HALF-INTEGRAL WEIGHT SIEGEL MODULAR FORMS, HECKE OPERATORS, AND THETA SERIES

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Abstract. We analyse the behavior of Siegel theta series attached to arbitrary rank lattices under the symplectic group, and define half-integral weight Siegel modular forms. Then we introduce Hecke operators for half-integral weight Siegel forms, explicitly describing the action on Fourier coefficients (and giving an explicit choice for the matrices giving the action of each Hecke operator). We introduce generators of the Hecke algebra whose action on Fourier coefficients is more transparant. Applying these operators to theta series, we show that the average Siegel theta series of half-integral weight are eigenforms for the Hecke operators attached to primes not dividing the level; we explicitly compute the eigenvalues.

§0. Introduction

Quadratic forms abound in mathematics, as they capture the geometric notions of distance and orthogonality. Siegel asked: Given a lattice $L = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_m$ with geometry given by a positive definite quadratic form Q, and given another quadratic form T, how many sublattices of L have their geometry given by T? To study this question, Siegel introduced generalised theta series, which have Fourier expansions supported on positive semi-definite quadratic forms T; the Tth Fourier coefficient carries the answer to Siegel's question. (Note that every quadratic form can be naturally associated with a symmetric matrix.) Siegel theta series are the prototypes for Siegel modular forms and, as in the classical case of elliptic modular forms, Hecke operators help us study the Fourier coefficients of these modular forms.

In this paper we focus on Siegel modular forms of half-integral weight and the action of Hecke operators on their Fourier coefficients; for this, we need a good understanding of Siegel theta series. Thus, after presenting terminology and notation in §1, in §2 we analyse the behavior of Siegel theta series under the appropriate subgroup of the symplectic group; alternate proofs can be found, for instance, in §1.3 and §1.4 of [1]. We also define Siegel modular forms of weight $k + 1/2$ in §2. In §3 we define Hecke operators on half-integral weight Siegel modular forms, and we evaluate their action on the Fourier coefficients of the Siegel modular forms (see Proposition 3.1 and Theorem 3.4). This action involves generalised "twisted" Gauss

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sums, which we evaluate in §4, using the theory of quadratic forms over finite fields (see Theorem 4.3); more general versions of these are evaluated in Theorem 1.3 of [9]. In §5, we present a different set of generators for the Hecke algebra so that, using the results of §3 and §4, the action of the operators on Fourier coefficients is simpler to describe (see Theorem 5.1). As an application, in §6 we apply these averaged operators to Siegel theta series (provided the Hecke operators are associated to primes not dividing the "level" of the Siegel theta series). We obtain a generalised Eichler Commutation Relation (Theorem 6.4), and from this we deduce that the "average" theta series is an eigenform, as we simultaneously compute its eigenvalues (Corollary 6.5). In particular, when $\theta^{(n)}(L)$ is a Siegel theta series of degree n, weight $k+1/2$, and level N, the eigenvalue of the average theta series $\theta^{(n)}(\text{gen}L)$ under $T_j'(p^2)$ is

$$p^{j(j-1)/2+j(k-n)}\beta(n,j)(p^k+\chi'(p))\cdots(p^{k-j+1}+\chi'(p)),$$

provided p is a prime with $p\nmid N$, and $1\le j\le k$; here $\beta(n,j)$ is the number of j-dimensional subspaces of an n-dimensional space over $\mathbb{Z}/p\mathbb{Z}$, and

$$\chi'(p)=\left(\frac{(-1)^{k+1}2\det Q}{p}\right).$$

In Theorem 6.6, we show that $T_j'(p^2)$ annihilates $\theta^{(n)}(L)$ when $p\nmid N$ and $j>k$.

In §7 we recount some theory of quadratic forms over the p-adics $\mathbb{Z}_p$; in §8 we recount some results on representation numbers of quadratic forms over $\mathbb{Z}/p\mathbb{Z}$, $p\ne 2$; in §9 we recount some technical lemmas on symmetric matrices and coprime symmetric pairs.

To a large extent this paper is self-contained. For a broader discussion of Siegel modular forms, the reader is referred, for instance, to [1], [2], [4], [7]; for a broader discussion of quadratic forms, the reader is referred, for instance, to [3], [5], [8].

§1. Terminology and notation

Throughout, we let Q be a positive definite quadratic form on a lattice $L=\mathbb{Z}x_\oplus\cdots\oplus\mathbb{Z}x_m$; for convenience, we assume Q is scaled so that $Q(L)\subseteq 2\mathbb{Z}$. Associated to Q is the symmetric bilinear form B_Q given by $Q(x+y)=Q(x)+Q(y)+2B_Q(x,y)$. Relative to the basis $(x_1,\dots,x_m)$, we can identify L with $\mathbb{Z}^{m,1}$, and Q can be represented by the matrix $\big(B_Q(x_i,x_j)\big)$ (which we sometimes refer to as Q); with $x\in\mathbb{Z}^{m,1}$ representing a vector in L, the action of Q is given by the matrix multiplication ${}^txQx$.

For $n\in\mathbb{Z}_+$, we attach to L a generalised theta series

$$\theta^{(n)}(L;\tau)=\sum_{C\in\mathbb{Z}^{m,n}}\mathrm{e}\{Q[C]\tau\}$$

where $\mathrm{e}\{*\}=\exp(\pi i Tr(*))$ and $Q[C]={}^tCQC$. As Siegel proved (cf. §1.1.1 and §1.3.3 of [1]), $\theta^{(n)}(L;\tau)$ is an analytic function in all the variables of τ where τ lies in Siegel's upper half-space

$$\mathcal{H}_{(n)}=\{X+iY:\ X,Y\in\mathbb{R}^{n,n}_{\text{sym}},\ Y>0\ \};$$

$Y > 0$ means that as a quadratic form, Y is positive definite. Note that

$$\theta^{(n)}(L;\tau) = \sum_T r(Q,T)\mathrm{e}\{T\tau\}$$

where T varies over all $n \times n$ symmetric matrices and

$$r(Q,T) = \#\{C \in \mathbb{Z}^{m,n} :\ {}^tCQC = T\ \}.$$

Since $Q(L) \subseteq 2\mathbb{Z}$, the matrix for Q is even integral, meaning it is integral with even diagonal entries; thus with $C \in \mathbb{Z}^{m,n}$, ${}^tCQC$ is also even integral. This representation number $r(Q,T)$ gives us information about the number of sublattices of L on which the quadratic form Q restricts to T; the sublattice associated to ${}^tCQC$ is $\mathbb{Z}y_1 + \cdots + \mathbb{Z}y_n$ where

$$(y_1\ \dots\ y_n) = (x_1\ \dots\ x_m)C.$$

Also, when $n > m$ (m the rank of L), then $\mathbb{Z}y_1 + \cdots + \mathbb{Z}y_n$ necessarily has rank less than n; for this reason, we only consider $n \le m$.

The symplectic group is defined by

$$Sp_n(\mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} :\ A\,{}^tB,\, C\,{}^tD \text{ are symmetric, } A\,{}^tD - B\,{}^tC = I \right\}$$

where ${}^tB$ denotes the transpose of B; we set

$$\Gamma_0^{(n)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} :\ N|C \right\}.$$

As discussed, for example in Proposition 1.2.1 of [1], $Sp_n(\mathbb{Z})$ acts on $\mathcal{H}_{(n)}$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\tau = (A\tau + B)(C\tau + D)^{-1}.$$

Using the Inversion Formula, one derives Siegel's Transformation Formula (see Theorem 2.2) which relates $\theta^{(n)}(L;\tau)$ to $\theta^{(n)}(L;\gamma\tau)$ where $\gamma \in \Gamma_0^{(n)}(N)$.

The quadratic form Q on L can be naturally extended to a quadratic form on the vector space $V = \mathbb{Q}L = \mathbb{Q}x_1 \oplus \cdots \oplus \mathbb{Q}x_m$. Identifying Q with an even integral matrix (as discussed above), Q has an invariant called the level, which is the smallest positive integer N so that NQ^{-1} is also even integral. Siegel's Inversion Formula (Lemma 1.3.15 of [1], and stated here as Theorem 2.1) relates $\theta^{(n)}(L;\tau)$ to $\theta^{(n)}(L^\#;-\tau^{-1})$ where $L^\#$ is the dual of L, defined by

$$L^\# = \{w \in V :\ B_Q(w,L) \subseteq \mathbb{Z}\ \}.$$

(Note that when we have identified L with $\mathbb{Z}^{m,1}$ and Q with a matrix relative to some $\mathbb{Z}$-basis for L, $L^\#$ is naturally identified with $Q^{-1}\mathbb{Z}^{m,1}$.)

We rely heavily on the arithmetic theory of quadratic forms over $\mathbb{Z}$ and over the p-adics $\mathbb{Z}_p$, as well as the arithmetic theory of quadratic forms over finite fields. When L is a free lattice over a Dedekind domain $\mathbb{D}$ with quadratic form Q and A is a matrix representing Q relative to some $\mathbb{D}$-basis for L, we write $L \simeq A$, and we set $\operatorname{disc} L = \det A$. (Thus $\operatorname{disc} L$ is well-defined up to squares of units of $\mathbb{D}$. Also note that since $\mathbb{Z}_p$ is a principle ideal domain, any $\mathbb{Z}_p$-lattice is free.) The lattice L is said to be regular if 0 is the only vector $x \in L$ so that $B_Q(x, L) = 0$. When A is a matrix with $L \simeq A$ relative to some $\mathbb{D}$-basis $(x_1, \dots, x_m)$, L is regular exactly when $\det A \neq 0$; in the case $\det A \neq 0$, $(y_1, \dots, y_m) = (x_1, \dots, x_m)A^{-1}$ is a dual basis for $(x_1, \dots, x_m)$ (where A^{-1} has entries in the quotient field of $\mathbb{D}$), and $(y_1, \dots, y_m)$ is a $\mathbb{D}$-basis for $L^{\#}$. (So $L^{\#} \simeq A^{-1}$.) We say L is unimodular if $L = L^{\#}$; a regular lattice L is unimodular exactly when A is invertible over $\mathbb{D}$. We say two lattices L, K are isometric if $L \simeq A$ and $K \simeq A$, A a symmetric matrix; the notation $L \simeq K$ means L and K are isometruc. An automorphism σ of L is an isometry of L when $B_Q(\sigma x, \sigma y) = B_Q(x, y)$ for all $x, y \in L$. We use $O(L)$ to denote the orthogonal group of L, meaning the group of isometries mapping L to itself; we use $o(L)$ to denote the order of this group. (Note that when Q is positive definite, $o(L)$ is known to be finite.)

A nonzero vector x is called isotropic if $Q(x) = 0$; a lattice (or vector space) is called isotropic if it contains a (nonzero) isotropic vector, and it is called anisotropic otherwise; it is called totally isotropic if all its vectors are isotropic.

For $\mu_i \in \mathbb{D}$, we use $\langle \mu_1, \dots, \mu_m \rangle$ to denote the diagonal matrix $\operatorname{diag}\{\mu_1, \dots, \mu_m\}$, and for square matrices A_1, A_2 we use $A_1 \perp A_2$ to denote the block diagonal matrix $\operatorname{diag}\{A_1, A_2\}$. For sublattices L_1, L_2 of L with $L_1 \cap L_2 = \{0\}$, we write $L_1 \perp L_2$ to denote the direct sum of L_1 and L_2 in the case that $B_Q(L_1, L_2) = 0$.

Suppose p is an odd prime and V is a vector space over $\mathbb{Z}/p\mathbb{Z}$ with quadratic form Q. The radical of V is

$$\operatorname{rad} V = \{w \in V : \ B_Q(w, V) = 0 \ \}.$$

When $\operatorname{rad} V = \{0\}$, we say V is regular. Note that we always have $V = \operatorname{rad} V \perp V'$ where V' is regular; while V' is not uniquely determined, its isometry class is. We use $\mathbb{H}$ to denote a hyperbolic plane in V, meaning $\mathbb{H} \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (relative to some basis). We use $\mathbb{A}$ to denote an anisotropic plane, meaning $\dim \mathbb{A} = 2$ and $\mathbb{A}$ contains no isotropic vectors. When V is regular and U is a totally isotropic subspace of V, there exists another subspace U' of V so that $U \oplus U' \simeq \mathbb{H}^{\dim U}$. Also, whenever V is a regular space with $\dim V \geq 3$, V is isotropic. A regular subspace U of V splits V, meaning $V = U \perp U'$ for some U'; consequently, when V is regular, $V \simeq \mathbb{H}^d \perp W$ where W is regular with dimension 0, 1, or 2. Thus when $\dim V = 2d$, either $V \simeq \mathbb{H}^d$ or $V \simeq \mathbb{H}^{d-1} \perp \mathbb{A}$. When $\dim V = 2d + 1$ then $V \simeq \mathbb{H}^d \perp \langle \nu \rangle$ for some $\nu \neq 0$; the square class of ν determines the isometry class of V.

For $\mathbb{F}$ a finite field and $Q \in \mathbb{F}_{\text{sym}}^{m,m}$, $T \in \mathbb{F}_{\text{sym}}^{\ell,\ell}$, we use $r(Q, T)$ to denote the number of representations of T by Q, and $r^*(Q, T)$ to denote the number of primitive representations of T by Q. Thus

$$r(Q,T) = \#\{C \in \mathbb{F}^{m,\ell} : {}^tCQC = T\ \},$$
$$r^*(Q,T) = \#\{C \in \mathbb{F}^{m,\ell} : {}^tCQC = T,\ \operatorname{rank} C = \ell\ \}.$$

We often write $r^*(Q,\alpha)$ for $r^*(Q,\langle\alpha\rangle)$. Note also that when $\alpha \neq 0$, we necessarily have $r(Q,\alpha) = r^*(Q,\alpha)$. With $o(T) = r^*(T,T)$, the order of the orthogonal group of T, we set

$$R^*(Q,T) = \frac{r^*(Q,T)}{o(T)};$$

so with V a dimension m vector space over $\mathbb{F}$ with quadratic form Q, $R^*(Q,T)$ is the number of dimension ℓ subspaces of V on which Q restricts to give T (relative to some basis for the subspace).

There are also several elementary functions we frequently encounter, and thus it is useful to give them names. For a fixed prime p and $m, r \in \mathbb{Z}$ we set

$$\delta(m,r) = \prod_{i=0}^{r-1}(p^{m-i}+1), \qquad \mu(m,r) = \prod_{i=0}^{r-1}(p^{m-i}-1),$$
$$\beta(m,r) = \frac{\mu(m,r)}{\mu(r,r)}, \qquad \gamma(m,r) = \frac{\mu\delta(m,r)}{\mu\delta(r,r)},$$
$$\eta(m,r) = \prod_{i=0}^{m-r-1}(p^m - p^{r+i}) = p^{m(m-1)/2-r(r-1)/2}\mu(m-r,m-r).$$

(Here we have written $\mu\delta(m,r)$ for the product $\mu(m,r)\delta(m,r)$; also note that when $m, r \geq 0$, $\beta(m,r)$ is the number of r-dimensional subspaces of an m-dimensional space over $\mathbb{Z}/p\mathbb{Z}$.) These functions satisfy some easily verified relations, which we will exploit frequently:

$$\delta(m,r)\delta(m-r,q) = \delta(m,r+q), \qquad \mu(m,r)\mu(m-r,q) = \mu(m,r+q),$$

and when $m \geq 1$,

$$\beta(m,r) = \beta(m,m-r), \qquad \gamma(m,r) = \gamma(m,m-r),$$
$$\beta(m,r) = p^r\beta(m-1,r) + \beta(m-1,r-1) = \beta(m-1,r) + p^{m-r}\beta(m-1,r-1),$$
$$\gamma(m,r) = p^{2r}\gamma(m-1,r) + \gamma(m-1,r-1).$$

§2. Siegel theta series and half-integral weight Siegel modular forms

Throughout this section, L is a rank m $\mathbb{Z}$-lattice equipped with the positive definite, even integral quadratic form Q of level N.

Here we use an "inversion formula" from [1] to give an alternate proof of the transformation formula of a Siegel theta series. To do this we generalise an argument of Eichler, where he first relies on the identity

$$\frac{a\tau+b}{c\tau+d}=\frac{b}{d}+\frac{\tau}{d(c\tau+d)}$$

for $\tau\in\mathcal{H}_{(1)}$ and $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$. Then we use the theory of quadratic forms, and lemmas based on the Elementary Divisor Theorem, to generalise Eichler's method to evaluate the character that arises in this transformation formula.

The following theorem is Lemma 1.3.15 of [1].

Theorem 2.1 (Inversion Formula). *With $L^{\#}$ the dual of L,*

$$\theta^{(n)}(L;\tau)=(\det Q)^{-n/2}(\det(-i\tau))^{-m/2}\theta^{(n)}(L^{\#};-\tau^{-1}).$$

More generally, take $G_0\in\mathbb{Q}^{m,n}$ and set

$$\theta^{(n)}(L,G_0;\tau)=\sum_{G\in\mathbb{Z}^{m,n}}e\{Q[G+G_0]\tau\}$$

where $Q[G']={}^tG'QG'$. Then

$$\theta^{(n)}(L,G_0;\tau)\\ =(\det Q)^{-n/2}(\det(-i\tau))^{-m/2}\sum_{G\in\mathbb{Z}^{m,n}}e\{-Q^{-1}[G]\tau^{-1}-2\,{}^tGG_0\}.$$

(Here $(\det(-i\tau))^{1/2}$ is taken to be positive when $\tau=iY$, $Y>0$; in general, the sign is found by analytic continuation in $\mathcal{H}_{(n)}$.)

