathfrak{H}$, the sum on j is actually finite. Hence $\sum_{j\geq 0}\frac{c_0(f,T^j)}{a_{j,0}(T^j)}\ P\psi_j(z)$ converges and defines a weight 0 automorphic form that agrees with f on a fundamental domain, so it must equal f.

The case where the weight is 1/2 is analogous; we first rewrite the Fourier coefficients of the Eisenstein series by making use of its functional equation. Let $F(s,\beta') = F_s(\beta') = L(s,\chi_{\beta_0})a(s,\beta')$. Write $\beta' = \beta_0\beta_1^2$ where β_0 is square-free. For $\deg\beta'$ odd,

$$F\left(2s - \frac{1}{2},\beta'\right) = (qu)^{\deg\beta'-1}\cdot F\left(2(1-s) - \frac{1}{2},\beta'\right)$$

and for $\deg\beta'$ even,

$$F\left(2s-\frac{1}{2},\beta'\right)=(qu)^{\deg\beta'-2}\frac{(1-\eta q^{1/2-2s})}{(1-\eta q^{-3/4+2s})}\cdot F\left(2(1-s)-\frac{1}{2},\beta'\right)$$

where $v\in\mathbb{F}$ is the lead coefficient of β' and $\eta=\mathrm{sgn}(-v)$ where $\mathrm{sgn}(-v)=\pm 1$, according to whether $-v$ is a square in $\mathbb{F}$. Note that $F(s,\beta')=\sum_{\ell\geq 0}R_\ell(\beta')q^{\ell s}$, and $R_\ell(\beta')=0$ when $\ell>\deg\beta'$. Consequently, for $\beta'\neq 0$ with $T^2\beta' y^2\in\mathcal{O}_\infty$,

$$\widetilde{b}'_\beta(y,s)=\gamma(y)(1-q^{1-4s})\cdot\sum_{t\geq 0}R_t(\beta')\Big(q^{1-2m+t/2-2s(t-m)}-q^{-1-3t/2-2s(m-t-2)}\Big).$$

Now we proceed as in the weight 0 case. □

As an application, we consider sums of squares. Notice that for $k\geq 2$,

$$\theta^k(z)=\gamma^k(y)|y|^{k/2}\sum_{\alpha\in\mathbb{A}}r_k(\alpha,m)\ \mathrm{e}\{\alpha T^2 x\}$$

where $-m=\deg y$ and

$$r_k(\alpha,m)=\#\{(\alpha_1,\dots,\alpha_k):\ \alpha_i\in\mathbb{A},\ \deg\alpha_i<m,\ \sum_i\alpha_i^2=\alpha\ \}.$$

Applying the preceding theorem, we immediately get

Corollary 2. *Fix $k,m\in\mathbb{Z}_+$ with $k\geq 2$. Let* $\epsilon=\begin{cases}\gamma(T)^k & \text{if } k \text{ is even,}\\ \gamma(T)^{k-1} & \text{if } k \text{ is odd.}\end{cases}$ *Then for $\alpha\neq 0$ and $m>\deg\alpha$,*

$$r_k(\alpha,m)=\begin{cases}q^{m(k-2)+1}(1-\epsilon q^{-k/2})\sigma_{1-k/2}(\alpha,\epsilon) & \text{if } k \text{ is even,}\\ q^{m(k-2)+1}(1-q^{1-k})F_{(k-1)/2}(\alpha,\epsilon) & \text{if } k \text{ is odd}\end{cases}$$

where

$$\sigma_s(\alpha,\epsilon)=\sum_{\text{monic }\alpha'|\alpha}(\epsilon q^s)^{\deg\alpha'}=\sum_{j\geq 0}\epsilon^j\tau_j(\alpha)q^{js}$$