Theorem 2.2 (Transformation Formula). *With L as above and $\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in\Gamma_0^{(n)}(N)$, we have*

$$\theta(L;(A\tau+B)(C\tau+D)^{-1})\\ =(\det(-i\tau(C\tau+D)^{-1}D)^{-m/2}(\det(-i\tau))^{m/2}\\ \cdot\left(\sum_{G\in\mathbb{Z}^{m,n}/\mathbb{Z}^{m,n}\,{}^tD}e\{Q[G]BD^{-1}\}\right)\theta(L;\tau).$$

Proof. Note that ${}^tDA-{}^tBC=I$, and so $(A\tau+B)(C\tau+D)^{-1}={}^tD^{-1}\,{}^tB+{}^tD^{-1}\tau(C\tau+D)^{-1}$ with ${}^tD^{-1}\,{}^tB$ symmetric. Using these observations, the fact that $e\{MM'\} = e\{M'M\}$, and the Inversion Formula (Theorem 2.1), we get

$$\begin{aligned}
&\theta(L;(A\tau+B)(C\tau+D)^{-1})\\
&\quad= \sum_{G_0\in\mathbb{Z}^{m,n}/\mathbb{Z}^{m,n}\,{}^tD} e\{Q[G_0]\,{}^tD^{-1}\,{}^tB\}\\
&\qquad\cdot \sum_{G\in\mathbb{Z}^{m,n}} e\{Q[G_0+G\,{}^tD]\,{}^tD^{-1}\tau(C\tau+D)^{-1}\}\\
&\quad= \sum_{G_0} e\{Q[G_0]BD^{-1}\}\theta(L,G_0\,{}^tD^{-1};\tau(C\tau+D)^{-1}D\}\\
&\quad= (\det Q)^{-n/2}(\det(-i\tau(C\tau+D)^{-1}D))^{-m/2}\sum_{G_0} e\{Q[G_0]BD^{-1}\}\\
&\qquad\cdot \sum_{G\in\mathbb{Z}^{m,n}} e\{-Q^{-1}[G]D^{-1}(C\tau+D)\tau^{-1} - 2\,{}^tGG_0\,{}^tD^{-1}\}.
\end{aligned}$$

Since $e\{MM'\} = e\{M'M\}$, $e\{M\} = e\{{}^tM\}$, and $D\,{}^tA - C\,{}^tB = I$, we have

$$\begin{aligned}
&e\{-Q[G_0B - Q^{-1}G]D^{-1}C\}\\
&\quad= e\{-Q[G_0]BD^{-1}C\,{}^tB + 2\,{}^tG_0GD^{-1}C\,{}^tB - Q^{-1}[G]D^{-1}C\}\\
&\quad= e\{Q[G_0]BD^{-1} - 2\,{}^tGG_0\,{}^tD^{-1} - Q^{-1}[G]D^{-1}C\}.
\end{aligned}$$

By Lemma 9.2, with G any element of $\mathbb{Z}^{m,n}$, $G_0B - Q^{-1}G$ varies over the quotient $Q^{-1}\mathbb{Z}^{m,n}/Q^{-1}\mathbb{Z}^{m,n}D$ as G_0 varies over $\mathbb{Z}^{m,n}/\mathbb{Z}^{m,n}\,{}^tD$; in particular (taking $G=0$), G_0B varies over $Q^{-1}\mathbb{Z}^{m,n}/Q^{-1}\mathbb{Z}^{m,n}D$ as G_0 varies over $\mathbb{Z}^{m,n}/\mathbb{Z}^{m,n}\,{}^tD$. Also note that since $N|C$ and NQ^{-1} is even integral,

$$e\{Q[Q^{-1}G + Q^{-1}G'D]D^{-1}C\} = e\{Q[Q^{-1}G]D^{-1}C\}$$

for any $G, G' \in \mathbb{Z}^{m,n}$, and hence

$$\sum_{G\in\mathbb{Z}^{m,n}/\mathbb{Z}^{m,n}D} e\{Q[Q^{-1}G]D^{-1}C\}$$

is well-defined. Thus

$$\begin{aligned}
&\sum_{G_0\in\mathbb{Z}^{m,n}/\mathbb{Z}^{m,n}\,{}^tD} e\{Q[G_0]BD^{-1} - 2\,{}^tGG_0\,{}^tD^{-1} - Q^{-1}[G]D^{-1}C\}\\
&\quad= \sum_{G_0\in\mathbb{Z}^{m,n}/\mathbb{Z}^{m,n}\,{}^tD} e\{Q[G_0]BD^{-1}\}.
\end{aligned}$$

So

$$\begin{aligned}
\theta(L;(A\tau+B)(C\tau+D)^{-1}) &= (\det Q)^{-n/2}(\det(-i\tau(C\tau+D)^{-1}D)^{-m/2}\\
&\qquad\cdot \sum_{G_0\in\mathbb{Z}^{m,n}/\mathbb{Z}^{m,n}\,{}^tD} e\{Q[G_0]BD^{-1}\}\ \theta(L^{\#};-\tau^{-1});
\end{aligned}$$

applying the Inversion Formula again yields the theorem. □

Definition. For $b, d \in \mathbb{Z}$ with $(b,d)=1$ and $d \neq 0$, we let $\mathcal{G}_b(d)$ denote the usual Gauss sum, meaning

$$\mathcal{G}_b(d) = \sum_{a \in \mathbb{Z}/d\mathbb{Z}} \mathrm{e}\{2a^2 b/d\}.$$

For $({}^tB, {}^tD)$ a coprime symmetric pair of $n \times n$ integral matrices with $\det D \neq 0$, and for Q an $m \times m$, symmetric, even integral matrix with $\det Q \neq 0$, we define a generalised Gauss sum

$$\mathcal{G}_B(D;Q) = \sum_{G \in \mathbb{Z}^{m,n}/\mathbb{Z}^{m,n}\,{}^tD} \mathrm{e}\{Q[G]BD^{-1}\}.$$

When $m=1$ and $Q=(2)$, we let $\mathcal{G}_B(D) = \mathcal{G}_B(D;(2))$.

In the following theorem we analyze $\mathcal{G}_B(D;Q)$ in terms of the classical Gauss sum and the Kronecker symbol; recall that the Kronecker symbol generalises the Legendre symbol using the rule

$$\left(\frac{a}{2}\right) = \begin{cases} 0 & \text{if } 2|a, \\ 1 & \text{if } a \equiv \pm 1 \pmod 8, \\ -1 & \text{if } a \equiv \pm 3 \pmod 8. \end{cases}$$

The proof below relies on some elementary, technical lemmas about symmetric matrices, which are stated and proved in §9.

Note also that by Proposition 7.4, if Q is an $m \times m$ even integral matrix with $2 \nmid \det Q$, then m is necessarily even.

Lemma 2.3. *Say $Q \in \mathbb{Z}^{m,m}$, $U' \in \mathbb{Z}^{r,r}$ so that Q, U' are symmetric with Q even, and p is a fixed prime such that $p \nmid \det Q$, $p \nmid \det U'$. Then*

$$\mathcal{G}_{U'}(p^a I_r; Q) = \begin{cases} \left(\frac{2^m \det Q}{p}\right)^{ar} \left(\frac{\det U'}{p}\right)^{am} \mathcal{G}_1(p^a)^{mr} & \text{if } p \neq 2, \\ \left(\frac{(-1)^k \det Q}{2}\right)^{ar} 2^{ark} & \text{if } p = 2,\ m = 2k. \end{cases}$$

Also, for $p \neq 2$ and $a > 1$, $\mathcal{G}_1(p^a) = p\mathcal{G}_1(p^{a-2})$.

Proof. Note that since Q, U' are symmetric, for $G, G' \in \mathbb{Z}^{m,r}$ we have

$$\mathrm{e}\{{}^tGQG'U'/p\} = \mathrm{e}\{U'\,{}^tG'QG/p\} = \mathrm{e}\{{}^tG'QGU'/p\}.$$

So for $a > 1$,

$$\sum_{\substack{G \in \mathbb{Z}^{m,r} \\ G(p^a)}} \mathrm{e}\{Q[G]U'/p^a\} = \sum_{\substack{G \in \mathbb{Z}^{m,r} \\ G(p^{a-1})}} \mathrm{e}\{Q[G]U'/p^a\} \cdot \sum_{\substack{G' \in \mathbb{Z}^{m,r} \\ G'(p)}} \mathrm{e}\{2\,{}^tGQG'U'/p\}.$$

For fixed G, the map $G' \mapsto \mathrm{e}\{2\,{}^tGQG'U'/p\}$ is a character on $\mathbb{Z}^{m,r}/p\mathbb{Z}^{m,r}$, and since $p \nmid \det U'$, $p \nmid \det Q$, this is the trivial character only when $G \in p\mathbb{Z}^{m,r}$; thus

$$\mathcal{G}_{U'}(p^a I_r; Q) = p^{mr}\mathcal{G}_{U'}(p^{a-2}I_r; Q);$$

this proves the theorem when $2|a$.

Next, suppose $a = 1$; first consider $p \neq 2$. Then by §92 [8] (see also Proposition 7.1), we know Q, U' can be diagonalized over $\mathbb{Z}_p$, so we can find $E_p \in SL_m(\mathbb{Z}_p)$, $E'_p \in SL_r(\mathbb{Z}_p)$ so that ${}^tE_pQE_p$ and $E'_pU'\,{}^tE'_p$ are diagonal. Since $SL_\ell(\mathbb{Z})$ maps onto $SL_\ell(\mathbb{Z}/M\mathbb{Z})$ for any $\ell, M \in \mathbb{Z}$ (see, for instance, p. 21 of [10]), we can find $E \in SL_m(\mathbb{Z})$ and $E' \in SL_r(\mathbb{Z})$ so that $E \equiv E_p\ (\mathrm{mod}\ p\mathbb{Z}_p)$ and $E' \equiv E'_p\ (\mathrm{mod}\ p\mathbb{Z}_p)$, and thus

$$\begin{aligned} {}^tEQE &\equiv \mathrm{diag}\{2\alpha_1, \dots, 2\alpha_m\}\ (\mathrm{mod}\ p), \\ E'U'\,{}^tE' &\equiv \mathrm{diag}\{\mu_1, \dots, \mu_r\}\ (\mathrm{mod}\ p). \end{aligned}$$

Since $E\mathbb{Z}^{m,r}E' = \mathbb{Z}^{m,r}$,

$$\mathcal{G}_{U'}(pI_r; Q) = \prod_{i=1}^{m}\prod_{j=1}^{r}\mathcal{G}_{\mu_j}(p; 2\alpha_i) = \left(\frac{2^m \det Q}{p}\right)^r \left(\frac{\det U'}{p}\right)^m \mathcal{G}_1(p)^{mr}.$$

Now say $p = 2$. Then by Proposition 7.4, m must be even, and we can find $E_p \in SL_m(\mathbb{Z}_p)$ so that ${}^tE_pQE_p$ is an orthogonal sum of matrices of the form $\begin{pmatrix} 2\alpha & 1 \\ 1 & 2\beta \end{pmatrix}$ where $\alpha, \beta \in \mathbb{Z}_p$, and we can find $E'_p \in SL_r(\mathbb{Z}_p)$ so that $E'_pU'\,{}^tE'_p$ is an orthogonal sum of a diagonal matrix and matrices of the form $\begin{pmatrix} 2\nu & 1 \\ 1 & 2\eta \end{pmatrix}$. Thus again using that $SL_\ell(\mathbb{Z})$ maps onto $SL_\ell(\mathbb{Z}/M\mathbb{Z})$, we can find $E \in SL_m(\mathbb{Z})$, $E' \in SL_r(\mathbb{Z})$ so that

$$\begin{aligned} {}^tEQE &\equiv \begin{pmatrix} 2\alpha_1 & 1 \\ 1 & 2\beta_1 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 2\alpha_k & 1 \\ 1 & 2\beta_k \end{pmatrix}\ (\mathrm{mod}\ 4), \\ E'U'\,{}^tE' &\equiv \mathrm{diag}\{\mu_1, \dots, \mu_t\} \perp \begin{pmatrix} 2\nu_1 & 1 \\ 1 & 2\eta_1 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 2\nu_s & 1 \\ 1 & 2\eta_s \end{pmatrix}\ (\mathrm{mod}\ 4). \end{aligned}$$

(Note that since $2 \nmid \det U'$, we have $2 \nmid \mu_i$.) Thus with $Q_i = \begin{pmatrix} 2\alpha_i & 1 \\ 1 & 2\beta_i \end{pmatrix}$ and $U_h = \begin{pmatrix} 2\nu_h & 1 \\ 1 & 2\eta_h \end{pmatrix}$,

$$\mathcal{G}_{U'}(2I_r; Q) = \prod_{i=1}^{k}\left(\prod_{j=1}^{t}\mathcal{G}_{\mu_j}(2; Q_i)\right)\left(\prod_{h=1}^{s}\mathcal{G}_{U_h}(2I_2; Q_i)\right).$$

Now, for $2 \nmid \mu$,

$$\mathcal{G}_\mu(2;Q_i) = \mathrm{e}\{0\} + \mathrm{e}\{\alpha_i\} + \mathrm{e}\{\beta_i\} + \mathrm{e}\{\alpha_i+\beta_i+1\} = \left(\frac{-\det Q_i}{2}\right)\cdot 2.$$

Somewhat similarly,

$$\mathcal{G}_{U_h}(2I_2;Q_i) = \sum_{a,b,c,d\in\mathbb{Z}/2\mathbb{Z}} \mathrm{e}\{ad+bc\} = 4.$$

Thus

$$\mathcal{G}_{U'}(pI_r;Q) = \left(\frac{(-1)^k\det Q}{2}\right)^r \cdot 2^{kr}.$$

The lemma now follows. □

Theorem 2.4. *Suppose* $({}^tB,\,{}^tD)$ *is a coprime symmetric pair of* $n\times n$ *integral matrices with* $\det D\neq 0$. *Take* $Q\in\mathbb{Z}^{m,m}$ *symmetric and even with* $\det Q\neq 0$ *and* $(\det Q,\det D)=1$. *Then*

$$\mathcal{G}_B(D;Q) = \begin{cases} \left(\frac{(-1)^k\det Q}{|\det D|}\right)|\det D|^k & \text{if } m=2k,\\ \left(\frac{(-1)^k 2\cdot\det Q}{|\det D|}\right)|\det D|^k\ \mathcal{G}_B(D) & \text{if } m=2k+1.\end{cases}$$

Also, for $q\in\mathbb{Z}$ *with* $(q,\det D)=1$, *we have* $\mathcal{G}_{qB}(D) = \left(\frac{q}{|\det D|}\right)\ \mathcal{G}_B(D)$ *and* $\left(\mathcal{G}_B(D)\right)^2 = \left(\frac{-1}{|\det D|}\right)|\det D|$.

Proof. By the Elementary Divisor Theorem, we can find $E_1,E_2\in SL_n(\mathbb{Z})$ so that

$$V = {}^tE_2^{-1}DE_1 = \mathrm{diag}\{d_1,\dots,d_n\}$$

with $d_i|d_{i+1}$. Notice that as G runs over $\mathbb{Z}^{m,n}/\mathbb{Z}^{m,n}\,{}^tD$, GE_2^{-1} runs over

$$\mathbb{Z}^{m,n}E_2^{-1}/\mathbb{Z}^{m,n}\,{}^tDE_2^{-1} = \mathbb{Z}^{m,n}/\mathbb{Z}^{m,n}\,{}^tE_1^{-1}V = \mathbb{Z}^{m,n}/\mathbb{Z}^{m,n}V,$$

so with $U=E_2BE_1$ we have

$$\mathcal{G}_B(D;Q) = \sum_{G\in\mathbb{Z}^{m,n}/\mathbb{Z}^{m,n}\,{}^tD} \mathrm{e}\{Q[G]BE_1V^{-1}\,{}^tE_2^{-1}\} = \mathcal{G}_U(V;Q).$$

For each prime $p|\det D$, set $V_p=\mathrm{diag}\{p^{e_1},\dots,p^{e_n}\}$ where $e_i=\mathrm{ord}_p(d_i)$, and set $W_p=VV_p^{-1}$. Then by Lemma 9.3,

$$\mathcal{G}_B(D;Q) = \prod_{p|\det D}\left(\sum_{G\in\mathbb{Z}^{m,n}W_p/\mathbb{Z}^{m,n}V} \mathrm{e}\{Q[G]UV^{-1}\}\right) = \prod_{p|\det D}\mathcal{G}_{W_pU}(V_p;Q).$$

Notice that this means $\mathcal{G}_B(D) = \prod_{p|\det D} \mathcal{G}_{W_pU}(V_p)$.

Fix a prime $p|\det D$; choose $a_1 < a_2 < \cdots < a_\ell$ and $r_1, \dots, r_\ell \in \mathbb{Z}$ so that $V_p = \mathrm{diag}\{p^{a_1} I_{r_1}, \dots, p^{a_\ell} I_{r_\ell}\}$. Then by Lemma 9.1 (where we use W_pU in place of U and V_p in place of V), there is some $Y \in SL_n(\mathbb{Z})$ so that

$$Y W_p U V_p^{-1}\, {}^tY \equiv \mathrm{diag}\{p^{-a_1} U_1', \dots, p^{-a_\ell} U_\ell'\} \pmod{\mathbb{Z}}$$

where U_i' is $r_i \times r_i$ and symmetric with $p \nmid \det U_i'$ unless $i = 1$ and $a_1 = 0$. Also, $V_p Y = Y' V_p$ where $Y' \in SL_n(\mathbb{Z})$. Hence

$$\mathbb{Z}^{m,n} Y / \mathbb{Z}^{m,n} V_p Y = \mathbb{Z}^{m,n} / \mathbb{Z}^{m,n} V_p;$$

so replacing G by GY we have

$$\mathcal{G}_{W_pU}(V_p; Q) = \prod_{i=1}^{\ell} \mathcal{G}_{U_i'}(p^{a_i} I_{r_i}; Q).$$

Fix i and let $r = r_i$, $a = a_i$, $U' = U_i'$. Note that if $a_1 = 0$ then the sum on G_1 is 1, so assume $i > 1$ if $a_1 = 0$.

Note that for $p \neq 2$, standard techniques show $\mathcal{G}_1(p)^2 = p\left(\frac{-1}{p}\right)$. So when $m = 2k$, Lemma 2.3 gives us

$$\begin{aligned}\mathcal{G}_{W_pU}(V_p; Q) &= \prod_{i=1}^{\ell} \left(\frac{(-1)^k \det Q}{p}\right)^{a_i r_i} \cdot p^{a_i r_i k} \\ &= \left(\frac{(-1)^k \det Q}{|\det V_p|}\right) \cdot |\det V_p|^k.\end{aligned}$$

Now suppose $m = 2k + 1$. Then

$$\mathcal{G}_B(D) = \prod_{p|\det D} \mathcal{G}_{W_pU}(V_p),$$

and by Lemma 2.3,

$$\mathcal{G}_{W_pU}(V_p) = \prod_{i=1}^{\ell} \left(\frac{\det U_i'}{p}\right)^{a_i} \mathcal{G}_1(p^{a_i})^{r_i}.$$

Thus

$$\begin{aligned}\mathcal{G}_{W_pU}(V_p; Q) &= \prod_{i=1}^{\ell} \left(\frac{(-1)^k 2 \det Q}{p}\right)^{a_i r_i} p^{a_i r_i k} \mathcal{G}_{W_pU}(V_p) \\ &= \left(\frac{(-1)^k 2 \det Q}{|\det V_p|}\right) |\det V_p|^k \mathcal{G}_{W_pU}(V_p).\end{aligned}$$

Since $\mathcal{G}_B(D;Q) = \prod_{p|\det D} \mathcal{G}_{W_pU}(V_p;Q)$, the first claim of the theorem now follows.

The product decomposition of $\mathcal{G}_B(D)$ above makes it clear that $\mathcal{G}_{qB}(D) = \left(\frac{q}{|\det D|}\right)\mathcal{G}_B(D)$ for $(q,\det D)=1$; also,

$$(\mathcal{G}_B(D))^2 = \mathcal{G}_B(D;2I_2) = \left(\frac{-1}{|\det D|}\right)|\det D|. \qquad \square$$

With $\theta^{(n)}(\tau) = \sum_{C\in\mathbb{Z}^{1,n}} \mathrm{e}\{2\,{}^tCC\tau\}$, we have $(\theta^{(n)}(\tau))^m = \theta^{(n)}(L';\tau)$ where $L' \simeq 2I_m$; the level of the quadratic form $2I_m$ is 4. Also, as discussed in §1, an even quadratic form on an odd rank lattice necessarily divisible by 4. Thus Theorems 2.2 and 2.4 give us the following.