and

$$F_s(\alpha,\epsilon)=\sum_{j\geq 0}\epsilon^j R_j(\alpha)q^{-js}.$$

For $\alpha \in \mathbb{A}$, $\alpha \neq 0$, with $m \leq deg\alpha \leq 2m-2$,

$$r_k(\alpha, m) = \begin{cases} q^{m(k-2)+1}(1-\epsilon q^{-k/2}) \sum_{j=0}^{m-1} \epsilon^j q^{j(1-k/2)} \Big(\tau_j(\alpha) - \tau_{2m-1-j}(\alpha)\Big) \\ \qquad\qquad \textit{if } k \textit{ is even,} \\ q^{m(k-2)+1}(1-q^{1-k}) \sum_{j=0}^{m-2} \epsilon^j q^{j(1-k)} \Big(R_j(\alpha) - q^{j+2-m} R_{2m-2-j}(\alpha)\Big) \\ \qquad\qquad \textit{if } k \textit{ is odd} \end{cases}$$

where $R_j(\alpha)$ is defined in the preceding theorem. Finally,

$$r_k(0, m) = \begin{cases} q^{m(k-2)+1} + \epsilon^m q^{\frac{mk}{2}-1}(q-1) & \\ \qquad +q^{\frac{mk}{2}-1} \cdot \frac{(q^2-1)(\epsilon^{m+1} q^{\frac{k}{2}-2} - q^{m(\frac{k}{2}-2)})}{1-\epsilon q^{k/2-2}} & \textit{if } 2|k,\ k \neq 4, \\ q^{m(k-2)+1} + (q-1) q^{\frac{m(k-1)}{2}-1} & \\ \qquad +q^{\frac{m(k-1)}{2}-1} \cdot \frac{(q^2-1)(q^{k-3} - q^{\frac{m(k-3)}{2}})}{1-q^{k-3}} & \textit{if } 2 \nmid k,\ k \neq 3,\ 2|m, \\ \gamma^k(T) q^{m(k-2)+1} & \\ \qquad +\gamma^k(T) q^{\frac{m(k-1)}{2}-1} \frac{(q^2-1)(q^{\frac{(k-3)}{2}} - q^{\frac{m(k-3)}{2}})}{1-q^{k-3}} & \textit{if } k \neq 3,\ 2 \nmid km, \\ q^{2m} + q^{2m-1} m (q^2-1) & \textit{if } k=4, \\ q^m + q^{m-1} m (\frac{q^2-1}{2}) & \textit{if } k=3,\ 2|m, \\ \overline{\gamma}(T)(q^{m+2} + q^{m-1}(\frac{(m-1)(q^2-1)}{2}) & \textit{if } k=3,\ 2 \nmid m. \end{cases}$$

Proof. Write $\theta^k(z) = \sum_{\beta' \in \mathbb{A}} c_{\beta'}(\theta^k, y) \mathrm{e}\{T^2 \beta' x\}$. We know that for any $z \in H$,

$$\theta^k(z) = \begin{cases} \sum_{j \geq 0} \big(c_0(\theta^k, T^j)/a_{j,0}(T^j)\big) P\psi_j(z) & \text{if } k \text{ is even,} \\ \sum_{j \geq 0} \big(c_0(\theta^k, T^j)/\widetilde{a}_{j,0}(T^j)\big) \widetilde{P\psi}_j(z) & \text{if } k \text{ is odd.} \end{cases}$$

Also, for fixed $z \in \mathfrak{H}$,

$$P\psi_j(z) = \sum_{\substack{\beta' \in \mathbb{A} \\ \deg\beta' \leq -2\deg y - 2}} a_{j,\beta'}(y) \mathrm{e}\{T^2 \beta' x\} \qquad \text{and} \qquad \widetilde{P\psi}_j(z) = \sum_{\substack{\beta' \in \mathbb{A} \\ \deg\beta' \leq -2\deg y - 2}} \widetilde{a}_{j,\beta'}(y) \mathrm{e}\{T^2 \beta' x\}.$$