Corollary 2.5. *Let L be a lattice equipped with a positive definite quadratic form Q with level N; take $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(N)$. If rank$L = 2k$, then*

$$\theta^{(n)}(L;\gamma\tau) = \det(C\tau+D)^k\,(sgn\,\det D)^k \left(\frac{(-1)^k\det Q}{|\det D|}\right)\cdot\theta^{(n)}(L;\tau).$$

If rank$L = 2k+1$, then $4|N$ and

$$\theta^{(n)}(L;\gamma\tau) = \left(\frac{2\det Q}{|\det D|}\right)\left(\frac{\theta^{(n)}(\gamma\tau)}{\theta^{(n)}(\tau)}\right)^{2k+1}\theta^{(n)}(L;\tau).$$

Note that in the course of proving Theorem 2.4, we have also evaluated $\mathcal{G}_B(D)$. In particular, we have the following corollary, which we will use in the following section.

Corollary 2.6. *Let $D = diag\{I_{r_0}, pI_{r_1}, p^2I_{r_2}, I_{n-j}\}$ where $r_0+r_1+r_2 = j$, and take $B\in\mathbb{Z}^{n,n}$ so that $({}^tB, {}^tD)$ is a coprime symmetric pair. Then*

$$\mathcal{G}_B(D) = p^{r_2}\left(\frac{\det Y_1}{p}\right)\mathcal{G}_1(p)^{r_1}$$

where $B = \begin{pmatrix} Y_0 & * & * \\ * & Y_1 & * \\ * & * & * \end{pmatrix}$ with Y_0 $r_0\times r_0$ and Y_1 $r_1\times r_1$.

Siegel's generalised theta series are the prototypes for Siegel modular forms. Here we are concerned with half-integral weight, so from now on we restrict our attention to this case.

Definition. With $n, k, N \in \mathbb{Z}_+$, $n > 1$, $4|N$, and χ a character modulo N, we say a function $F : \mathcal{H}_{(n)} \to \mathbb{C}$ is a degree n, weight $k + 1/2$ Siegel modular form of level N and character χ if $F(\tau)$ is analytic (in each variable of τ), and

$$F(\gamma\tau) = \chi(\det D) \left(\frac{\theta^{(n)}(\gamma\tau)}{\theta^{(n)}(\tau)} \right)^{2k+1} F(\tau)$$

for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(N)$; note that by Theorems 2.2 and 2.4,

$$\left| \frac{\theta^{(n)}(\gamma\tau)}{\theta^{(n)}(\tau)} \right|^2 = |\det(C\tau + D)|.$$

We let $\mathcal{M}_{k+1/2}(\Gamma_0^{(n)}(N), \chi)$ denote the complex vector space of all such functions. Take $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n^+(\mathbb{Q})$ and $\phi_\gamma : \mathcal{H}_{(n)} \to \mathbb{C}$ so that

$$|\phi_\gamma(\tau)|^2 = |(\det \gamma)^{-1/2} \det(C\tau + D)|;$$

with $4|N$ and $F \in \mathcal{M}_{k+1/2}(\Gamma_0^{(n)}(N), \chi)$, we define

$$\big(F|[\gamma, \phi_\gamma]\big)(\tau) = \phi_\gamma(\tau)^{-(2k+1)} F(\gamma\tau).$$

For $\gamma \in \Gamma_0^{(n)}(4)$, we set $\widetilde{\gamma} = [\gamma, \theta^{(n)} \circ \gamma/\theta^{(n)}]$; then $F|\widetilde{\gamma} = \chi(\gamma)\, F$ for $\gamma \in \Gamma_0^{(n)}(N)$. Also, for $\gamma, \gamma' \in Sp_n^+(\mathbb{Q})$,

$$F|[\gamma, \phi_\gamma]|[\gamma', \phi_{\gamma'}] = \mathbb{F}|[\gamma\gamma', (\phi_\gamma \circ \gamma') \cdot \phi_{\gamma'}]$$

(where $\big((\phi_\gamma \circ \gamma') \cdot \phi_{\gamma'}\big)(\tau) = \phi_\gamma(\gamma'\tau) \cdot \phi_{\gamma'}(\tau)$.) Thus for $\gamma, \gamma' \in \Gamma_0^{(n)}(4)$, $F|\widetilde{\gamma}|\widetilde{\gamma}' = F|\widetilde{\gamma\gamma'}$.

Remark. As discussed on pp. 44-46 of [7], when $F : \mathcal{H}_{(n)} \to \mathbb{C}$ is analytic with $F(\tau + B) = F(\tau)$ for all $B \in \mathbb{Z}_{\mathrm{sym}}^{n,n}$, we have

$$F(\tau) = \sum_T c(T) \mathrm{e}\{T\tau\}$$

where T varies over even integral $n \times n$ matrices. Then, by work of Koecher (Theorem 1, p. 45 [7]), those T in the support of F must be positive semi-definite, and in addition, for any $\epsilon > 0$, F is bounded on the subset $\{\tau \in \mathcal{H}_{(n)} :\ Y - \epsilon I \geq 0\ \}$.

Notice that for $G \in GL_n(\mathbb{Z})$ and $F \in \mathcal{M}_{k+1/2}(\Gamma_0^{(n)}(N), \chi)$, we have

$$\gamma = \begin{pmatrix} G^{-1} & \\ & {}^tG \end{pmatrix} \in \Gamma_0^{(n)}(N), \text{ and so } \chi(\det G)\, F = \big(F|\widetilde{\gamma}\big);$$

since $\theta^{(n)}(G^{-1}\tau\,{}^tG^{-1}) = \theta^{(n)}(\tau)$, we get $c(T) = \chi(\det G)\,c({}^tGTG)$. We interpret T as the matrix for a quadratic form on a rank n lattice Λ (oriented if $\chi(-1) = -1$). Then we set $c(\Lambda) = c(T)$ and

$$\mathrm{e}^*\{\Lambda\tau\} = \sum_G \mathrm{e}\{\,{}^tGTG\tau\}$$

where G varies over $O(T)\backslash GL_n(\mathbb{Z})$ (or over $O^+(T)\backslash SL_n(\mathbb{Z})$ if $\chi(-1) = -1$), and so

$$F(\tau) = \sum_{\mathrm{cls}\Lambda} c(\Lambda)\,\mathrm{e}^*\{\Lambda\tau\}$$

where $\mathrm{cls}\Lambda$ varies over all isometry classes of even integral, positive semi-definite lattices of rank n (oriented if $\chi(-1) = -1$).

§3. Hecke operators on half-integral weight Siegel modular forms

We begin by defining Hecke operators acting on half-integral weight Siegel modular forms; then we analyse their action on Fourier coefficients.

Fix N so that $4|N$, and set $\widetilde{\Gamma} = \{\widetilde{\gamma} :\ \gamma \in \Gamma_0^{(n)}(N)\ \}$; let $\widetilde{\delta} = \left[\begin{pmatrix} pI & \\ & I\end{pmatrix}, p^{-n/2}\right]$. Similar to the case of integral weight, we define

$$F|T(p) = \sum_{\widetilde{\gamma}} \overline{\chi}(\gamma)\ F|\widetilde{\delta}^{-1}\widetilde{\gamma}$$

where $\widetilde{\gamma}$ runs over a complete set of representatives for $\widetilde{\Gamma} \cap \widetilde{\delta}\widetilde{\Gamma}\widetilde{\delta}^{-1}\backslash\widetilde{\Gamma}$.

Proposition 3.1. *For $F \in \mathcal{M}_{m/2}(N,\chi)$ and p prime, $F|T(p) = 0$.*

Proof. We will show that with $\Gamma' = \delta\Gamma\delta^{-1}$,

$$\left[\widetilde{\Gamma} \cap \widetilde{\Gamma}' : \widetilde{\Gamma} \cap \widetilde{\delta}\widetilde{\Gamma}\widetilde{\delta}^{-1}\right] = 2,$$

and that for $\widetilde{\gamma}_0 \in \widetilde{\Gamma} \cap \widetilde{\Gamma}'$, $\widetilde{\gamma}_0 \notin \widetilde{\delta}\widetilde{\Gamma}\widetilde{\delta}^{-1}$, we have

$$\overline{\chi}(\gamma_0)\ F|\widetilde{\delta}^{-1}\widetilde{\gamma}_0 = -F|\widetilde{\delta}^{-1}.$$

Consequently, for $\widetilde{\gamma}$ a set of coset representatives for $\widetilde{\Gamma} \cap \widetilde{\Gamma}'\backslash\widetilde{\Gamma}$,

$$F|T(p) = \left(\sum_\gamma \overline{\chi}(\gamma)\ F|\widetilde{\delta}^{-1}\widetilde{\gamma}\right) + \left(\sum_\gamma \overline{\chi}(\gamma_0\gamma)\ F|\widetilde{\delta}^{-1}\widetilde{\gamma}_0\widetilde{\gamma}\right) = 0.$$

To show $\left[\widetilde{\Gamma} \cap \widetilde{\Gamma}' : \widetilde{\Gamma} \cap \widetilde{\delta}\widetilde{\Gamma}\widetilde{\delta}^{-1}\right] = 2$, suppose $\gamma = \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \Gamma$ so that $\gamma' = \delta\gamma\delta^{-1} \in \Gamma$. (Notice this means $p|C$ and hence $(p, \det D) = 1$.) We have

$$\widetilde{\delta}\widetilde{\gamma}\widetilde{\delta}^{-1} = \left[\gamma', \frac{\theta^{(n)} \circ \gamma \circ \delta^{-1}}{\theta^{(n)} \circ \delta^{-1}}\right]$$

and from the Transformation Formula (Theorem 2.1) and Theorems 2.2 and 2.4,

$$\frac{\theta^{(n)} \circ \gamma \circ \delta^{-1}(\tau)}{\theta^{(n)} \circ \delta^{-1}(\tau)} = \frac{\theta^{(n)}(\gamma\tau/p)}{\theta^{(n)}(\tau/p)} = \left(\frac{p}{|\det D|}\right) \frac{\theta^{(n)}(\gamma'\tau)}{\theta^{(n)}(\tau)}.$$

If we have $\gamma_0, \gamma_1 \in \Gamma$ such that $\gamma_i' = \delta\gamma_i\delta^{-1} \in \Gamma$ and

$$\frac{\theta^{(n)}(\gamma_i\tau/p)}{\theta^{(n)}(\tau/p)} = -\frac{\theta^{(n)}(\gamma_i'\tau)}{\theta^{(n)}(\tau)},$$

then

$$\frac{\theta^{(n)}(\gamma_0\gamma_1^{-1}\tau/p)}{\theta^{(n)}(\tau/p)} = \frac{\theta^{(n)}(\gamma_0\gamma_1^{-1}\tau/p)}{\theta^{(n)}(\gamma_1^{-1}\tau/p)} \frac{\theta^{(n)}(\gamma_1^{-1}\tau/p)}{\theta^{(n)}(\tau/p)} = \frac{\theta^{(n)}(\gamma_0'(\gamma_1')^{-1}\tau)}{\theta^{(n)}(\tau)},$$

and hence $\widetilde{\delta}\widetilde{\gamma}_0\widetilde{\delta}^{-1}$ and $\widetilde{\delta}\widetilde{\gamma}_1\widetilde{\delta}^{-1}$ lie in the same coset. Thus $\left[\widetilde{\Gamma} \cap \widetilde{\Gamma}' : \widetilde{\Gamma} \cap \widetilde{\delta}\widetilde{\Gamma}\widetilde{\delta}^{-1}\right] \leq 2$. To show this index is 2, we show some γ_0 as above exists. To see this, choose a prime $q \nmid N$ so that $\left(\frac{p}{q}\right) = -1$. Then with $D = \begin{pmatrix} q & \\ & I_{n-1} \end{pmatrix}$, (pNI, D) is a coprime symmetric pair, so there are matrices A, B so that

$$\gamma_0 = \begin{pmatrix} A & B \\ pNI & D \end{pmatrix} \in \Gamma,\ \gamma_0' = \delta\gamma_0\delta^{-1} \in \Gamma \cap \Gamma',\ \text{and}\ \left(\frac{p}{|\det D|}\right) = -1.$$

Furthermore, $\chi(\gamma_0) = \chi(\det D) = \chi(\gamma_0')$, but

$$F|\widetilde{\delta}^{-1}\widetilde{\gamma}_0 = \left(\frac{p}{q}\right) F|\widetilde{\gamma}_0'\widetilde{\delta}^{-1} = -F|\widetilde{\delta}^{-1}.$$

Hence $F|T(p) = 0$, as claimed. □

We now define the Hecke operators $T_j(p^2)$. For $1 \leq j \leq n$, set

$$X = \begin{pmatrix} pI_j & \\ & I_{n-j} \end{pmatrix},\ \delta_j = \begin{pmatrix} X & \\ & X^{-1} \end{pmatrix},\ \text{and}\ \widetilde{\delta}_j = [\delta_j, p^{-j/2}].$$

For $F \in \mathcal{M}_{k+1/2}(\Gamma_0^{(n)}(N), \chi)$, define

$$F|T_j(p^2) = \sum_{\widetilde{\gamma}} \overline{\chi}(\gamma)\, F|\widetilde{\delta}_j^{-1}\widetilde{\gamma}$$

where $\widetilde{\gamma}$ runs over a complete set of representatives for $\widetilde{\Gamma} \cap \widetilde{\delta}_j\widetilde{\Gamma}\widetilde{\delta}_j^{-1}\backslash\widetilde{\Gamma}$.

Lemma 3.2. *Let $\Gamma=\Gamma_0(N)$ and let $\delta_j,\widetilde{\delta}_j$ be as above. Set $\Gamma'_j=\delta_j\Gamma\delta_j^{-1}$; then for $p\nmid N$,*

$$\widetilde{\Gamma}'_j\cap\widetilde{\Gamma}=\widetilde{\delta}_j\widetilde{\Gamma}\widetilde{\delta}_j^{-1}\cap\widetilde{\Gamma}.$$

Proof. Say $\gamma=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\Gamma$ so that $\gamma'=\delta_j\gamma\delta_j^{-1}\in\Gamma$. (Note that this means $(X\,{}^tBX,X\,{}^tDX^{-1})$ is a coprime symmetric pair, so $p\nmid\det D$.) So by Theorem 2.2,

$$\begin{aligned}\frac{\theta(\gamma'\tau)}{\theta(\tau)}&=(\det(-i\tau))^{1/2}(\det(-i\tau(X^{-1}CX^{-1}\tau+X^{-1}DX)^{-1}X^{-1}DX))^{-1/2}\\&\quad\cdot\mathcal{G}_{XBX}(X^{-1}DX)\\&=(\det(-i\tau))^{1/2}(\det(-i\tau(CX^{-1}\tau X^{-1}+D)^{-1}D))^{-1/2}\mathcal{G}_{XBX}(X^{-1}DX),\end{aligned}$$

and

$$\frac{\theta^{(n)}(\gamma\delta_j^{-1}\tau)}{\theta^{(n)}(\delta_j^{-1}\tau)}=(\det(-i\tau))^{1/2}(\det(-i\tau(CX^{-1}\tau X^{-1}+D)^{-1}D))^{-1/2}\mathcal{G}_B(D).$$

Also,

$$\mathcal{G}_{XBX}(X^{-1}DX)=\sum_G \mathrm{e}\{2(X\,{}^tGGX)BD^{-1}\}$$

where G varies over $\mathbb{Z}^{1,n}/\mathbb{Z}^{1,n}X\,{}^tDX^{-1}$, and so GX varies over $\mathbb{Z}^{1,n}X/\mathbb{Z}^{1,n}X\,{}^tD$. We argue that when G varies over $\mathbb{Z}^{1,n}/\mathbb{Z}^{1,n}\,{}^tD$, pG varies over $\mathbb{Z}^{1,n}/\mathbb{Z}^{1,n}\,{}^tD$ and over $\mathbb{Z}^{1,n}X/\mathbb{Z}^{1,n}X\,{}^tD$, and hence $\mathcal{G}_{XBX}(X^{-1}DX)=\mathcal{G}_B(D)$. Take $G_0,G'_0\in\mathbb{Z}^{1,n}$. Clearly, if $G_0-G'_0\in\mathbb{Z}^{1,n}\,{}^tD$ then $p(G_0-G'_0)\in\mathbb{Z}^{1,n}X\,{}^tD\subseteq\mathbb{Z}^{1,n}\,{}^tD$. So suppose $p(G_0-G'_0)\in\mathbb{Z}^{1,n}X\,{}^tD$. Then

$$(G_0-G'_0)\,{}^tD^{-1}\in\mathbb{Z}^{1,n}\frac{1}{p}X\cap\mathbb{Z}^{1,n}\,{}^tD^{-1}.$$

Locally at each prime q, either $\frac{1}{p}X$ or ${}^tD^{-1}$ lies in $\mathbb{Z}_q^{1,n}$, so $(G_0-G'_0)\,{}^tD^{-1}\in\mathbb{Z}_q^{1,n}$ for all primes q. Therefore $G_0-G'_0\in\mathbb{Z}^{1,n}\,{}^tD$ whenever $p(G_0-G'_0)\in\mathbb{Z}^{1,n}X\,{}^tD$. Similarly, $G_0-G'_0\in\mathbb{Z}^{1,n}\,{}^tD$ whenever $p(G_0-G'_0)\in\mathbb{Z}^{1,n}\,{}^tD$. □

Theorem 3.3. *Take p a prime and $F\in\mathcal{M}_{k+1/2}\left(\Gamma_0^{(n)}(N),\chi\right)$.*

(a) If $p\nmid N$, then

$$F|T_j(p^2)=\sum_{\Omega,\Lambda'_1,Y}\chi(\det D)F|\widetilde{\delta}_j^{-1}\left[\begin{pmatrix}D&{}^tY\\&D^{-1}\end{pmatrix},\frac{\mathcal{G}_Y(D)}{(\det D)}\right]\widetilde{\begin{pmatrix}G^{-1}&\\&{}^tG\end{pmatrix}}$$

where Ω,Λ'_1 vary subject to $p\Lambda\subseteq\Omega\subseteq\frac{1}{p}\Lambda$, $\overline{\Lambda}'_1$ is a codimension $n-j$ subspace of $\Lambda\cap\Omega/p(\Lambda+\Omega)$, and $G=G(\Omega,\Delta_1)\in SL_n(\mathbb{Z})$,

$$D=D(\Omega)=diag\{I_{r_0},pI_{r_1},p^2I_{r_2},I_{n-j}\}$$

so that

$$\Omega = \Lambda G D^{-1} X,\ \Lambda_1' = \Lambda G \begin{pmatrix} 0_{r_0} & & \\ & I_{r_1} & \\ & & 0 \end{pmatrix} \text{ where } X = \begin{pmatrix} pI_j & \\ & I_{n-j} \end{pmatrix}.$$

Also,

$$ {}^tY = \begin{pmatrix} Y_0 & Y_2 & 0 & Y_3 \\ p\,{}^tY_2 & Y_1 & 0 & \\ 0 & 0 & I & \\ {}^tY_3 & & & \end{pmatrix}$$

with Y_0 symmetric, $r_0 \times r_0$, varying modulo p^2, Y_1 symmetric, $r_1 \times r_1$ varying modulo p with the restriction that $p \nmid \det Y_1$, Y_2 $r_0 \times r_1$, varying modulo p, Y_3 $r_0 \times (n-j)$, varying modulo p.

(b) If $p|N$, then

$$F|T_j(p^2) = \sum_{\Omega,Y} F|\widetilde{\delta}_j^{-1} \widetilde{\begin{pmatrix} I & Y \\ & I \end{pmatrix}} \widetilde{\begin{pmatrix} G^{-1} & \\ & {}^tG \end{pmatrix}}$$

where Ω varies subject to $p\Lambda \subseteq \Omega \subseteq \Lambda$, $[\Lambda : \Omega] = p^j$, $G = G(\Omega, \Lambda_1) \in GL_n(\mathbb{Z})$ so that $\Omega = \Lambda G X$ and

$$Y = \begin{pmatrix} Y_0 & Y_3 \\ {}^tY_3 & 0 \end{pmatrix}$$

with Y_0 symmetric, $j \times j$, varying modulo p^2, Y_3 $j \times (n-j)$, varying modulo p.

Proof. First suppose $p \nmid N$. By Lemma 5.2, a set of coset representatives for the action of the half-integral weight Hecke operator $T_j(p^2)$ is $\{\widetilde{\gamma}\}$ where $\{\gamma\}$ is a set of coset representatives for the integral weight Hecke operator $T_j(p^2)$, and a set of representatives for this was given in Proposition 2.1 of [6] in the case $N = 1$; note that the matrices G presented there can be chosen from $SL_n(\mathbb{Z})$. The representatives presented there are

$$\left\{ \begin{pmatrix} D & {}^tY \\ U & W \end{pmatrix} \begin{pmatrix} G^{-1} & \\ & {}^tG \end{pmatrix} \right\}$$

where D, G, Y vary as in the statement of the theorem, and U, W are any $n \times n$ matrices so that $\begin{pmatrix} D & {}^tY \\ U & W \end{pmatrix} \in Sp_n(\mathbb{Z})$. The only modification we need to make here is to ensure $\begin{pmatrix} D & {}^tY \\ U & W \end{pmatrix} \in \Gamma_0^{(n)}(N)$. To do this, we note that with $D' = \operatorname{diag}\{I_{r_0}, pI_{r_1}, p^2 I_{r_2}\}$ and

$$Y' = \begin{pmatrix} Y_0 & Y_2 & 0 \\ p\,{}^tY_2 & Y_1 & 0 \\ 0 & 0 & I_{r_2} \end{pmatrix},$$

(D', NY') is a coprime symmetric pair, and so we can find U', W' so that $N|U'$ and $\begin{pmatrix} D' & Y' \\ U' & W' \end{pmatrix} \in Sp_j(\mathbb{Z})$. Setting $U = \begin{pmatrix} U' & \\ & 0 \end{pmatrix}$, $W = \begin{pmatrix} W' & \\ & I_{n-j} \end{pmatrix}$, we have $\begin{pmatrix} D & {}^tY \\ U & W \end{pmatrix} \in \Gamma_0^{(n)}(N)$. Then by the construction in the proof of Proposition 2.1 of [6] and Lemma 7.1, we have

$$M \begin{pmatrix} G & \\ & {}^tG^{-1} \end{pmatrix} \begin{pmatrix} D & {}^tY \\ U & W \end{pmatrix}^{-1} \in \delta_j \Gamma_0^{(n)}(1) \delta_j^{-1} \cap \Gamma_0^{(n)}(N) = \delta_j \Gamma_0^{(n)}(N) \delta_j^{-1} \cap \Gamma_0^{(n)}(N).$$

Thus a set of coset representatives corresponding to $T_j(p^2)$ on $\mathcal{M}_{k+1/2}(\Gamma_0^{(n)}(N), \chi)$ is

$$\left\{ \widetilde{\begin{pmatrix} D & {}^tY \\ U & W \end{pmatrix}} \widetilde{\begin{pmatrix} G^{-1} & \\ & {}^tG \end{pmatrix}} \right\}$$

where $G = G(\Omega, \Lambda_1)$, $D = D(\Omega)$, Y vary as claimed.

Set

$$\beta = \begin{pmatrix} D & {}^tY \\ & D^{-1} \end{pmatrix}, \ \gamma' = \delta_j^{-1} \gamma \beta^{-1} \delta_j = \begin{pmatrix} I & \\ XUD^{-1}X & I \end{pmatrix}.$$

We have

$$XU = \begin{pmatrix} pU' & \\ & 0_{n-j} \end{pmatrix},$$

and hence $XUD^{-1}X$ is integral and divisible by N. Thus $\gamma' \in \Gamma_0^{(n)}(N)$. We will show that

$$\widetilde{\gamma}' = \widetilde{\delta}_j^{-1} \widetilde{\gamma} [\beta^{-1}, (\det D)(\mathcal{G}_Y(D))^{-1}] \widetilde{\delta}_j,$$

and hence

$$F | \widetilde{\delta}_j^{-1} \widetilde{\gamma} = F | \widetilde{\gamma}' \widetilde{\delta}_j^{-1} [\beta, (\det D)^{-1} \mathcal{G}_Y(D)] = F | \widetilde{\delta}_j^{-1} [\beta, (\det D)^{-1} \mathcal{G}_Y(D)].$$

Note that by Corollary 2.3 and Lemma 7.4 (a), $\mathcal{G}_Y(D) = \mathcal{G}_{Y_1}(pI_{r_1}) p^{r_2}$.

We have

$$\begin{aligned} &\widetilde{\delta}_j^{-1} \widetilde{\gamma} [\beta^{-1}, (\det D)(\mathcal{G}_Y(D))^{-1}] \widetilde{\delta}_j \\ &= \left[\gamma', (\det D)(\mathcal{G}_Y(D))^{-1} \cdot \frac{\theta(\gamma \beta^{-1} \delta_j \tau)}{\theta(\beta^{-1} \delta_j \tau)} \right]. \end{aligned}$$

Also,

$$\beta^{-1} \delta_j = \begin{pmatrix} D^{-1}X & -YX^{-1} \\ & DX^{-1} \end{pmatrix}, \ \gamma \beta^{-1} \delta_j = \begin{pmatrix} X & \\ UD^{-1}X & X^{-1} \end{pmatrix}.$$

Therefore, using the Inversion Formula,

$$\begin{aligned}
&\theta^{(n)}(\gamma\beta^{-1}\delta_j\tau)\\
&\quad= \sum_{g\in\mathbb{Z}^{1,n}} \mathrm{e}\{2\,{}^tggX\tau(UD^{-1}X\tau+X^{-1})^{-1}\}\\
&\quad= \frac{1}{\sqrt{2}^n}(\det X)^{-1}\left(\det(-i\tau(XUD^{-1}X\tau+I)^{-1})\right)^{-1/2}\\
&\qquad\cdot\sum_{g\in\mathbb{Z}^{1,n}} \mathrm{e}\{-\frac{1}{2}\,{}^tgg(UD^{-1}X\tau+X^{-1})\tau^{-1}X^{-1}\}\\
&\quad= \frac{1}{\sqrt{2}^n}(\det X)^{-1}\left(\det(-i\tau(XUD^{-1}X\tau+I)^{-1})\right)^{-1/2}\\
&\qquad\cdot\sum_{g_0\,(D)} \mathrm{e}\left\{-\frac{1}{2}\,{}^tg_0g_0UD^{-1}\right\}\theta^{(n)}\left(\frac{1}{2}\mathbb{Z}^{1,n}, g_0D^{-1}; -DX^{-1}\tau^{-1}X^{-1}D\right)\\
&\quad= (\det X)^{-1}(\det(-iD^{-1}X\tau XD^{-1}))^{1/2}\left(\det(-i\tau(XUD^{-1}X\tau+I)^{-1})\right)^{-1/2}\\
&\qquad\cdot\sum_{g_0\,(D)} \mathrm{e}\left\{-\frac{1}{2}\,{}^tg_0g_0UD^{-1}\right\}\sum_{g\in\mathbb{Z}^{1,n}} \mathrm{e}\{2\,{}^tggD^{-1}X\tau XD^{-1}-2D^{-1}\,{}^tg_0g\}.
\end{aligned}$$

(By $g_0\,(D)$ we really mean $g\in\mathbb{Z}^{1,n}/\mathbb{Z}^{1,n}\,D$.) Since $UY=WD-I$ and $\mathrm{e}\{MM'\}=\mathrm{e}\{M'M\}$,

$$\begin{aligned}
&\mathrm{e}\{2\,{}^t(g_0U/2+g)(g_0U/2+g)YD^{-1}\}\\
&\quad= \mathrm{e}\left\{-\frac{1}{2}\,{}^tg_0g_0UD^{-1}-2D^{-1}\,{}^tg_0g\}\ \mathrm{e}\{2\,{}^tggYD^{-1}\right\}
\end{aligned}$$

(recall that $N|U$ and $4|N$). Thus

$$\begin{aligned}
\theta(\gamma\beta^{-1}\delta_j\tau) &= (\det D)^{-1}(\det(-i\tau))^{1/2}\left(\det(-i\tau(XUD^{-1}X\tau+I)^{-1})\right)^{-1/2}\\
&\quad\cdot\sum_{\substack{g,g_0\in\mathbb{Z}^{1,n}\\ g_0\,(D)}} \mathrm{e}\{2\,{}^t(g_0U/2+g)(g_0U/2+g)YD^{-1}\}\\
&\quad\cdot\mathrm{e}\{2\,{}^tgg(D^{-1}X\tau XD^{-1}-YD^{-1})\}.
\end{aligned}$$

For fixed $g\in\mathbb{Z}^{1,n}$, $g_0U/2+g$ varies over $\mathbb{Z}^{1,n}/\mathbb{Z}^{1,n}D$ as g_0 does (recall that $(D,\,{}^tU)$ are coprime and $4|U$). Thus the sum on g_0 is independent of the choice of g, and so

$$\begin{aligned}
\theta(\gamma\beta^{-1}\delta_j\tau) &= (\det D)^{-1}\mathcal{G}_Y(D)(\det(-i\tau))^{1/2}\left(\det(-i\tau(XUD^{-1}X\tau+I)^{-1})\right)^{-1/2}\\
&\quad\cdot\theta(\beta^{-1}\delta_j\tau)\\
&= (\det D)^{-1}\mathcal{G}_Y(D)\frac{\theta(\gamma'\tau)}{\theta(\tau)}\cdot\theta(\beta^{-1}\delta_j\tau).
\end{aligned}$$

This completes the proof in the case that $p \nmid N$.

In the case $p|N$, the coset representatives for $(\Gamma'_j \cap \Gamma^{(n)})\backslash\Gamma^{(n)}$ are those representatives as above where $D = I$. Since $\Omega = \Lambda G X$, we have $p\Lambda \subseteq \Omega \subseteq \Lambda$ with $[\Lambda : \Omega] = p^j$. $\square$

When $p \nmid N$, the generalised Gauss sums $\mathcal{G}_Y(D)$ in the automorphy factors in Theorem 3.3 contribute to the Fourier coefficients of $F|T_j(p^2)$ a term that we call a generalised twisted Gauss sum, defined as follows.

Definition. Let p be an odd prime, and $2T$ a symmetric, even integral $t \times t$ matrix. Define the twisted Gauss sum by

$$\mathcal{G}^*_T(pI_t) = \sum_{Y\ (p)} \left(\frac{\det Y}{p}\right) \mathrm{e}\{2YT/p\}$$

where Y varies over all symmetric, integral, $t \times t$ matrices modulo p. Also, define the normalised twisted Gauss sum by

$$\widetilde{\mathcal{G}}_T(pI_t) = p^{-t}\, \mathcal{G}_1(p)^t\, \mathcal{G}^*_T(pI_t).$$

For U a dimension t quadratic space over $\mathbb{Z}/p\mathbb{Z}$ with $U \simeq 2T$ modulo p, we set

$$\mathcal{G}^*(U) = \mathcal{G}^*_T(pI_n), \text{ and } \widetilde{\mathcal{G}}(U) = \widetilde{\mathcal{G}}_T(pI_n).$$

In [9] a more general version of such sums are evaluated: There the quadratic character $\left(\frac{*}{p}\right)$ is replaced by an arbitrary character. However, it is (unsurprisingly) simpler – and somewhat amusing – to evaluate these with quadratic characters; we do this in the next section, using the theory of quadratic forms over finite fields.

We complete this section by evaluating the action of $T_j(p^2)$ on the Fourier coefficients of a modular form.

Theorem 3.4. *Take $F \in \mathcal{M}_{k+1/2}(\Gamma_0^{(n)}(N), \chi)$ where $4|N$, and let p be a prime. Let χ' be the character modulo N defined by*

$$\chi'(d) = \chi(d)\left(\frac{(-1)^{k+1}}{|d|}\right)(sgn\, d)^{k+1}.$$

(a) Suppose $p \nmid N$. Given

$$\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2,\ \Omega = p\Lambda_0 \oplus \Lambda_1 \oplus \frac{1}{p}\Lambda_2$$

with $n_i = rank \Lambda_i$, $r = n_0 + n_2$, set

$$A_j(\Lambda, \Omega) = \chi'(p^{j-r}) p^{j/2 + k(n_2 - n_0) + n_0(n - n_2)} \sum_{\substack{\mathrm{cls}\, U \\ \dim U = j - r}} R^*(\Lambda_1/p\Lambda_1, U)\widetilde{\mathcal{G}}(U)$$

if Λ, Ω are even integral, and set $A_j(\Lambda,\Omega) = 0$ otherwise. Then the Λth coefficient of $F|T_j(p^2)$ is

$$\sum_{p\Lambda\subseteq\Omega\subseteq\frac{1}{p}\Lambda} A_j(\Lambda,\Omega)c(\Omega).$$

(b) Suppose $p|N$. The Λth coefficient of $F|T_j(p^2)$ is

$$p^{j(n-k+1/2)} \sum_{\substack{p\Lambda\subseteq\Omega\subseteq\Lambda \\ [\Lambda:\Omega]=p^j}} c(\Omega).$$

Proof. Let $X = \begin{pmatrix} pI_j & \\ & I_{n-j}\end{pmatrix}$. Suppose first that $p \nmid N$; then

$$\begin{aligned} F(\tau)|T_j(p^2) = p^{-j(k+1/2)} &\sum_{D,Y,G} \chi(\det D)(\det D)^{2k+1}\mathcal{G}_Y(D)^{-2k-1} \\ &\cdot \sum_T c(T)\mathrm{e}\{TX^{-1}DG^{-1}\tau\,{}^tG^{-1}DX^{-1}\}\,\mathrm{e}\{TX^{-1}\,{}^tYDX^{-1}\} \end{aligned}$$

where D, Y, G vary as in Theorem 3.3, and T varies over all $n\times n$ even integral, positive semi-definite matrices.

Fix T, G and $D = \mathrm{diag}\{I_{r_0}, pI_{r_1}, p^2I_{r_2}, I_{n-j}\}$, and let Y vary. As described in Theorem 3.3, we have

$${}^tY = \begin{pmatrix} Y_1 & Y_2 & 0 & Y_3 \\ p\,{}^tY_2 & Y_1 & 0 & \\ 0 & 0 & I & \\ {}^tY_3 & & & \end{pmatrix};$$

correspondingly, we write

$$T = \begin{pmatrix} T_0 & T_2 & * & T_3 \\ {}^tT_2 & T_1 & * & * \\ * & * & * & * \\ {}^tT_3 & * & * & * \end{pmatrix}.$$

By Corollary 3.4, $\mathcal{G}_Y(D) = p^{r_2}\left(\frac{\det Y_1}{p}\right)\mathcal{G}_1(p)^{r_1}$; so

$$\begin{aligned} &\sum_Y \mathcal{G}_Y(D)^{-2k-1}\mathrm{e}\{TX^{-1}\,{}^tYDX^{-1}\} \\ &\quad = p^{-r_2(2k+1)}\mathcal{G}_1(p)^{-r_1(2k+1)} \sum_{Y_0\,(p^2)} \mathrm{e}\{T_0Y_0/p^2\}\cdot \sum_{Y_2\,(p)} \mathrm{e}\{2T_2Y_2/p\} \\ &\quad\cdot \sum_{Y_3\,(p)} \mathrm{e}\{2T_3Y_3/p\}\cdot \sum_{Y_1\,(p)} \left(\frac{\det Y_1}{p}\right)\mathrm{e}\{T_1Y_1/p\}. \end{aligned}$$

If $T_0 \equiv 0 \,(\mathrm{mod}\ p^2)$, $T_2 \equiv 0 \,(\mathrm{mod}\ p)$, $T_3 \equiv 0 \,(\mathrm{mod}\ p)$ then the sum on Y is

$$p^{r_0(n+1-r_2)}\mathcal{G}^*_{\frac{1}{2}T_1}(pI_{r_1}),$$

and otherwise the sum on Y is 0. Therefore, using the fact that $\mathcal{G}_1(p)^2 = \left(\frac{-1}{p}\right)p$, we have

$$F(\tau)|T_j(p^2) = \sum_{D,G} \chi'(\det D) p^{(2k+1)(r_2-j/2)+(k+1)r_1+r_0(n+1-r_2)} \cdot \sum_{\substack{T \\ T[X^{-1}D] \text{ integral}}} \widetilde{\mathcal{G}}_{\frac{1}{2}T_1}(pI_{r_1})c(T)\mathrm{e}\{T[X^{-1}DG^{-1}]\tau\}.$$

Let $\Lambda = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_n$ be equipped with the quadratic form $T[X^{-1}DG^{-1}]$ (relative to the given basis); then $\Omega = \Lambda GD^{-1}X \simeq T$ relative to $(x_1\ \dots\ x_n)GD^{-1}X$. Also, relative to these bases we have splittings

$$\Lambda = \Lambda_0 \oplus \Lambda_1' \oplus \Lambda_2, \Lambda_1'',\ \Omega = p\Lambda_0 \oplus \Lambda_1' \oplus \frac{1}{p}\Lambda_2 \oplus \Lambda_1''$$

where $r_0 = \mathrm{rank}\Lambda_0$, $r_1 = \mathrm{rank}\Lambda_1'$, $r_2 = \mathrm{rank}\Lambda_2$, and $\Lambda_1' \simeq T_1$. Also, with $\Lambda_1 = \Lambda_1' \oplus \Lambda_1''$ and Ω fixed, $G = G(\Omega, \Lambda_1')$ varies to vary $\overline{\Lambda}_1'$ over all dimension r_1 subspaces of $\Lambda_1/p\Lambda_1 \approx \Lambda \cap \Omega/p(\Lambda + \Omega)$. Thus

$$\sum_{\Lambda_1'} \widetilde{\mathcal{G}}_{\frac{1}{2}T_1}(pI_{r_1}) = \sum_{\substack{\mathrm{cls} U \\ \dim U = r_1}} R^*(\Lambda_1/p\Lambda_1, U)\widetilde{\mathcal{G}}(U).$$

This proves the theorem in the case $p \nmid N$.