Thus

$$c_{\beta'}(\theta^k, y) = \begin{cases} \sum_{j=0}^{-2\deg y} \big(c_0(\theta^k, T^j)/a_{j,0}(T^j)\big)\, a_{j,\beta}(y) & \text{if } k \text{ is even,} \\ \sum_{j=0}^{-2\deg y} \big(c_0(\theta^k, T^j)/\widetilde{a}_{j,0}(T^j)\big)\, \widetilde{a}_{j,\beta}(y) & \text{if } k \text{ is odd.} \end{cases}$$

Note that for $j \geq 0$ and k even,

$$\frac{c_0(\theta^k, T^j)}{a_{j,0}(T^j)} = \begin{cases} \frac{1}{q+1} & \text{if } j = 0, \\ \epsilon^j q^{jk/2} & \text{if } j > 0. \end{cases}$$

Similarly, for $j \geq 0$ and k odd,

$$\frac{c_0(\theta^k, T^j)}{\widetilde{a}_{j,0}(T^j)} = \begin{cases} \frac{1}{q+1} & \text{if } j = 0, \\ \epsilon^j q^{jk/2} & \text{if } j > 0. \end{cases}$$

The theorem now easily follows by using the formulas for the $a_{j,\beta'}$ and the $\widetilde{a}_{j,\beta'}$. $\square$

Specializing to the case $k = 4$ and α irreducible or an element of $\mathbb{F}$, we see that the restricted representation number is nonzero for any $m > \frac{1}{2}\deg\alpha$; since the product of two sums of four squares is again a sum of four squares, we obtain the following analogue of the classical result on sums of four squares.

Corollary 3. *For any $\alpha \in \mathbb{A}$ and $m > \frac{1}{2}\deg\alpha$, $r_k(\alpha, m) \neq 0$.*

§**7. Spectral Decomposition.** Since it is not clear whether $r_k(\alpha, m) \neq 0$ when $m \leq \deg\alpha < 2m - 1$ and $k \geq 3$, we use spectral decomposition to show that $r_k(\alpha, m)$ is (essentially) asymptotic to the Fourier coefficients of the Eisenstein series $E_{k/4}(z)$ as $\deg\alpha \to \infty$. (When $k = 3$ or 4, we need to remove the singularity from the Eisenstein series to get this result.)

To develop the spectral decomposition, we follow classical arguments (see, for example, Theorem 1 in §3.7 of [T]).

Suppose $f, g : \mathfrak{H} \to \mathbb{C}$ such that for all $\delta \in \Gamma$,

$$\frac{f(\delta z)}{f(z)} = \left(\frac{\theta(\delta z)}{\theta(z)}\right)^k = \frac{g(\delta z)}{g(z)},$$

where $k \in \mathbb{Z}_+$. Then we define

$$\langle f, g \rangle = \int_{\Gamma \backslash \mathfrak{H}} f(z)\overline{g(z)} d\mu(z)$$

where $d\mu$ is the measure on $\Gamma\backslash\mathfrak{H}$ giving the γ-orbit of $z = \begin{pmatrix} T^m & 0 \\ 0 & T^{-m} \end{pmatrix}$ $(m \geq 0)$ pointmass

$$\frac{q(q-1)}{|\mathrm{Stab}_\Gamma z|} = \begin{cases} \frac{1}{q+1} & \text{if } m = 0, \\ q^{-2m} & \text{if } m > 0. \end{cases}$$

Lemma 6. *Let $\psi : \mathbb{K}_\infty^\times / \mathcal{O}_\infty{}^\times \to \mathbb{C}$ have finite support; extend ψ in the natural way to a function on $\mathbb{K}_\infty^\times$. Then*

$$\langle P\psi, E_s \rangle = M\psi(\overline{s} - 1) + \frac{g(1 - \overline{s})}{g(\overline{s})} \, M\psi(-\overline{s})$$

and

$$\langle \widetilde{P\psi}, \widetilde{E_s} \rangle = M\psi(\overline{s}-1) + \frac{\widetilde{g}(1-\overline{s})}{\widetilde{g}(\overline{s})}\ M\psi(-\overline{s}).$$

Here $g(s), \widetilde{g}(s)$ are as defined in Corollary 1 and Lemma 4.