In the case $p|N$, the analysis is simpler, as D is always I (and hence $r_0 = j$, $r_1 = r_2 = 0$); following the above reasoning yields the theorem in this case. □

§4. Evaluating twisted Gauss sums

Here we use the theory of quadratic spaces over finite fields to evaluate the twisted Gauss sums defined in the previous section (cf. [9]). We begin with some elementary lemmas.

Fix an odd prime p. Set $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$; we will often write GL_t for $GL_t(\mathbb{F})$, and $\mathcal{G}^*_T$ for $\mathcal{G}^*_T(pI_t)$ where $2T$ is a $t \times t$ symmetric, even integral matrix. Note that $\mathcal{G}^*_T = \mathcal{G}^*_{T'}$ if $T \equiv T' \,(\mathrm{mod}\ p)$, so we can think of T as lying in $\mathbb{F}^{t,t}_{\mathrm{sym}}$. Write $\langle\alpha\rangle^d$ to denote the $d \times d$ matrix $\mathrm{diag}\{\alpha, \dots, \alpha\}$, and $A \perp B$ to denote $\mathrm{diag}\{A, B\}$. We fix ω so that $\left(\frac{\omega}{p}\right) = -1$, and set $J_t = I_{t-1} \perp \langle\omega\rangle = \mathrm{diag}\{I_{t-1}, \omega\}$.

Note that GL_t acts by conjugation on $\mathbb{F}^{t,t}_{\mathrm{sym}}$; we write $T \sim U$ when T, U lie in the same GL_t-orbit. The distinct GL_t-orbits are represented by

$$\langle 0\rangle^t,\ I_d \perp \langle 0\rangle^{t-d},\ J_d \perp \langle 0\rangle^{t-d}\ (1 \le d \le t).$$

Lemma 4.1. *Fix integers d, ℓ with $0 < \ell \le d$, and set $U_\ell = I_\ell \perp \langle 0\rangle^{d-\ell}$, $\overline{U}_\ell = J_\ell \perp \langle 0\rangle^{d-\ell}$. Then*

$$\sum_{Y\sim U_\ell} e\{2Y/p\} - \sum_{Y\sim \overline{U}_\ell} e\{2Y/p\} = \sum_{\mathrm{cls} W\in \mathbb{F}^{\ell,\ell}_{\mathrm{sym}}} R^*(I_d, W)\mathcal{G}^*_W.$$

Proof. Note that with $W \in \mathbb{F}^{\ell,\ell}$ and $W' = \begin{pmatrix} W & * \\ * & * \end{pmatrix} \in \mathbb{F}^{d,d}$, we have

$$\mathrm{e}\{2U_\ell W'/p\} = \mathrm{e}\{2W/p\} \text{ and } \mathrm{e}\{2\overline{U}_\ell W'/p\} = \mathrm{e}\{2J_\ell W/p\};$$

also, since the number of ways to extend a rank ℓ matrix from $F^{d,\ell}$ to a matrix in $GL_d(\mathbb{F})$ is $\eta(d, \ell)$, the number of $G \in GL_d$ so that the upper left $\ell \times \ell$ block of $G\,{}^tG$ is W is $\eta(d, \ell)r^*(I_d, W)$. Also, Lemma 8.1 gives the values of $o(U_\ell)$ and $o(\overline{U}_\ell)$. Thus we have

$$\begin{aligned}
&\sum_{Y\sim U_\ell} \mathrm{e}\{2Y/p\} - \sum_{Y\sim \overline{U}_\ell} \mathrm{e}\{2Y/p\} \\
&\quad= \frac{1}{o(U_\ell)} \sum_{G\in GL_d} \mathrm{e}\{2\,{}^tGU_\ell G/p\} - \frac{1}{o(\overline{U}_\ell)} \sum_{G\in GL_d} \mathrm{e}\{2\,{}^tG\overline{U}_\ell G/p\} \\
&\quad= \frac{1}{o(U_\ell)} \sum_{G\in GL_d} \mathrm{e}\{2U_\ell G\,{}^tG/p\} - \frac{1}{o(\overline{U}_\ell)} \sum_{G\in GL_d} \mathrm{e}\{2\overline{U}_\ell G\,{}^tG/p\} \\
&\quad= \eta(d,\ell) \sum_{W\in \mathbb{F}^{\ell,\ell}_{\mathrm{sym}}} r^*(I_d, W) \left(\frac{1}{o(U_\ell)}\mathrm{e}\{2W/p\} - \frac{1}{o(\overline{U}_\ell)}\mathrm{e}\{2J_\ell W/p\} \right) \\
&\quad= \sum_{\mathrm{cls} W\in \mathbb{F}^{\ell,\ell}_{\mathrm{sym}}} R^*(I_d, W) \\
&\qquad\cdot \left(\frac{1}{o(I_\ell)} \sum_{G\in GL_\ell} \mathrm{e}\{2\,{}^tGWG/p\} - \frac{1}{o(J_\ell)} \sum_{G\in GL_\ell} \mathrm{e}\{2J_\ell\,{}^tGWG/p\} \right) \\
&\quad= \sum_{\mathrm{cls} W\in \mathbb{F}^{\ell,\ell}_{\mathrm{sym}}} R^*(I_d, W) \left(\sum_{Y\sim I_\ell} \mathrm{e}\{2YW/p\} - \sum_{Y\sim J_\ell} \mathrm{e}\{2YW/p\} \right) \\
&\quad= \sum_{\mathrm{cls} W\in \mathbb{F}^{\ell,\ell}_{\mathrm{sym}}} R^*(I_d, W)\mathcal{G}^*_{2W}. \qquad \square
\end{aligned}$$

Lemma 4.2.

(a) We have $A(c, s) = (-1)^c$ where

$$A(c,q) = \sum_{\ell=0}^{c} (-1)^\ell p^{2\ell(q-2c+\ell)} \mu\delta(q-c, c-\ell)\gamma(c,\ell).$$

(b) We have $B(s,q)=1$ where

$$B(s,q)=\sum_{a=0}^{s}(-1)^a p^{a(a+q)}\beta(s,a)\delta(s+q,s-a).$$

(c) We have $C(s)=1$ where

$$C(s)=\sum_{\ell=0}^{s}\sum_{a=0}^{s-\ell}(-1)^{a+\ell}p^{a(a-2s+2\ell)+2\ell^2-2s\ell-\ell}\mu\delta(s,\ell)\gamma(s,\ell)\beta\delta(s-\ell,s-\ell-a).$$

Proof. (a) Using the identity

$$\gamma(c,\ell)=p^{2\ell}\gamma(c-1,\ell)+\gamma(c-1,\ell-1),$$

we get

$$\begin{aligned}A(c,q)=&\sum_{\ell=0}^{c-1}(-1)^\ell p^{2\ell(q-2c+\ell+1)}\mu\delta(q-c,c-\ell)\gamma(c-1,\ell)\\&+\sum_{\ell=1}^{c}(-1)^\ell p^{2\ell(q-2c+\ell)}\mu\delta(q-c,c-\ell)\gamma(c-1,\ell-1)\end{aligned}$$

Replacing a by $a+1$ in the latter sum, and then simplifying, we get $A(c,q)=-A(c-1,q-1)$. Consequently, $A(c,q)=(-1)^cA(0,q-c)=(-1)^c$.

(b) Using that

$$\beta(s,a)=p^a\beta(s-1,a)+\beta(s-1,a-1)$$

we get $B(s,q)=B(s-1,q+1)=B(0,q+s)=1$.

(c) To evaluate $C(s)$, we first replace a by $s-\ell-a$. Then we use that

$$\gamma(s,\ell)\beta\delta(s-\ell,a)=\frac{\mu\delta(s,\ell+a)}{\mu\delta(\ell,\ell)\mu(a,a)}=\beta\delta(s,a)\gamma(s-a,\ell)$$

to get

$$C(s)=\sum_{a=0}^{s}(-1)^{a+s}p^{a^2-s^2}\beta\delta(s,a)D(s-a,s)$$

where

$$D(q,s)=\sum_{\ell=0}^{q}p^{\ell(\ell-1)}\mu\delta(s,\ell)\gamma(q,\ell).$$

As when evaluating $A(c,q)$ above, we use our basic identity on $\gamma(q,\ell)$ to get $D(q,s)=p^{2s}D(q-1,s)$. Thus $D(q,s)=p^{2qs}$, and $C(s)=B(s,0)$ where $B(s,q)$ was evaluated above in (b). Thus $C(s)=1$. □

Theorem 4.3. *Let $T \in \mathbb{F}_{\mathrm{sym}}^{t,t}$; set $\varepsilon = \left(\frac{-1}{p}\right)$.*

(a) Suppose $t = 2s$, and $T \sim I_d \perp \langle 0 \rangle^{t-d}$ or $J_d \perp \langle 0 \rangle^{t-d}$. Then with $d = 2c$ or $d = 2c+1$,

$$\mathcal{G}_T^* = (-1)^c \varepsilon^s p^{s^2} \cdot \prod_{i=1}^{s-c} (p^{2i-1} - 1).$$

(b) Suppose $t = 2s+1$, $T \sim I_d \perp \langle 0 \rangle^{t-d}$, and $\overline{T} \sim J_d \perp \langle 0 \rangle^{t-d}$. If $d = 2c$ then $\mathcal{G}_T^* = \mathcal{G}_{\overline{T}}^* = 0$. If $d = 2c+1$ then

$$\mathcal{G}_T^* = -\mathcal{G}_{\overline{T}}^* = (-1)^c \varepsilon^{s+c} p^{s^2+2s-c} \mathcal{G}_1(p) \cdot \prod_{i=1}^{s-c} (p^{2i-1} - 1).$$

Proof. First note that, by the definition, $\mathcal{G}_T^*$ only depends on the GL_t-orbit containing T; as noted earlier, the orbits are represented by $I_d \perp \langle 0 \rangle^{t-d}$, $J_d \perp \langle 0 \rangle^{t-d}$. Also, in this proof we will use the fact that

$$\prod_{i=1}^{q} (p^{2i-1} - 1) = \frac{\mu(2q, 2q)}{\mu\delta(q, q)}.$$

(a) Set $t = 2s$; we argue by induction on s. When $s = 0$, we agree that $\mathcal{G}_T^* = 1$. Now suppose that $0 < s$, and that the value of $\mathcal{G}_W^*$ is as claimed for all $W \in \mathbb{F}_{\mathrm{sym}}^{2\ell,2\ell}$, $\ell < s$. Suppose first that $0 \le d < t$ and $T \sim I_d \perp \langle 0 \rangle^{t-d}$. Then

$$\begin{aligned}
\mathcal{G}_T^* &= \sum_{Y \sim I_t} \mathrm{e}\{2YT/p\} - \sum_{Y \sim J_t} \mathrm{e}\{2YT/p\} \\
&= \frac{p^s(p^s + \varepsilon^s)}{o(I_{t+1})} \sum_{G \in GL_t} \mathrm{e}\{2\,{}^tGGT/p\} - \frac{p^s(p^s - \varepsilon^s)}{o(I_{t+1})} \sum_{G \in GL_t} \mathrm{e}\{2\,{}^tGJ_tGT/p\}.
\end{aligned}$$

Note that $\mathrm{e}\{2YT/p\} = \mathrm{e}\{2Y'/p\}$ where Y' is the upper left $d \times d$ block of Y, and that the number of $G \in GL_t$ so that ${}^tGXG = \begin{pmatrix} Y' & * \\ * & * \end{pmatrix}$ is $\eta(t,d)\, r^*(X, Y')$.

Set $U_\ell = I_\ell \perp \langle 0 \rangle^{d-\ell}$, and $\overline{U}_\ell = J_\ell \perp \langle 0 \rangle^{d-\ell}$. Then when ℓ is odd,

$$r^*(I_{t+1}, \langle 1 \rangle \perp U_\ell) = r^*(J_{t+1}, \langle 1 \rangle \perp \overline{U}_\ell)$$

and when ℓ is even,

$$r^*(I_{t+1}, \langle 1 \rangle \perp U_\ell) - r^*(J_{t+1}, \langle 1 \rangle \perp U_\ell) = r^*(J_{t+1}, \langle 1 \rangle \perp \overline{U}_\ell) - r^*(I_{t+1}, \langle 1 \rangle \perp \overline{U}_\ell).$$

Thus, writing d as $2c$ or $2c+1$ and using Lemma 4.1,

$$\begin{aligned}
\mathcal{G}_T^* &= \frac{\eta(t,d)}{o(I_{t+1})}\sum_{\ell=0}^{c}\big(r^*(I_{t+1},\langle 1\rangle \perp U_{2\ell}) - r^*(J_{t+1},\langle 1\rangle \perp U_{2\ell})\big)\\
&\qquad \cdot \left(\sum_{Y\sim U_{2\ell}} \mathrm{e}\{2Y/p\} - \sum_{Y\sim \overline{U}_{2\ell}} \mathrm{e}\{2Y/p\}\right)\\
&= \frac{\eta(t,d)}{o(I_{t+1})}\sum_{\ell=0}^{c}\big(r^*(I_{t+1},\langle 1\rangle \perp U_{2\ell}) - r^*(J_{t+1},\langle 1\rangle \perp U_{2\ell})\big)\\
&\qquad \cdot \sum_{\mathrm{cls} W\in\mathbb{F}_{\mathrm{sym}}^{2\ell,2\ell}} R^*(I_d,W)\mathcal{G}_W^*.
\end{aligned}$$

Then again using our formulas for representation numbers, as well as our induction hypothesis,

$$\sum_{\mathrm{cls} W\in\mathbb{F}_{\mathrm{sym}}^{2\ell,2\ell}} R^*(I_d,W)\mathcal{G}_W^* = \varepsilon^\ell p^{\ell^2}\sum_{a=0}^{\ell}(-1)^a p^{2a(d-c-2\ell+a)}\frac{\mu\delta(c,\ell)\mu\delta(d-c-\ell,\ell-a)}{\mu\delta(a,a)\mu\delta(\ell-a,\ell-a)}.$$

Multiply by $\mu\delta(\ell,a)/\mu\delta(\ell,a)$ and use that $\mu\delta(\ell,a)\mu\delta(\ell-a,\ell-a)=\mu\delta(\ell,\ell)$; then with $A(*,*)$ defined as in Lemma 4.2 and recalling that $\gamma(c,\ell)=\mu\delta(c,\ell)/\mu\delta(\ell,\ell)$, we get

$$\sum_{\mathrm{cls} W\in\mathbb{F}_{\mathrm{sym}}^{2\ell,2\ell}} R^*(I_d,W)\mathcal{G}_W^* = \varepsilon^\ell p^{\ell^2}\gamma(c,\ell)A(\ell,d-c) = (-1)^\ell\varepsilon^\ell p^{\ell^2}\gamma(c,\ell).$$

Now using our formulas for representation numbers, we get

$$\mathcal{G}_T^* = \frac{2\eta(t,d)}{o(I_{t+1})}\varepsilon^s p^{s+c(2d-2c-1)}\mu(s,d-c)A(c,s-d+2c),$$

where again $A(*,*)$ is defined and evaluated in Lemma 4.2. Simplifying yields the result in this case.

The argument to evaluate $\mathcal{G}_T^*$ where $T=J_d\perp\langle 0\rangle^{t-d}$ with $d<t$ is virtually identical, and thus is left to the reader.

Now take $d=t=2s$, $T=I_t$. By Theorem 2.4 we know

$$p^s(p^s+\varepsilon^s)\,\mathcal{G}_{I_t}(pI_t;2I_t) - p^s(p^s-\varepsilon^s)\,\mathcal{G}_{J_t}(pI_t;2I_t) = 2\varepsilon^s p^{2s^2+s}.$$

Then, using the definitions of the Gauss sums, we get

$$2\varepsilon^s p^{2s^2+s} = \sum_{Y\in\mathbb{F}_{\mathrm{sym}}^{t,t}}\left(p^s(p^s+\varepsilon^s)\,r(I_t,Y) - p^s(p^s-\varepsilon^s)\,r(J_t,Y)\right)\mathrm{e}\{2Y/p\}.$$

Thus arguing as before to evaluate the right-hand side of the equation, we have

$$2\varepsilon^s p^{2s^2+s} = \sum_{\ell=0}^{s} \big(r(I_{t+1}, \langle 1\rangle \perp U_{2\ell}) - r(J_{t+1}, \langle 1\rangle \perp U_{2\ell})\big) \cdot \sum_{W\in\mathbb{F}_{\rm sym}^{2\ell,2\ell}} R^*(I_t, W)\mathcal{G}^*_{2W}.$$

Note that a representation of I_ℓ by I_{t+1} is necessarily a primitive representation; thus

$$r(I_{t+1}, \langle 1\rangle \perp U_{2\ell}) = r^*(I_{t+1}, I_{2\ell+1})r(I_{t-2\ell}, \langle 0\rangle^{t-2\ell}).$$

Then using Lemmas 8.1 and 8.2 we find

$$r(I_{t+1}, \langle 1\rangle \perp U_{2\ell}) - r(J_{t+1}, \langle 1\rangle \perp U_{2\ell}) == 2\varepsilon^{s-\ell} p^{s(2s-2\ell+1)+\ell(\ell-1)}\mu\delta(s,\ell)\sum_{d=0}^{s-\ell}(-1)^d p^{d(d-2s+2\ell)}\beta\delta(s-\ell, s-\ell-d).$$

Therefore, using Lemma 4.2 (c),

$$2\varepsilon^s p^{2s^2+s} = 2p^{s^2}\mu\delta(s,s)\,\mathcal{G}^*_{I_t} + 2\varepsilon^s p^{2s^2+s}\big(C(s) - (-1)^s p^{-s}\mu\delta(s,s)\big)$$

and, since $C(s) = 1$, this gives us $\mathcal{G}^*_{I_t} = (-1)^s\varepsilon^s p^{s^2}$.

To evaluate $\mathcal{G}^*_{J_t}$, we begin with the equality

$$\begin{aligned} 2\varepsilon^s p^{2s^2+s} &= p^s(p^s+\varepsilon^s)\mathcal{G}_{I_t}(pI_t; 2J_t) - p^s(p^s-\varepsilon^s)\mathcal{G}_{J_t}(pI_t; 2J_t) \\ &= \sum_{\ell=0}^{s}\Big(r(I_{2s+1}, I_{2\ell+1}\perp\langle 0\rangle^{2(s-\ell)}) - r(J_{t+1}, I_{2\ell+1}\perp\langle 0\rangle^{2(s-\ell)})\Big) \\ &\quad \cdot \sum_{{\rm cls}W\in\mathbb{F}_{\rm sym}^{2\ell,2\ell}} R^*(J_{2s}, W)\mathcal{G}^*_{2W}. \end{aligned}$$

The evaluation now proceeds as for $\mathcal{G}^*_{I_t}$.

(b) Now set $t = 2s+1$. When $s = 0$, $\mathcal{G}^*_b = \mathcal{G}_b(p)$, whose value is well-known, and as claimed in the theorem.