Proof. We have

$$\begin{aligned}\langle P\psi, E_s \rangle &= \int_{\Gamma\backslash\mathfrak{H}} \sum_{\delta\in\Gamma_\infty\backslash\Gamma} \psi(\mathrm{Im}\delta z)\overline{E_s(z)}d\mu(z)\\ &= \int_{\Gamma\backslash\mathfrak{H}} \sum_{\delta\in\Gamma_\infty\backslash\Gamma} \psi(\mathrm{Im}\delta z)\overline{E_s(\delta z)}d\mu(z).\end{aligned}$$

Since $|\mathrm{Stab}_\Gamma z| < \infty$ for each $z \in \mathfrak{H}$ and $\psi(\mathrm{Im}z) \neq 0$ for only finitely many $z \in \Gamma_\infty\backslash\mathfrak{H}$, there are finitely many pairs $z \in \Gamma\backslash\mathfrak{H}$, $\delta \in \Gamma_\infty\backslash\Gamma$ with $\psi(\mathrm{Im}\delta z) \neq 0$; thus

$$\langle P\psi, E_s \rangle = \sum_{\delta\in\Gamma_\infty\backslash\Gamma} \int_{\delta\mathcal{D}} \psi(\mathrm{Im}z)\overline{E_s(z)}d\mu(z)$$

where $\mathcal{D} = \left\{ \begin{pmatrix} T^m & 0 \\ 0 & T^{-m} \end{pmatrix} : m \geq 0 \right\}$ is a fundamental domain for $\Gamma\backslash\mathfrak{H}$. For $z \in \mathfrak{H}$ with $|\mathrm{Im}z| < 1$ there are elements of $\Gamma - \Gamma_\infty$ that stabilize z; so for such z, $z \in \delta\mathcal{D}$ for several choices of $\delta \in \Gamma_\infty\backslash\Gamma$. Thus we define $d\nu$ to be the measure on $\Gamma_\infty\backslash\mathfrak{H}$ giving the Γ_∞-orbit of $z = \begin{pmatrix} T^m & 0 \\ 0 & T^{-m} \end{pmatrix}$ pointmass

$$\frac{q(q-1)}{|\mathrm{Stab}_{\Gamma_\infty} z|} = \begin{cases} q & \text{if } m < 0, \\ q^{-2m} & \text{if } m \geq 0. \end{cases}$$

(So we multiply the μ-pointmass of z by $\frac{|\mathrm{Stab}_\Gamma z|}{|\mathrm{Stab}_{\Gamma_\infty} z|} = |\mathrm{Stab}_{\Gamma_\infty\backslash\Gamma}(z)|$.) Hence

$$\begin{aligned}\langle P\psi, E_s \rangle &= \int_{\Gamma_\infty\backslash\mathfrak{H}} \psi(\mathrm{Im}z)\overline{E_s(z)}d\nu(z)\\ &= \sum_{m\geq 0} \psi(T^m)q^{-2m}\overline{E_s}\begin{pmatrix} T^m & 0 \\ 0 & T^{-m} \end{pmatrix}\\ &\quad + q \cdot \sum_{m<0} \psi(T^m) \sum_{x\in T^{2m-1}\mathbb{A}\cap\mathfrak{P}_\infty} \overline{E_s}\begin{pmatrix} T^m & xT^{-m} \\ 0 & T^{-m} \end{pmatrix}\end{aligned}$$

and recognizing the sum on x as Fourier transform,

$$\begin{aligned}\langle P\psi, E_s\rangle &= \sum_{m\in\mathbb{Z}} \psi(T^m) q^{-2m} \overline{b_0}(T^m, s)\\ &= M\psi(\overline{s}-1) + \frac{g(1-\overline{s})}{g(\overline{s})}\ M\psi(-\overline{s}).\end{aligned}$$