So suppose $s > 0$, and that and that the value of $\mathcal{G}^*_W$ is as claimed for all $W \in \mathbb{F}_{\rm sym}^{2\ell+1,2\ell+1}$, $\ell < s$. Suppose first that $0 \le d < t$ and $T \sim I_d \perp \langle 0\rangle^{t-d}$. Then, much as in the previous case,

$$\begin{aligned} \mathcal{G}^*_T &= \frac{1}{o(I_t)}\sum_{G\in GL_t}\big(\mathrm{e}\{{}^tGGT/p\} - \mathrm{e}\{{}^tGJ_tGT/p\}\big) \\ &= \frac{\eta(t,d)}{o(I_t)}\sum_{Y\in\mathbb{F}_{\rm sym}^{d,d}}(r^*(I_t,Y) - r^*(J_t,Y))\,\mathrm{e}\{2Y/p\} \\ &= \frac{\eta(t,d)}{o(I_t)}\sum_{\ell=0}^{s}\big(r^*(I_t,U_\ell) - r^*(J_t,\overline{U}_\ell)\big) \\ &\quad \cdot \sum_{{\rm cls}W\in\mathbb{F}_{\rm sym}^{\ell,\ell}} R^*(I_d,W)\mathcal{G}^*_W \end{aligned}$$

where $U_\ell = I_\ell \perp \langle 0\rangle^{d-\ell}$, $\overline{U}_\ell = J_\ell \perp \langle 0\rangle^{d-\ell}$. As we saw before,

$$r^*(I_{2s+1}, I_{2\ell} \perp \langle 0\rangle^{2s-2\ell+1}) = r^*(J_{2s+1}, I_{2\ell} \perp \langle 0\rangle^{2s-2\ell+1}),$$

so we only need to consider odd ℓ.

Fix ℓ and let $W_a = I_a \perp \langle 0\rangle^{2\ell+1-a}$, $\overline{W}_a = J_a \perp \langle 0\rangle^{2\ell+1-a}$. By our induction hypothesis, $\mathcal{G}^*_{W_a} = -\mathcal{G}^*_{\overline{W}_a}$, and $\mathcal{G}^*_{W_{2a}} = 0$. Also, with d even and a odd,

$$R^*(I_d, W_a) = R^*(I_d, \overline{W}_a).$$

Hence when d is even,

$$R^*(I_d, W_a)\mathcal{G}^*_{W_a} + R^*(I_d, \overline{W}_a)\mathcal{G}^*_{\overline{W}_a} = 0$$

for all a. Hence $\mathcal{G}^*_T = 0$ when d is even.

So suppose $d = 2c+1$. Then, using our induction hypothesis,

$$\sum_{\text{cls}W \in \mathbb{F}^{2\ell+1,2\ell+1}_{\text{sym}}} R^*(I_d, W)\mathcal{G}^*_W = \varepsilon^{c+\ell} p^{c+\ell^2} \mathcal{G}_1(p)\gamma(c,\ell)A(\ell,c)$$

where $A(\ell, c)$ is defined and shown to be $(-1)^\ell$ in Lemma 4.2 (a). Also, as we saw in (a),

$$\begin{aligned} &r^*(I_{2s+1}, I_{2\ell} \perp \langle 0\rangle^{2s-2\ell+1}) - r^*(J_{2s+1}, I_{2\ell} \perp \langle 0\rangle^{2s-2\ell+1}) \\ &\quad = r^*(J_{2s+1}, J_{2\ell} \perp \langle 0\rangle^{2s-2\ell+1}) - r^*(I_{2s+1}, J_{2\ell} \perp \langle 0\rangle^{2s-2\ell+1}) \\ &\quad = 2\varepsilon^{s+\ell} p^{s+c+2c^2+\ell(2s-4c+\ell)}\mu\delta(s, 2c-\ell). \end{aligned}$$

Thus

$$\mathcal{G}^*_T = \frac{2\eta(t,d)}{o(I_t)}\mathcal{G}_1(p)\varepsilon^{c+s}p^{s+2c+2c^2}\mu\delta(s,c)A(c,s).$$

We know $A(c,s) = (-1)^c$; substituting for $\eta(t,d)$ and $o(I_t)$ now yields the result (recall that $o(I_t)$ is computed in Lemma 8.1).

To evaluate $\mathcal{G}^*_{I_t}$, we begin with the expression

$$\mathcal{G}_{I_t}(pI_t; 2I_t) - \mathcal{G}_{J_t}(pI_t; 2I_t),$$

which by Theorem 2.3 is $2\mathcal{G}_1(p)p^{2s(s+1)}$. To evaluate $\mathcal{G}^*_{J_t}$, we begin with

$$\mathcal{G}_{I_t}(pI_t; 2J_t) - \mathcal{G}_{J_t}(pI_t; 2J_t),$$

which by Theorem 2.3 is $-2\mathcal{G}_1(p)p^{2s(s+1)}$.

Then we proceed exactly as before, and thus the details are left to the reader. □

§5. Averaging Hecke operators for weight $k + 1/2$

When evaluating the action on Fourier coefficients of the integral weight Hecke operators with $p \nmid N$ (see [6]), we encountered incomplete character sums; we completed these by replacing $T_j(p^2)$ with $\widetilde{T}_j(p^2)$, a weighted average of the operators $1, T_1(p^2), \dots, T_j(p^2)$. In the half-integral weight case, we have the generalised twisted Gauss sums instead of the incomplete character sums, and as discussed in the preceding section, we know the values of these twisted Gauss sums. However, the action on Fourier coefficients of the half-integral weight Hecke operators is much nicer when we replace $T_j(p^2)$ with $\widetilde{T}_j(p^2)$, a weighted average of $1, T_1(p^2), \dots, T_j(p^2)$, as we now define.

Definition. For $1 \le j \le n$ and p a prime, $N \in \mathbb{Z}_+$ with $p \nmid N$, define $\widetilde{T}_j(p^2)$ acting on $\mathcal{M}_{k+1/2}(\Gamma_0^{(n)}(N), \chi)$ by

$$\widetilde{T}_j(p^2) = p^{j(k-n)} \sum_{\ell=0}^{j} p^{-\ell/2} \chi'(p^{j-\ell}) \beta(n-\ell, j-\ell) T_\ell(p^2)$$

where $\chi'(d) = \chi(d) \left(\frac{(-1)^{k+1}}{|d|}\right) (\operatorname{sgn} d)^{k+1}$.

Theorem 5.1. *Take $F \in \mathcal{M}_{k+1/2}(\Gamma_0^{(n)}(N), \chi)$ where $4|N$, and let p be a prime such that $p \nmid N$; let χ' be defined as in the above definition. Given*

$$\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2, \ \Omega = p\Lambda_0 \oplus \Lambda_1 \oplus \frac{1}{p}\Lambda_2$$

with $n_i = \operatorname{rank}\Lambda_i$, $r = n_0 + n_2$, set

$$E_j(\Lambda, \Omega) = j(k-n) + k(n_2 - n_0) + n_0(n - n_2) + (j-r)(j-r-1)/2;$$

set

$$\widetilde{A}_j(\Lambda, \Omega) = \chi'(p^{j-r}) p^{E_j(\Lambda,\Omega)} R^*(\Lambda_1/p\Lambda_1 \perp \langle 2 \rangle, \langle 0 \rangle^{j-r})$$

if Λ, Ω are even integral, and set $\widetilde{A}_j(\Lambda, \Omega) = 0$ otherwise. Then the Λth coefficient of $F|\widetilde{T}_j(p^2)$ is

$$\sum_{p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda} \widetilde{A}_j(\Lambda, \Omega) c(\Omega).$$

Proving this theorem comes down to proving the next proposition, as we now discuss.

Fix Λ and Ω, with Λ_i, n_i, r defined as above; let $V = \Lambda_1/p\Lambda_1$ (so V is a quadratic space over $\mathbb{Z}/p\mathbb{Z}$). From the definition of $\widetilde{T}_j(p^2)$ and Theorem 5.4,

$$\widetilde{A}_j(\Lambda, \Omega) = \chi'(p^{j-r}) p^{j(k-n)+k(n_2-n_0)+n_0(n-n_2)} \cdot \sum_{\ell=0}^{j} \sum_{\substack{\operatorname{cls} U \\ \dim U = \ell - r}} \beta(n-\ell, j-\ell) R^*(V, U) \widetilde{\mathcal{G}}(U).$$

Notice that the number of ways to extend an $\ell-r$ dimensional subspace U of V to a $j-r$ dimensional subspace W of V is

$$\beta\big((n-r)-(\ell-r),(j-r)-(\ell-r)\big)=\beta(n-\ell,j-\ell);$$

thus

$$\beta(n-\ell,j-\ell)\,R^*(V,U)=\sum_{\substack{\mathrm{cls}W\\ \dim W=j-r}}R^*(V,W)\,R^*(W,U).$$

Hence

$$\sum_{\ell=0}^{j}\beta(n-\ell,j-\ell)\sum_{\substack{\mathrm{cls}U\\ \dim U=\ell-r}}R^*(V,U)\,\widetilde{\mathcal{G}}(U)$$
$$=\sum_{\substack{\mathrm{cls}W\\ \dim W=j-r}}R^*(V,W)\sum_{\substack{\mathrm{cls}U\\ \dim U\le j-r}}R^*(W,U)\,\widetilde{\mathcal{G}}(U).$$

Set $m=j-r$; let W' be a dimension m totally isotropic subspace of $V\perp\mathbb{F}v$ where $\mathbb{F}v\simeq\big\langle 2\big\rangle$, and let W be the projection of W' onto V. Then either

$$W=W'\simeq\big\langle 0\big\rangle^m\text{ or }W\simeq\big\langle 0\big\rangle^{m-1}\perp\big\langle -2\big\rangle;$$

also, there are exactly 2 subspaces of $V\perp\mathbb{F}v$ that project onto a given subspace $W\simeq\big\langle 0\big\rangle^{m-1}\perp\big\langle -2\big\rangle$ in V. Thus

$$R^*(V\perp\mathbb{F}v,\big\langle 0\big\rangle^m)=R^*(V,\big\langle 0\big\rangle^m)+2R^*(V,\big\langle 0\big\rangle^{m-1}\perp\big\langle -2\big\rangle).$$

Hence proving Theorem 5.1 reduces to proving the following.

Proposition 5.2. *With W of dimension $m\ge 1$ over $\mathbb{F}=\mathbb{Z}/p\mathbb{Z}$ $(p\ne 2)$, we have*

$$\sum_{q=0}^{m}\sum_{\substack{\mathrm{cls}U\\ \dim U=q}}R^*(W,U)\,\widetilde{\mathcal{G}}(U)=\begin{cases}p^{m(m-1)/2} & \text{if } W\simeq\big\langle 0\big\rangle^m,\\ 2p^{m(m-1)/2} & \text{if } W\simeq\big\langle 0\big\rangle^{m-1}\perp\big\langle -2\big\rangle,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Notice that the proposition is proved once we prove that

$$\sum_{\substack{\mathrm{cls}U\\ \dim U\le m}}R^*(W,U)\widetilde{\mathcal{G}}(U)=p^{m(m-1)/2}R^*(W\perp\langle 2\rangle,\langle 0\rangle^m).$$

To do this, we argue by induction on $m=\dim W$.

For $m=1$, the proposition is easily verified. (Recall that when $\dim U=0$, $\widetilde{\mathcal{G}}(U)=1$.)

So now suppose $m = \dim W \geq 1$. One easily verifies that for $0 \leq q < m$,

$$\sum_{a=q}^{m-1} (-1)^{m+a+1} p^{(m-a)(m-a-1)/2} \beta(m-q, a-q) = 1.$$

Thus with U a quadratic space over $\mathbb{F}$ with dimension q, we have

$$\beta(m-q, a-q) R^*(W,U) = \sum_{\substack{\mathrm{cls}\, Y \\ \dim Y = a}} R^*(W,Y) R^*(Y,U)$$

and hence

$$R^*(W,U) = \sum_{a=q}^{m-1} (-1)^{m+a+1} p^{(m-a)(m-a-1)/2} \sum_{\substack{\mathrm{cls}\, Y \\ \dim Y = a}} R^*(W,Y) R^*(Y,U).$$

So

$$\begin{aligned}
&\sum_{\substack{\mathrm{cls}\, U \\ \dim U < m}} R^*(W,U) \widetilde{\mathcal{G}}(U) \\
&\quad = -\sum_{a=0}^{m-1} (-1)^{m-a} p^{(m-a)(m-a-1)/2} \sum_{\substack{\mathrm{cls}\, Y \\ \dim Y = a}} R^*(W,Y) \sum_{\substack{\mathrm{cls}\, U \\ \dim U \leq a}} R^*(Y,U) \widetilde{\mathcal{G}}(U).
\end{aligned}$$

The implicit induction hypothesis tells us that for $a < m$ and $\dim Y = a$,

$$\sum_{\substack{\mathrm{cls}\, U \\ \dim U \leq a}} R^*(Y,U) \widetilde{\mathcal{G}}(U) = \begin{cases} p^{a(a-1)/2} & \text{if } Y \simeq \langle 0 \rangle^a, \\ 2p^{a(a-1)/2} & \text{if } Y \simeq \langle 0 \rangle^{a-1} \perp \langle -2 \rangle, \\ 0 & \text{otherwise.} \end{cases}$$

Thus for $a < m$,

$$\begin{aligned}
&\sum_{\substack{\mathrm{cls}\, Y \\ \dim Y = a}} \sum_{\substack{\mathrm{cls}\, U \\ \dim U \leq a}} R^*(W,Y) R^*(Y,U) \widetilde{\mathcal{G}}(U) \\
&\quad = p^{a(a-1)/2} \big(R^*(W, \langle 0 \rangle^a) + 2R^*(W, \langle 0 \rangle^{a-1} \perp \langle -2 \rangle) \big) \\
&\quad = p^{a(a-1)/2} R^*(W \perp \langle 2 \rangle, \langle 0 \rangle^a).
\end{aligned}$$

So to prove the proposition, we need to show

$$\widetilde{\mathcal{G}}(W) = (-1)^m p^{m(m-1)/2} \sum_{a=0}^{m} (-1)^a p^{a(a-m)} R^*(W \perp \langle 2 \rangle, \langle 0 \rangle^a).$$

Standard theory tells us that W splits as $W_0 \perp R$ where W_0 is regular and $R \simeq \langle 0 \rangle^s$ for some s. While R, the radical of W, is uniquely determined by W,

W_0 is not; however, the isometry class of W_0 is uniquely determined by W. So either $W_0 \simeq 2I_{m-s}$ or $W_0 \simeq 2J_{m-s}$. Also, with $\mathbb{F}w \simeq \langle 2\rangle$, any totally isotropic subspace U of $W \perp \mathbb{F}w$ splits as $U_0 \perp U_1$ where $U_1 \subseteq R$ and $U_0 \cap R = \{0\}$. Given a dimension t subspace of U_1 of R, the number of distinct totally isotropic, dimension a subspaces U of $W \perp \mathbb{F}w$ with $U \cap R = U_1$ is

$$p^{(s-t)(a-t)}R^*(W_0 \perp \langle 2\rangle, \langle 0\rangle^{a-t}).$$

Since there are $\beta(s,t)$ subspaces U_1 of R with $\dim U_1 = t$,

$$\begin{aligned}&\sum_{a=0}^{m}(-1)^a p^{a(a-m)}R^*(W \perp \langle 2\rangle, \langle 0\rangle^{a})\\ &\qquad = \sum_{\ell+t\le m}(-1)^{\ell+t}p^{(\ell+t)(\ell+t-m)+(s-t)\ell}\beta(s,t)R^*(W_0 \perp \langle 2\rangle, \langle 0\rangle^{\ell}).\end{aligned}$$

First suppose $W_0 \perp \langle 2\rangle \simeq \mathbb{H}^c$. (So $m = 2c+s-1$ and $W \simeq \mathbb{H}^{c-1} \perp \langle -2\rangle \perp \langle 0\rangle^{s}$.) Then

$$R^*(W_0 \perp \langle 2\rangle, \langle 0\rangle^{\ell}) = \beta(c,\ell)\delta(c-1,\ell)$$

and hence

$$\sum_a (-1)^a p^{a(a-m)}R^*(W \perp \langle 2\rangle, \langle 0\rangle^{a}) = (-1)^c p^{c(1-c)}X_s(c,-1)$$

where for $s,t \ge 0$, we define

$$\begin{aligned}X_s(c,q) &= \sum_{t=0}^{s}(-1)^t p^{t(t-c-s-q)}\beta(s,t)S_t(c,q),\\ S_t(c,q) &= (-1)^c p^{c(c-t+q)}\sum_{\ell=0}^{c}(-1)^{\ell}p^{\ell(\ell-2c+t-q)}\beta(c,\ell)\delta(c+q,\ell).\end{aligned}$$

Clearly $S_t(0,q) = 1$; using the definitions of β and δ, when $c > 0$ we have

$$\begin{aligned}&S_t(c,q) + (p^c-1)(p^{c+q}+1)S_t(c-1,q)\\ &\quad = (-1)^c p^{c(c-t+q)}\sum_{\ell=0}^{c}(-1)^{\ell}p^{\ell(\ell-2c+t-q)}\frac{\mu(c,\ell)\delta(c+q,\ell)}{\mu(\ell,\ell)}\\ &\qquad + (-1)^c p^{c(c-t+q)-2c-q+t+1}\sum_{\ell=1}^{c}(-1)^{\ell}p^{(\ell-1)(\ell+1-2c+t-q)}\frac{\mu(c,\ell)\delta(c+q,\ell)}{\mu(\ell-1,\ell-1)}\\ &\quad = (-1)^c p^{c(c-t+q)}\left[1+\sum_{\ell=1}^{c}(-1)^{\ell}p^{\ell(\ell-2c+t+1-q)}\beta(c,\ell)\delta(c+q,\ell)\right]\\ &\quad = p^c S_{t+1}(c,q).\end{aligned}$$

Taking $S_t(c,q)=0$ when $c<0$, we have

$$S_t(c,q)+(p^c-1)(p^{c+q}+1)S_t(c-1,q)=p^cS_{t+1}(c,q)$$

for all c. Also, replacing ℓ by $c-\ell$ and remembering that $\beta(s,a)=\beta(s,s-a)$, Lemma 4.2 (b) tells us that $S_0(c,q)=B(c,q)=1$.

Using and the identity $\beta(s+1,t)=p^t(\beta(s,t)+\beta(s,t-1)$, the recursion relation for $S_t(c,q)$, we have

$$\begin{aligned}
&X_{s+1}(c,q)\\
&\quad=\sum_{t=0}^{s}(-1)^tp^{t(t-c-s-q)}\beta(s,t)S_t(c,q)\\
&\quad\ +\sum_{t=1}^{s+1}(-1)^tp^{t(t-c-s-1-q)}\beta(s,t-1)S_t(c,q)\\
&\quad=X_s(c,q)-p^{-2c-s-q}X'_s(c,q)-p^{-2c-s-q}(p^c-1)(p^{c+q}+1)X_s(c-1,q)
\end{aligned}$$

where

$$X'_s(c,q)=\sum_{t=0}^{s}(-1)^tp^{t(t-c-s-q+1)}\beta(s,t)S_t(c,q).$$

Similarly, using the identity $\beta(s+1,t)=\beta(s,t)+p^{s+1-t}\beta(s,t-1)$,

$$X'_{s+1}(c,q)=X_s(c,q)-p^{-2c+1-q}X'_s(c,q)-p^{-2c+1-q}(p^c-1)(p^{c+q}+1)X_s(c-1,q).$$

We claim that

$$X_s(c,-1)=\begin{cases}p^{2x-2cx-x^2}\prod_{i=1}^{x}(p^{2i-1}-1) & \text{if } s=2x,\\ -p^{-c-2cx-x^2}\prod_{i=1}^{x+1}(p^{2i-1}-1) & \text{if } s=2x+1,\end{cases}$$

and

$$X'_s(c,-1)=\begin{cases}1 & \text{if } s=0,\\ X_{s-1}(c,-1)+(p^{c-1}+1)X_s(c-1,-1) & \text{if } s=2x>0,\\ (p^{c-1}+1)X_s(c-1,-1) & \text{if } s=2x+1\end{cases}$$

where we agree $\prod_{i=1}^{x}(p^{2i-1}-1)=1$ if $x=0$. These identities are easily proved using induction on x, the recursion relations for X and X', and the fact that $S_0(c,q)=1$; thus the details are left to the reader. Consequently, noting that

$$\mathbb{H}^{c-1}\perp\big\langle-2\big\rangle\simeq\begin{cases}2I_{2c-1} & \text{if } \varepsilon^c=1,\\ 2J_{2c-1} & \text{if } \varepsilon^c=-1,\end{cases}$$

we find that

$$\widetilde{\mathcal{G}}(W)=(-1)^mp^{m(m-1)/2}\sum_{a=0}^{m}(-1)^ap^{a(a-m)}R^*(W\perp\langle2\rangle,\langle0\rangle^a)$$

in the case that $W\simeq\mathbb{H}^{c-1}\perp\big\langle -2\big\rangle\perp\big\langle 0\big\rangle^s$.

Next suppose $W_0\perp\big\langle 2\big\rangle\simeq\mathbb{H}^{c-1}\perp\mathbb{A}$, $c\ge 1$. (So $m=2c+s-1$ and $W\simeq\mathbb{H}^{c-1}\perp\big\langle -2\omega\big\rangle\perp\big\langle 0\big\rangle^s$.) Then

$$R^*(W_0\perp\big\langle 2\big\rangle,\big\langle 0\big\rangle^\ell)=\beta(c-1,\ell)\delta(c,\ell)$$

and hence

$$\sum_a(-1)^a p^{a(a-m)}R^*(W\perp\big\langle 2\big\rangle,\big\langle 0\big\rangle^a)=(-1)^c p^{c(1-c)}X_s(c-1,1).$$

By induction on x, one finds

$$X_s(c-1,1)=\begin{cases} p^{2x-2cx-x^2}\prod_{i=1}^x(p^{2i-1}-1) & \text{if } s=2x,\\ p^{-c-2cx-x^2}\prod_{i=1}^{x+1}(p^{2i-1}-1) & \text{if } s=2x+1,\end{cases}$$

and

$$X_s'(c-1,1)=\begin{cases} 1 & \text{if } s=0,\\ (X_{s-1}(c-1,1)-(p^{c-1}-1)X_s(c-2,1) & \text{if } s=2x>0,\\ -(p^{c-1}-1)X_s(c-2,1) & \text{if } s=2x+1.\end{cases}$$

Consequently, noting that

$$\mathbb{H}^{c-1}\perp\big\langle -2\omega\big\rangle\simeq\begin{cases} 2I_{2c-1} & \text{if } \varepsilon^c=-1,\\ 2J_{2c-1} & \varepsilon^c=-1,\end{cases}$$

we find that

$$\widetilde{\mathcal{G}}(W)=(-1)^m p^{m(m-1)/2}\sum_{a=0}^m(-1)^a p^{a(a-m)}R^*(W\perp\big\langle 2\big\rangle,\big\langle 0\big\rangle^a)$$

when $W\simeq\mathbb{H}^{c-1}\perp\big\langle -2\omega\big\rangle\perp\big\langle 0\big\rangle^s$.

Finally suppose $W_0\perp\big\langle 2\big\rangle$ has dimension $2c+1$. (So $m=2c+s$ and $W\simeq 2I_{2c}\perp\big\langle 0\big\rangle^s$ or $2J_{2c}\perp\big\langle 0\big\rangle^s$.) Then

$$R^*(W_0\perp\big\langle 2\big\rangle,\big\langle 0\big\rangle^\ell)=\beta\delta(c,\ell)$$

and hence

$$\sum_a(-1)^a p^{a(a-m)}R^*(W\perp\big\langle 2\big\rangle,\big\langle 0\big\rangle^a)=(-1)^c p^{-c^2}X_s(c,0).$$

By induction on x, we get

$$X_s(c,0)=\begin{cases} p^{-2cx-x^2}\prod_{i=1}^x(p^{2i-1}-1) & \text{if } s=2x,\\ 0 & \text{if } s=2x+1,\end{cases}$$

and

$$X'_s(c,0) = \begin{cases} X_s(c-1,0) & \text{if } s = 2x > 0, \\ -p^{-2cx-x^2}\prod_{i=1}^{x+1}(p^{2i-1}-1) & \text{if } s = 2x+1. \end{cases}$$

Consequently, noting that

$$(\mathbb{H}^c, \mathbb{H}^{c-1} \perp \mathbb{A}) \simeq \begin{cases} (2I_{2c}, 2J_{2c}) & \text{if } \varepsilon^c = 1, \\ (2J_{2c}, 2I_{2c}) & \text{if } \varepsilon^c = -1, \end{cases}$$

we find that

$$\widetilde{\mathcal{G}}(W) = (-1)^m p^{m(m-1)/2} \sum_{a=0}^{m} (-1)^a p^{a(a-m)} R^*(W \perp \langle 2 \rangle, \langle 0 \rangle^a)$$

when $W \simeq W_0 \perp \langle 0 \rangle^s$ with $\dim W_0 = 2c$.

This proves the proposition. □

§6. Hecke operators on Siegel theta series of weight $k + 1/2$

Throughout this section, we assume L is a rank $2k+1$ lattice with an even integral, positive definite quadratic form Q; we fix $n \le 2k+1$. We use B_Q to denote the symmetric bilinear form associated to Q, N the level of L, χ the character of $\theta^{(n)}(L)$, and χ' the character defined by $\chi'(d) = \chi(d)\left(\frac{(-1)^{k+1}}{|d|}\right)(\operatorname{sgn} d)^{k+1}$.

At the end of §2 we defined the exponential $\mathrm{e}^*\{\Lambda\tau\}$. When working with theta series, it is convenient to have the exponential $\mathrm{e}\{\Lambda\tau\}$, which we define below.

As we have seen,

$$\theta^{(n)}(L;\tau) = \sum_{x_1,\dots,x_n \in L} \mathrm{e}\left\{\big(B_Q(x_h,x_i)\big)\tau\right\}.$$

Thus $\big(B_Q(x_h,x_i)\big)$ is the matrix for the quadratic form Q restricted to the (external) direct sum

$$\Lambda = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_n.$$

However, as a sublattice of L, we may have $d = \operatorname{rank}(\mathbb{Z}x_1 + \ldots + \mathbb{Z}x_n) < n$. In such a case there exists some $G \in GL_n(\mathbb{Z})$ so that

$$(x_1 \, \ldots \, x_n)G = (x'_1 \, \ldots \, x'_d \, 0 \, \ldots \, 0).$$

Still, we can consider Λ as a sublattice of L with "formal rank" n.

Given a sublattice $\Lambda' = \mathbb{Z}x'_1 + \cdots + \mathbb{Z}x'_d$ of L with $\operatorname{rank}\Lambda' = d$ and $T' = \big(B_Q(x'_h,x'_i)\big)$ (a $d \times d$ matrix),

$$\sum_{\substack{x_1,\dots,x_n \in L \\ \mathbb{Z}x_1+\cdots+\mathbb{Z}x_n = \Lambda'}} \mathrm{e}\left\{\big(B_Q(x_h,x_i)\big)\tau\right\} = \sum_G \mathrm{e}\left\{ {}^tG \begin{pmatrix} T' & \\ & 0_{n-d} \end{pmatrix} G\tau \right\}$$

where G varies over

$$\left\{\begin{pmatrix} I_d & 0 \\ * & * \end{pmatrix} \in GL_n(\mathbb{Z})\right\}\Big\backslash GL_n(\mathbb{Z}).$$

Thus with $x'_{d+1} = \cdots = x'_n = 0$, $\Lambda = \mathbb{Z}x'_1 \oplus \cdots \oplus \mathbb{Z}x'_n$ (the external direct sum), we define

$$\mathrm{e}\{\Lambda\tau\} = \sum_G \mathrm{e}\left\{ {}^tG \begin{pmatrix} T' & \\ & 0_{n-d} \end{pmatrix} G\tau \right\}$$

where G varies as above. Then

$$\theta^{(n)}(L;\tau) = \sum_{\Lambda \subseteq L} \mathrm{e}\{\Lambda\tau\}$$

where Λ varies over all distinct sublattices of L with formal rank n. (When $x_i, y_i \in L$, we say $\mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_n$ and $\mathbb{Z}y_1 \oplus \cdots \oplus \mathbb{Z}y_n$ are distinct sublattices of L with formal rank n when $\mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n \neq \mathbb{Z}y_1 + \cdots + \mathbb{Z}y_n$.)

Remark. For $x_i \in L$, $\Lambda = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_n$, and $\Lambda' = \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n$, we have $\mathrm{e}\{\Lambda\tau\} = o(\Lambda')\mathrm{e}^*\{\Lambda\tau\}$ since, with $d = \mathrm{rank}\Lambda'$ (as a sublattice of L),

$$O(\Lambda' \perp \langle 0\rangle^{n-d}) = \left\{ \begin{pmatrix} E' & 0 \\ * & * \end{pmatrix} \in GL_n(\mathbb{Z}) : \ E' \in O(\Lambda') \right\}.$$

Proposition 6.1. *For p a prime not dividing N and $1 \leq j \leq n$, we have*

$$\theta^{(n)}(L;\tau)|\widetilde{T}_j(p^2) = \sum_{\Omega} \widetilde{c}_j(\Omega)e\{\Omega\tau\}$$

where Ω varies over all even integral sublattices of $\frac{1}{p}L$ that have (formal) rank n. For given Ω, decompose Ω as $\frac{1}{p}\Omega_0 \oplus \Omega_1 \oplus p\Omega_2$ where $\Omega_i \subseteq L$ and $\Omega_0 \oplus \Omega_1$ is primitive in L modulo p, meaning that the formal rank of $\Omega_0 \oplus \Omega_1$ is the rank of its image in L, which is also the dimension of $\overline{\Omega_0 \oplus \Omega_1}$ in L/pL. Let r_i be the (formal) rank of Ω_i; set

$$E(\ell, t, \Omega) = t(k-n) + t(t-1)/2 + \ell(k - r_0 - r_1) + \ell(\ell-1)/2.$$

Then, if $\chi'(p) = 1$,

$$\widetilde{c}_j(\Omega) = \sum_{\ell,t} p^{E(\ell,t,\Omega)} R^*(\Omega_1/p\Omega_1 \perp \langle 2\rangle, \langle 0\rangle^{\ell}) \cdot \delta(k - r_0 - \ell, t)\beta(r_2, t)\beta(n - r_0 - \ell - t, n - j);$$

if $\chi'(p) = -1$,

$$\widetilde{c}_j(\Omega) = \sum_{\ell,t} (-1)^{\ell} p^{E(\ell,t,\Omega)} R^*(\Omega_1/p\Omega_1 \perp \langle 2\rangle, \langle 0\rangle^{\ell}) \cdot \mu(k - r_0 - \ell, t)\beta(r_2, t)\beta(n - r_0 - \ell - t, n - j).$$

Proof. By the definitions of $T_j(p^2)$ and $\widetilde{T}_j(p^2)$, we have

$$\theta^{(n)}(L;\tau)|\widetilde{T}_j(p^2) = \sum_{\substack{\Lambda\subseteq L\\ p\Lambda\subseteq\Omega\subseteq\frac{1}{p}\Lambda}} \widetilde{A}_j(\Omega,\Lambda)\mathrm{e}\{\Omega\tau\}$$

where $\widetilde{A}_j(\Omega,\Lambda)$ is defined in Theorem 3.4. (Note that at the end of the proof of Theorem 3.4 we made a change of variables that we do not make here.) Since $p\neq 2$ and $\Omega\subseteq\frac{1}{p}L$, Ω is even integral exactly when it is integral, so $\widetilde{A}_j(\Omega,\Lambda)=0$ when Ω is not integral. Interchanging the order of summation, we have

$$\theta^{(n)}(L;\tau)|\widetilde{T}_j(p^2) = \sum_{\substack{\Omega\subseteq\frac{1}{p}L\\ \Omega\text{ integral}}} \sum_{p\Omega\subseteq\Lambda\subseteq(\frac{1}{p}\Omega\cap L)} \widetilde{A}_j(\Omega,\Lambda)\mathrm{e}\{\Omega\tau\}.$$

Fix integral $\Omega\subseteq\frac{1}{p}L$ so that Ω has formal rank n; decompose Ω as in the statement of the Proposition. (Note that Ω_0 is only well-determined up to $p(\Omega_1\oplus\Omega_2)$, and $\Omega_0\oplus\Omega_1$ is only well-determined up to $p\Omega_2$.) Set $\Delta=\frac{1}{p}\Omega\cap L=\Omega_0\oplus\Omega_1\oplus\Omega_2$. Note that $\Omega\cap\Delta=\Omega_0\oplus\Omega_1\oplus p\Omega_2$, and $p(\Omega+\Delta)=\Omega_0\oplus p\Omega_1\oplus p\Omega_2$.

Fix d_i, d_i', d_2'' so that $d_1+d_1'=r_1$, $d_2+d_2'+d_2''=r_2$. As in the proof of Proposition 1.4 of [11] and Proposition 2.1 [12], we construct all

$$\Lambda=\Omega_0\oplus(\Lambda_1\oplus p\Lambda_1')\oplus(\Lambda_2\oplus p\Lambda_2'\oplus p^2\Lambda_2'')$$

where $\Lambda_1\oplus\Lambda_1'=\Omega_1$, $\Lambda_2\oplus\Lambda_2'\oplus\Lambda_2''=\Omega_2$, d_i is the (formal) rank of Λ_i, d_i' is the (formal) rank of Λ_i', d_2'' is the formal rank of Λ_2'', although here we need to weight each Λ by

$$R^*\big((\Lambda_1/p\Lambda_1\oplus p\Lambda_2'/p^2\Lambda_2')\perp\langle 2\rangle,\langle 0\rangle^{d_1+d_2'-n+j}\big).$$

(So then varying d_i, d_i', d_2'', these Λ vary over all lattices subject to $p\Omega\subseteq\Lambda\subseteq\Delta$.)

In $\Delta/p\Delta$, extend $\overline{\Omega\cap\Delta}=\overline{\Omega_0\oplus\Omega_1}$ to $\overline{\Omega_0\oplus\Omega_1\oplus\Delta_2}$ with $\dim\Delta_2=d_2+d_2'$; we have $\beta(r_2,d_2+d_2')=\beta(r_2,d_2'')$ choices. Let

$$\begin{aligned}\Delta'&=\text{preimage in }\Delta\text{ of }\overline{\Omega_0\oplus\Omega_1\oplus\Delta_2}\\ &=\Omega_0\oplus\Omega_1\oplus(\Delta_2\oplus p\Lambda_2'').\end{aligned}$$

In $\Delta'/p\Delta'$, extend $\overline{p\Omega}=\overline{\Omega_0}$ to $\overline{\Omega_0\oplus U}$ so that $\overline{\Omega_0\oplus U}\subseteq\overline{\Omega\cap\Delta}=\overline{\Omega_0\oplus\Omega_1\oplus p\Lambda_2''}$, $\dim\overline{U}=\ell$, $\overline{U}$ is independent of $\overline{p\Delta}=\overline{p\Lambda_2''}$, and either

$$\overline{U}\simeq\langle 0\rangle^{\ell}\text{ or }\overline{U}\simeq\langle 0\rangle^{\ell-1}\perp\langle -2\rangle.$$

When $\overline{U}\simeq\langle 0\rangle^{\ell}$, we weight $\overline{U}$ by 1; when $\overline{U}\simeq\langle 0\rangle^{\ell-1}\perp\langle -2\rangle$, we weight $\overline{U}$ by 2. So we have $p^{\ell d_2''}R^*(\Omega_1/p\Omega_1\perp\langle 2\rangle,\langle 0\rangle^{\ell})$ (weighted) choices for $\overline{U}$. Next, extend $\overline{\Omega_0\oplus U}$ to $\overline{\Omega_0\oplus\Lambda_1}$ so that $\overline{\Omega_0\oplus\Lambda_1}\subseteq\overline{\Omega_0\oplus\Omega_1\oplus p\Lambda_2''}$ with Λ_1 independent of $\overline{p\Lambda_2''}$;

we have $p^{(d_1-\ell)d_2''}\beta(r_1-\ell,d_1-\ell)$ choices. Now extend $\overline{\Omega_0\oplus\Lambda_1}$ to $\overline{\Omega_0\oplus\Lambda_1\oplus\Lambda_2}$ so that $\dim\overline{\Lambda_2}=d_2$ and $\overline{\Lambda}_2$ is independent of $\overline{\Omega_0\oplus\Omega_1\oplus p\Lambda_2''}$; we have $p^{d_2(d_1'+d_2'')}\beta(d_2+d_2',d_2)$ choices. So $\overline{\Delta'}=\overline{\Omega_0\oplus(\Lambda_1\oplus\Lambda_1')\oplus(\Lambda_2\oplus\Lambda_2'\oplus p\Lambda_2'')}$; let

$$\Lambda=\text{preimage in }\Delta'\text{ of }\overline{\Omega_0\oplus\Lambda_1\oplus\Lambda_2}.$$

Note that with $d=d_1+d_2'-n+j$, there are $p^{\ell(d_2'-d+\ell)}\beta(d_2',d-\ell)$ dimension d subspaces of $\Omega\cap\Lambda/p(\Omega+\Lambda)\approx\Lambda_1/p\Lambda_1\oplus p\Lambda_2'/p^2\Lambda_2'$ that project onto a given choice of $\overline{U}$. Thus with $x=d+d_2-\ell$,

$$\begin{aligned}\widetilde{c}_j(\Omega)&=\sum_{p\Omega\subseteq\Lambda\subseteq\Delta}\widetilde{A}_j(\Omega,\Lambda)\\&=\sum_{d_i,d_i',d_2'',\ell}\chi'(p^d)p^{j(k-n)+k(d_2-r_0-d_1'-d_2'')+(r_0+d_1'+d_2'')(n-d_2)+d(d-1)/2}\\&\qquad\cdot p^{d_2''(d_1-\ell)+d_2(d_1'+d_2'')+\ell(\ell-d+r_2-d_2)}R^*(\Omega_1/p\Omega_1\perp\langle 2\rangle,\langle 0\rangle^\ell)\\&\qquad\cdot\beta(r_2,d_2'')\beta(r_1-\ell,d_1-\ell)\beta(d_2+d_2',d_2)\beta(d_2',d-\ell).\end{aligned}$$

Thus with $x=d+d_2-\ell=j-r_0-t-\ell$,

$$\begin{aligned}\beta(r_2,d_2'')\beta(d_2+d_2',d_2)\beta(d_2',d-\ell)\frac{\mu(x,d_2)}{\mu(x,d_2)}&=\frac{\mu(r_2,x)\mu(r_2-x,d_2'')\mu(x,d_2)}{\mu(d_2'',d_2'')\mu(d_2,d_2)\mu(x,x)}\\&=\beta(r_2,x)\beta(r_2-x,d_2'')\beta(x,d_2).\end{aligned}$$

Also, recall that $r_1=d_1+d_1'$. For fixed t,

$$\sum_{d_1'+d_2''=t}p^{d_2''(r_1-d_1'-\ell)}\beta(r_2-x,d_2'')\beta(r_1-\ell,d_1')=\beta(r_1+r_2-x-\ell,t)$$

since, with V and V' $\mathbb{Z}/p\mathbb{Z}$-vector spaces of dimensions r_2-x and $r_1-\ell$, we can construct all dimension t subspaces of $V\oplus V'$ by first constructing a dimension d_1' subspace U' of V', and then extending this to a dimension t subspace $U\oplus U'$ of $V\oplus V'$, where U is independent of V'. Therefore

$$\widetilde{c}_j(\Omega)=\sum_{\ell,t}\chi'(p^{j-r_0-t})p^ER^*(\overline{\Omega}_1\perp\langle 2\rangle,\langle 0\rangle^\ell)\beta(n-j+t,t)\beta(r_2,x)S(x,k+1-j+t)$$

where

$$E=(k-n)(j-r_0-t)+(j-r_0-t)(j-r_0-t-1)/2+\ell(\ell+n-j-r_1+t),$$

and

$$S(x,y)=\sum_{d_2=0}^{x}\chi'(p^{d_2})p^{d_2y+d_2(d_2-1)/2}\beta(x,d_2).$$

Using that $\beta(m,q) = p^q\beta(m-1,q) + \beta(m-1,q-1)$ and $x = j - r_0 - t - \ell$, we find that

$$S(x,y) = (\chi'(p)p^y + 1)S(x-1,y+1);$$

consequently

$$S(x,y) = \begin{cases} \delta(x+y-1,x) & \text{if } \chi'(p) = 1, \\ (-1)^x\mu(x+y-1,x) & \text{if } \chi'(p) = -1. \end{cases}$$

Finally, we replace t by $j - r_0 - t - \ell$, and use that $\beta(m,q) = \beta(m,m-q)$. □

For Proposition 6.3, we will need to use Lemma 4.1 from [12], which is the following.