Similarly one computes $\langle \widetilde{P\psi}, \widetilde{E_s}\rangle$. □

Lemma 7. *$\langle P\psi, 1\rangle = M\psi(-1)$ and $\langle \widetilde{P\psi}, \theta\rangle = M\psi(-3/4)$.*

Proof. This is similar to the proof of the preceding lemma.

$$\begin{aligned}\langle P\psi, 1\rangle &= \int_{\Gamma\backslash\mathfrak{H}} \sum_{\delta\in\Gamma_\infty\backslash\Gamma} \psi(\mathrm{Im}\delta z) d\mu(z)\\ &= \int_{\Gamma_\infty\backslash\mathfrak{H}} \psi(\mathrm{Im} z) d\mu(z)\\ &= \sum_{m\in\mathbb{Z}} \psi(T^m) q^{-2m}.\end{aligned}$$

Similarly,

$$\langle \widetilde{P\psi}, \theta\rangle = \int_{\Gamma\backslash\mathfrak{H}} \sum_{\delta\in\Gamma_\infty\backslash\Gamma} \gamma(\mathrm{Im}\delta z) \frac{\theta(z)}{\theta(\delta z)} \psi(\mathrm{Im}\delta z) \overline{\theta(z)} d\mu(z)$$

and since $\frac{\theta(z)}{\theta(\delta z)} = \pm 1$,

$$\begin{aligned}\langle \widetilde{P\psi}, \theta\rangle &= \int_{\Gamma\backslash\mathfrak{H}} \sum_{\delta\in\Gamma_\infty\backslash\Gamma} \gamma(\mathrm{Im}\delta z) d\mu(z) \psi(\mathrm{Im}\delta z) \overline{\theta(\delta z)} d\mu(z)\\ &= \int_{\Gamma_\infty\backslash\mathfrak{H}} \gamma(\mathrm{Im} z) \psi(\mathrm{Im} z) \overline{\theta(z)} d\mu(z)\\ &= \sum_{m\in\mathbb{Z}} \gamma(T^m) \psi(T^m) \overline{c_0}(\theta, T^m) q^{-2m}\\ &= \sum_{m\in\mathbb{Z}} \psi(T^m) q^{-3m/2}. \qquad \square\end{aligned}$$

Theorem 3. *Let $\mathcal{C}$ be the contour $s = 1/2 + it$, $0 \le t < \pi/\ln q$. With $u = q^{-2s}$,*

$$P\psi(z) = \frac{1}{4\pi i}\int_{\mathcal{C}} \langle P\psi, E_s\rangle \; E_s(z)du \;+\; \langle P\psi, 1\rangle Res_{s=1}E_s(z)$$

and

$$\widetilde{P\psi}(z) = \frac{1}{4\pi i}\int_{\mathcal{C}} \langle \widetilde{P\psi}, \widetilde{E_s}\rangle \; \widetilde{E_s}du \;+\; \langle \widetilde{P\psi}, \theta\rangle Res_{s=3/4}\widetilde{E_s}.$$

Proof. Let $\mathcal{C}'$ be the contour $s = \sigma + it$, $0 \le t < \pi/\ln q$ with $\sigma > 1$ fixed. Then using inverse Mellin transform,

$$\begin{aligned} P\psi(z) &= \sum_{\delta\in\Gamma_\infty\backslash\Gamma} \frac{1}{2\pi i}\int_{\mathcal{C}'} M\psi(-s)|\mathrm{Im}\delta z|^{2s}\frac{du}{u} \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}'} M\psi(-s)E_s(z)\frac{du}{u}. \end{aligned}$$

Now move the contour from $\mathcal{C}'$ to $\mathcal{C}$, running over the pole of E_s at $s = 1$. Thus

$$P\psi(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} M\psi(-s)E_s(z)\frac{du}{u} + M\psi(-1)\mathrm{Res}_{s=1}E_s(z).$$

The functional equation of E_s gives us

$$\int_{\mathcal{C}} M\psi(-s)E_s(z)\frac{du}{u} = \frac{1}{2}\int_{\mathcal{C}} M\psi(-s)\left(E_s(z) + \frac{g(1-s)}{g(s)}E_{1-s}(z)\right)\frac{du}{u}$$

and replacing s by $1-s$ in the second part of the integral ,

$$\int_{\mathcal{C}} M\psi(-s)E_s(z)\frac{du}{u} = \frac{1}{2}\int_{\mathcal{C}}\left(M\psi(-s) + \frac{g(s)}{g(1-s)}M\psi(1-s)\right)E_s(z)\frac{du}{u}.$$

(Note that $s \mapsto 1-s$ takes $\mathcal{C}$ onto $\mathcal{C}$, reversing orientation.) Noting that $s = 1-\overline{s}$ yields the result. A similar computation holds for $\widetilde{P\psi}$. □

Standard arguments now yield

Corollary 4. *For $f \in L^2(\Gamma\backslash\mathfrak{H})$ an automorphic form of weight 0,*

$$f(z) = \frac{1}{4\pi i}\int_{c} \langle f, E_s\rangle \; E_s(z)\frac{du}{u} \;+\; \langle f, 1\rangle Res_{s=1}E_s(z).$$

For $f \in L^2(\Gamma\backslash\mathfrak{H})$ *an automorphic form of weight* $1/2$,

$$f(z) = \frac{1}{4\pi i}\int_c \langle f, \widetilde{E_s}\rangle\ \widetilde{E_s}(z)\frac{du}{u}\ +\ \langle f, \theta\rangle Res_{s=3/4}\widetilde{E_s}(z).$$

When $k \geq 5$, the difference between θ^k and the Eisenstein series at $s = k/4$ is an L^2-function. However, when $k = 3$ or 4 the Eisenstein series has a removable singularity at $s = k/4$. Thus we let $R = \lim_{s\to 1}(1 - q^{2-2s})E_s$ and set

$$E' = \lim_{s\to 1}\left(E_s - \frac{1}{1 - q^{2-2s}}R\right).$$

Similarly, we let $\widetilde{R} = \lim_{s\to 3/4}(1 - q^{3-4s})\widetilde{E}_s$ and set

$$\widetilde{E}' = \lim_{s\to 3/4}\left(E_s - \frac{1}{1 - q^{3-4s}}\widetilde{R}\right).$$

For $k \geq 5$, we set

$$f(z) = \begin{cases} \theta^k(z) - \gamma^k(\mathrm{Im}z)E_{k/4}(z) & \text{if } k \text{ is even,} \\ \theta^k(z) - \gamma^{k-1}(\mathrm{Im}z)\widetilde{E}_{k/4}(z) & \text{if } k \text{ is odd.} \end{cases}$$

When $k = 4$ we set $f(z) = \theta^4(z) - E'(z)$ and when $k = 3$ we set $f(z) = \theta^3(z) - \gamma^2(\mathrm{Im}z)\widetilde{E}'(z)$. Thus f is an automorphic form of weight 0 or 1/2, and one easily sees that f is L^2.

Lemma 8. *Set*

$$\epsilon = \begin{cases} \gamma^k(T) & \text{if } k \text{ is even,} \\ \gamma^{k-1}(T) & \text{if } k \text{ is odd.} \end{cases}$$

When k *is even,* $k \geq 4$,

$$\langle f, E_s\rangle = \frac{\epsilon q^{1+k/2-2\overline{s}}(1 - q^{1-2\overline{s}})(1 - q^{-k/2})(1 + \epsilon q^{1-k/2})(1 - \epsilon q^{2-k/2})}{(1 - q^{2-2\overline{s}})(1 - \epsilon q^{k/2-2\overline{s}})(1 - \epsilon q^{2-k/2-2\overline{s}})(1 - q^{2-k/2})}.$$

(To evaluate this when $k = 4$*, one first considers the case that* $\epsilon = 1$ *and reduces* $(1 - \epsilon q^{2-k/2})/(1 - q^{2-k/2})$ *to 1; then one sets* $k = 4$*.) When* k *is odd,* $k \geq 3$,

$$\langle f, \widetilde{E}_s\rangle = \frac{q^{1+k/2-2\overline{s}}(1 - q^{2-4\overline{s}})(1 - q^{1-k})}{(1 - q^{3-4\overline{s}})(1 - q^{k/2-2\overline{s}})(1 - q^{2-k/2-2\overline{s}})}.$$

Proof. When k is even, $\langle f, E_s\rangle = \frac{1}{q+1}c_0(f,T^0)b_0(T^0,\overline{s})+\sum_{m\geq 1} q^{-2m}c_0(f,T^m)b_0(T^m,\overline{s})$ where c_0 denotes the 0th coefficient of f and b_0 that of E_s. Note that when $k=4$, $m \geq 0$,

$$c_0(f,T^m) = m - \frac{m+1}{q^2},$$

and when $k \geq 6$ with k even,

$$c_0(f,T^m) = \frac{-\epsilon^m q^{1+m(2-k/2)}(1-q^{-k/2})}{(1-q^{2-k/2})}.$$

Similarly, when $k=3$, $m \geq 0$,

$$c_0(f,T^m) = \epsilon^m \left(\frac{m}{2}q^{m/2} - \frac{m+2}{2q}\right),$$

and when $k \geq 5$ with k odd,

$$c_0(f,T^m) = \frac{-\epsilon^m \gamma(T^m) q^{1+m(2-k/2)}(1-q^{1-k})}{(1-q^{3-k})}.$$

The lemma easily follows. □

In Corollary 2 we found precise formulas for $r_k(\alpha,m)$. We now show that, as in the classical case, $r_k(\alpha,m)$ is asymptotic to the Fourier coefficient of the Eisenstein series (with the singularity removed in the case $k=3$ or 4). This yields another proof that, at least for m sufficiently large, the restricted representation numbers are nonzero for any $m > \frac{1}{2}\deg\alpha$.

Theorem 4. *For α a square-free element of $\mathbb{A}$, $k \geq 3$, and $m \geq 0$ so that $n = \deg\alpha \leq 2m-2$,*

$$r_k(\alpha,m) = \begin{cases} q^{mk/2}b_{T^2\alpha}(T^{-m},k/4) + O(q^{m(k/2-1)}4^m) & \text{if } k \text{ is even,} \\ \epsilon^m q^{mk/2}\tilde{b}_{T^{-m}}(T^2\alpha,k/4) + O(q^{m(k/2-1)}4^m) & \text{if } k \text{ is odd.} \end{cases}$$

In particular, except for $q=3$ when $k=3$ or 4, $r_k(\alpha,m) \neq 0$ for m sufficiently large.

Proof. Let $\beta = \beta' T^2$ where $\beta' \in \mathbb{A}$ has degree n. For k even and $\Re s = 1/2$,

$$|\langle f, E_s\rangle| \leq \frac{2q(1+\epsilon q^{1-k})(1-\epsilon q^{2-k/2})}{(q-1)(1-q^{1-k/2})(1-q^{2-k/2})}.$$

Also,

$$\left|b_\beta(T^{-m},k/4)\right| \geq q^{1+m(k/2-2)}(1-q^{-k/2})(1-q^{1-k/2})\sigma_{1-k/2}(\beta').$$

Except when $q=3$ and $k=4$,

$$\left|b_\beta(T^{-m},k/4)\right| > \left|c_\beta(T^{-m})\right|$$

whenever m is large enough.

Now consider k odd; suppose β' is square-free. Then for k odd and $\Re s = 1/2$,

$$|\langle f, \widetilde{E}_s\rangle| \leq \frac{2q(1-q^{1-k})}{(q-1)(1-q^{1-k/2})^2}.$$

Using the Riemann hypothesis over function fields, we have

$$L\left(\frac{4s-1}{2},\chi_{\beta'}\right) = (1-\eta q^{1/2-2s})\prod_{i=1}^{D}(1-\alpha_i q^{1/2-2s})$$

where $\eta=\pm 1$, $D=n-2$ if n is even and $D=n-1$ if n is odd, and $|\alpha_i| = q^{1/2}$. Thus

$$|L\left(\frac{4s-1}{2},\chi_{\beta'}\right)| \leq \begin{cases} (1+q^{1-2s})^{n-2}(1+q^{1/2-2s}) & \text{if } n \text{ is even,}\\ (1+q^{1-2s})^{n-1} & \text{if } n \text{ is odd}\end{cases}$$

and

$$|L\left(\frac{4s-1}{2},\chi_{\beta'}\right)| \geq \begin{cases} (1-q^{1/2-2s})(1-q^1-2s)^{n-2} & \text{if } n \text{ is even,}\\ (1-q^{1-2s})^{n-1} & \text{if } n \text{ is odd.}\end{cases}$$

So for $\Re s = t \geq 1/2$,

$$(1-q^{1/2-2t})(1-q^{1-2t})^{n-2} \leq |L\left(\frac{4s-1}{2},\chi_{\beta'}\right)| \leq (1+q^{1/2-2t})(1-q^{1-2t})^{n-2}$$

when n is even and

$$(1-q^{1-2t})^{n-1} \leq |L\left(\frac{4s-1}{2},\chi_{\beta'}\right)| \leq (1-q^{1-2t})^{n-1}$$

when n is odd. Also,

$$(1-q^{-1})^2 \leq \left|W(\beta' y^2)(1-q^{2-4s})\right| \leq 2(1+q^{-1/2})^2$$

when n is even, and

$$(1-q^{-2}) \leq \left|W(\beta' y^2)(1-q^{2-4s})\right| \leq 2(1+q^{-1})$$

when n is odd. Hence

$$\left|\frac{1}{4\pi i}\int_{\mathbb{C}} \langle f, \widetilde{E}_s\rangle b_\beta(y,s)\frac{du}{u}\right| \leq \begin{cases} \frac{2^{n-2}q^{1-m}(1+q^{1-k})(1+q^{-1/2})^2(1+q^{1-k/2})}{(1-q^{1-k/2})(1-q^{-1/2})} & \text{if } n \text{ is even,} \\ \frac{2^{n-2}q^{1-m}(1+q^{1-k})(1+q^{1-k/2})}{(1-q^{1-k/2})} & \text{if } n \text{ is odd.} \end{cases}$$

Also, for $s = k/4$, $k \geq 3$,

$$\begin{aligned} &\left|b_\beta(y,k/4)\right| \\ &\quad \geq \begin{cases} q^{1+m(k/2-2)}(1-q^{1/2-k/2})(1-q^{1-k/2})^{n-2}(1-q^{-1})^2 & \text{if } n \text{ is even,} \\ q^{1+m(k/2-2)}(1-q^{1-k/2})^{n-1}(1-q^{-2}) & \text{if } n \text{ is odd,} \end{cases} \\ &\quad \geq \begin{cases} q^{1+m(k/2-2)}(1-q^{-1})^3(1-q^{-1/2})^{n-2} & \text{if } n \text{ is even,} \\ q^{1+m(k/2-2)}(1-q^{-2})(1-q^{-1/2})^{n-1} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

(Note that for $k = 3$, $b_\beta(y, 3/4)$ is the β coefficient of $\widetilde{E}'$ since β' is square-free.) So except when $k = 3$ and $q = 3$, $r_k(\beta', m) = q^{mk/2}c_\beta(T^{-m}, k/4) \neq 0$ for m sufficiently large. $\square$

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