Lemma 6.2 (Reduction Lemma). *Let U be a dimension d space over $\mathbb{Z}/p\mathbb{Z}$, $\ell \geq 0$ and $t \in \mathbb{Z}$ so that $U \perp \mathbb{H}^t$ (resp. $U \perp \mathbb{H}^t \perp \mathbb{A}$) is defined.*
(a) $\varphi_\ell(U \perp \mathbb{H}^t) = \sum_{r=0}^{\ell} p^{r(t-\ell+r)}\delta(d-1+t-r,\ell-r)\beta(t,\ell-r)\varphi_r(U)$.
(b) $\varphi_\ell(U \perp \mathbb{H}^t \perp \mathbb{A}) = \sum_{r=0}^{\ell}(-1)^r p^{r(t+1-\ell+r)}\beta(d+t-r,\ell-r)\delta(t+1,\ell-r)\varphi_r(U)$.

Proposition 6.3. *Let L be as above, and fix a prime $p \nmid N$; choose j so that $1 \leq j \leq n$ and $j \leq k$. We say a lattice K is a p^j-neighbor of L if $K \in genL$ and*

$$L = L_0 \oplus L_1 \oplus L_2, \quad K = \frac{1}{p}L_0 \oplus L_1 \oplus pL_2$$

with $rankL_0 = rankL_2 = j$.

(a) The number of p^j-neighbors of L is $p^{j(j-1)/2}\beta\delta(k,j)$.

(b) We have

$$\sum_{K_j} \theta^{(n)}(K_j;\tau) = \sum_{\Omega} b_j(\Omega)e\{\Omega\}$$

where K_j varies over all p^j-neighbors of L, and Ω varies over all even integral sublattices of $\frac{1}{p}L$ with (formal) rank n. For such Ω, decompose Ω as $\frac{1}{p}\Omega_0 \oplus \Omega_1 \oplus p\Omega_2$ where $\Omega_i \subseteq L$ and $\Omega_0 \oplus \Omega_1$ is primitive in L modulo p; let r_i denote the (formal) rank of Ω_i. Then, if $\chi'(p) = 1$,

$$b_j(\Omega) = p^{(j-r_0)(j-r_0-1)/2} \sum_{\ell=0}^{j-r_0} p^{\ell(k-j-r_1+\ell)} R^*(\Omega_1/p\Omega_1 \perp \langle 2\rangle, \langle 0\rangle^{\ell}) \cdot \delta(k-r_0-\ell, j-r_0-\ell)\beta(k-r_0-r_1, j-r_0-\ell);$$

if $\chi'(p) = -1$,

$$b_j(\Omega) = p^{(j-r_0)(j-r_0-1)/2} \sum_{\ell=0}^{j-r_0} (-1)^{\ell} p^{\ell(k-j-r_1+\ell)} R^*(\Omega_1/p\Omega_1 \perp \langle 2\rangle, \langle 0\rangle^{\ell}) \cdot \beta(k-r_0-\ell, j-r_0-\ell)\delta(k-r_0-r_1, j-r_0-\ell).$$

Proof. For this proof we use the arithmetic theory of quadratic forms over $\mathbb{Z}/p\mathbb{Z}$ and over $\mathbb{Z}_p$. (a) To construct all K_j, choose a dimension j totally isotropic subspace $\overline{C}$ of L/pL; let K' be the preimage in L of $\overline{C}$. (Note that there are $R^*(L/pL, \langle 0\rangle^j) = \beta\delta(k,j)$ choices for $\overline{C}$.) Since L/pL is regular, there is some $\overline{D} \subseteq L/pL$ so that $\overline{C} \oplus \overline{D} \simeq \begin{pmatrix} 0 & I_j \\ I_j & 0 \end{pmatrix}$ (over $\mathbb{Z}/p\mathbb{Z}$). So $\overline{C} \oplus \overline{D}$ is regular and hence splits L/pL as $(\overline{C} \oplus \overline{D}) \perp \overline{J}$; we know $L/pL \simeq \mathbb{H}^k \perp \langle 2\nu\rangle$ for some $\nu \in (\mathbb{Z}/p\mathbb{Z})^\times$, and hence $\overline{J} \simeq \mathbb{H}^{k-j} \perp \langle 2\nu\rangle$. Thus $K' = (C \oplus pD) \oplus pJ$, and in K'/pK' (scaled by $1/p$), $\overline{pJ} = \mathrm{rad}K'/pK'$, and $\overline{C \oplus pD} \simeq \mathbb{H}^j$ with $\overline{pD}$ totally isotropic. So $\overline{C \oplus pD} = \overline{C' \oplus pD}$ for some totally isotropic $\overline{C'}$. The number of such $\overline{C'}$ is the number of dimension j totally isotropic subspaces of $\overline{C \oplus pD}$ that are independent of $\overline{pD}$. To construct these $\overline{C'}$, choose an isotropic vector $\overline{x_1} \in \overline{C \oplus pD}$ so that $\overline{x_1} \notin \overline{pD}$; by the formula on p. 146 of [3], we have $(p^j-1)(p^{j-1}+1)-(p^j-1)$ choices. Since $\overline{C} \oplus \overline{pD} \simeq \mathbb{H}^j$ with $\overline{pD}$ totally isotropic of dimension j, $\overline{x_1}$ cannot be orthogonal to $\overline{pD}$; so we can choose $\overline{py_1} \in \overline{pD}$ so that $\overline{py_1}$ is not orthogonal to $\overline{x_1}$. Thus $\overline{x_1}, \overline{py_1}$ span a hyperbolic plane, which splits $\overline{C \oplus pD}$. So with $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$, we have $\overline{C \oplus pD} = (\mathbb{F}\overline{x_1} \oplus \mathbb{F}\overline{py_1}) \perp (\overline{C_1 \oplus pD_1})$ where $\mathbb{F}\overline{py_1} \oplus \overline{pD_1} = \overline{pD}$ and $\overline{C_1 \oplus pD_1} \simeq \mathbb{H}^{j-1}$. One then finds there are $p[(p^{j-1}-1)(p^{j-2}+1)-(p^{j-1}-1)]$ choices for isotropic $\overline{x_2} \in \overline{C \oplus pD}$ so that $\overline{x_2}$ is orthogonal to $\overline{x_1}$ and independent of $\mathbb{F}\overline{x_1} \oplus \overline{pD_1}$. Continuing, we find there are $p^{j(j-1)}\mu(j,j)$ choices for a basis of a dimension j totally isotropic subspace of $\overline{C \oplus pD}$ that is independent of $\overline{pD}$; since a space of dimension j over $\mathbb{Z}/p\mathbb{Z}$ has $p^{j(j-1)/2}\mu(j,j)$ bases, we find there are $p^{j(j-1)/2}$ choices for $\overline{C'}$.

Let pK_j be the preimage in K' of $\overline{C'}^\perp$. Thus $K_j = \frac{1}{p}C' \oplus J \oplus pD$, and K_j is integral with $\mathbb{Z}_q K_j = \mathbb{Z}_q L$ for all primes $q \neq p$; also, $\mathbb{Z}_p K_j$ is unimodular with $\mathrm{disc}K_j = \mathrm{disc}L_j$, and hence by Proposition 7.1 (c), $\mathbb{Z}_p K_j \simeq \mathbb{Z}_p L$. Therefore $K_j \in \mathrm{gen}L$.

Note that, given a p^j-neighbor K_j of L, we can construct K_j through this process by choosing $\overline{C} = \overline{L}_0 = p(K_j + L)$ and $\overline{C'}^\perp = \overline{L}_0 = pK_j$ where $L = L_0 \oplus L_1 \oplus L_2$, $K_j = \frac{1}{p}L_0 \oplus L_1 \oplus pL_2$. Hence there are $p^{j(j-1)/2}\beta\delta(k,j)$ p^j-neighbors of L.

(b) Now choose integral $\Omega = \frac{1}{p}\Omega_0 \oplus \Omega_1 \oplus p\Omega_2$ where Ω has formal rank n, $\Omega_i \subseteq L$, and $\Omega_0 \oplus \Omega_1$ is primitive in L modulo p. Note that since Ω is integral, Ω_0 is totally isotropic modulo p^2, and orthogonal to Ω_1 modulo p.

Take some K', K_j constructed as above; so $K' = C + pL = C \oplus pD \oplus pJ$ for some C, D, J with $Q(C) \equiv 0 \pmod{p}$ and $B_Q(C \oplus D, J) \equiv 0 \pmod{p}$. To get $\Omega \subseteq K_j$, we must have $p\Omega \subseteq K'$, or in other words, in L/pL we must have $\overline{\Omega_0} (= \overline{p\Omega})$ contained in $\overline{C}$. So suppose this is the case. Then to get $\Omega \subseteq K_j$, in K'/pK' (scaled by $1/p$), we must have $\overline{\Omega_0 \oplus p\Omega_1} (= \overline{p\Omega})$ contained in $\overline{C'} \perp \mathrm{rad}K'/pK'$, which requires we have $\overline{p\Omega_1} (= \overline{p\Omega \cap pL})$ contained in $\mathrm{rad}K'/pK' (\subseteq \overline{pL})$. To have $\overline{p\Omega_1} \subseteq \mathrm{rad}K'/pK'$, we need

$$B_Q(\Omega_1, K') = B_Q(\Omega_1, C + pL) \equiv 0 \pmod{p},$$

which is equivalent to $\overline{C} \subseteq \overline{\Omega_1}^\perp (= \overline{\Omega \cap L}^\perp)$ in L/pL. Thus to have $\Omega \subseteq K_j$, we need to have $\overline{\Omega_0} \subseteq \overline{C} \subseteq \overline{\Omega_1}^\perp$ in L/pL, and $\overline{\Omega_0} \subseteq \overline{C'} \perp \mathrm{rad}K'/pK'$ in K'/pK'.

So to construct K_j so that $\Omega \subseteq K_j$, we begin by decomposing L/pL as

$$L/pL = (\overline{\Omega_0 \oplus \Omega_0'}) \perp V$$

where $\overline{\Omega_0 \oplus \Omega_0'} \simeq \mathbb{H}^{r_0}$; since $L/pL \simeq \mathbb{H}^k \perp \langle 2\nu \rangle$, we have $V \simeq \mathbb{H}^{k-r_0} \perp \langle 2\nu \rangle$. Also, we can (and do) choose V so that $\overline{\Omega_1} \subseteq V$. Then in L/pL we extend $\overline{\Omega_0}$ to $\overline{C} \subseteq \overline{\Omega_1}^{\perp}$; thus we have $R^*(\overline{\Omega_1}^{\perp} \cap \overline{J}, \langle 0 \rangle^{j-r_0})$ choices for $\overline{C}$. Then in K'/pK' (scaled by $1/p$), we extend $\overline{\Omega_0} \oplus \mathrm{rad} K'/pK' (= \overline{p\Omega} + \mathrm{rad} K'/pK')$ to $\overline{C'} \oplus \mathrm{rad} K'/pK'$; thus we have $p^{(j-r_0)(j-r_0-1)/2}$ choices for $\overline{C'} \oplus \mathrm{rad} K'/pK'$.

Hence $b_j(\Omega) = p^{(j-r_0)(j-r_0-1)/2} R^*(\overline{\Omega_1}^{\perp} \cap \overline{J}, \langle 0 \rangle^{j-r_0})$. We now relate $R^*(\overline{\Omega_1}^{\perp} \cap \overline{J}, \langle 0 \rangle^{j-r_0})$ to $R^*(\overline{\Omega_1} \perp \langle 2 \rangle, \langle 0 \rangle^{\ell})$ for varying ℓ.

Say V is a vector space over $\mathbb{Z}/p\mathbb{Z}$ with $V \simeq \mathbb{H}^{k-r_0} \perp \langle 2\nu \rangle$, and U is a subspace of V with dimension r_1. In V, we can decompose U as $W \perp R$ where W is regular of some dimension $d \leq r_1$, and $R \simeq \langle 0 \rangle^{r_1-d}$. Thus

$$V = (R \oplus R') \perp V'$$

where $R \oplus R' \simeq \mathbb{H}^{r_1-d}$, $W \subseteq V'$, and $V' \simeq \mathbb{H}^{k-r_0-r_1+d} \perp \langle 2\nu \rangle$. Since W is regular, W splits V' as $V' = W \perp W'$; note that $U^{\perp} \cap V = R \perp W'$.

When $W \simeq \mathbb{H}^c$, we have $d = 2c$;

$$W' \simeq \mathbb{H}^{k-r_0-r_1+c} \perp \langle 2\nu \rangle \simeq W \perp \langle -2\nu \rangle \perp \mathbb{H}^{k-r_0-r_1},$$

and

$$W' \simeq \mathbb{H}^{k-r_0-r_1-1+c} \perp \mathbb{A} \perp \langle 2\omega\nu \rangle \simeq W \perp \langle -2\omega\nu \rangle \perp \mathbb{H}^{k-r_0-r_1-1} \perp \mathbb{A}.$$

By Lemma 8.1, we know that for any $\nu' \neq 0$,

$$R^*(\mathbb{H}^a \perp \langle 2\nu' \rangle, \langle 0 \rangle^{\ell}) = R^*(\mathbb{H}^a \perp \langle 2 \rangle, \langle 0 \rangle^{\ell})$$

and

$$R^*(\mathbb{H}^a \perp \mathbb{A} \perp \langle 2\nu' \rangle, \langle 0 \rangle^{\ell}) = R^*(\mathbb{H}^a \perp \mathbb{A} \perp \langle 2 \rangle, \langle 0 \rangle^{\ell}).$$

Consequently, with Lemma 6.2,

$$\begin{aligned}
R^*(U^{\perp} \cap V, \langle 0 \rangle^{j-r_0}) &= R^*(U \perp \langle 2 \rangle \perp \mathbb{H}^{k-r_0-r_1}, \langle 0 \rangle^{j-r_0}) \\
&= R^*(U \perp \langle 2 \rangle \perp \mathbb{H}^{k-r_0-r_1-1} \perp \mathbb{A}, \langle 0 \rangle^{j-r_0}) \\
&= \sum_{\ell=0}^{j-r_0} p^{\ell(k-r_1-j+\ell)} R^*(U \perp \langle 2 \rangle, \langle 0 \rangle^{\ell}) \\
&\qquad \cdot \beta(k-r_0-r_1, j-r_0-\ell)\delta(k-r_0-\ell, j-r_0-r) \\
&= \sum_{\ell=0}^{j-r_0} (-1)^{\ell} p^{\ell(k-r_1-j+\ell)} R^*(\overline{\Omega_1} \perp \langle 2 \rangle, \langle 0 \rangle^{\ell}) \\
&\qquad \cdot \delta(k-r_0-r_1, j-r_0-\ell)\beta(k-r_0-\ell, j-r_0-r).
\end{aligned}$$

When $W \simeq \mathbb{H}^{c-1} \perp \mathbb{A}$, the argument is the same as above.

So suppose $W \simeq \mathbb{H}^c \perp \langle 2\nu' \rangle$. Then

$$\begin{aligned} W' &\simeq \mathbb{H}^{k-r_0-r_1+c} \perp \langle -2\nu', 2\nu \rangle \\ &\simeq \mathbb{H}^{k-r_0-r_1+c-1} \perp \langle 2, -2, 2\nu', -2\nu \rangle \\ &\simeq W \perp \langle 2 \rangle \perp \langle 2, 2\nu \rangle. \end{aligned}$$

Since $\chi'(p) = \left(\frac{-\nu}{p}\right)$, $\langle 2, 2\nu \rangle \simeq \mathbb{H}$ if $\chi'(p) = 1$, and $\langle 2, 2\nu \rangle \simeq \mathbb{A}$ otherwise. Applying Lemma 8.2 now yields the result. □

Theorem 6.4 (Generalised Eichler Commutation Relation). *L is a lattice of rank $2k+1$ equipped with a positive definite quadratic form Q of level N, and $1 \le n \le 2k+1$; fix a prime p so that $p \nmid N$. Take j so that $1 \le j \le n$ and $j \le k$; for $0 \le q \le j$, set*

$$u_q(j) = (-1)^q p^{q(q-1)/2} \beta(n-j+q, q), \qquad T_j'(p^2) = \sum_{q=0}^{j} u_q(j) \widetilde{T}_{j-q}(p^2),$$

$$v_q(j) = \begin{cases} (-1)^q \beta(k-n+q-1, q) \delta(k-j+q, q) & \text{if } \chi'(p) = 1, \\ (-1)^q \delta(k-n+q-1, q) \beta(k-j+q, q) & \text{if } \chi'(p) = -1. \end{cases}$$

Then

$$\theta^{(n)} | T_j'(p^2) = \sum_{q=0}^{j} v_q(j) \left(\sum_{K_{j-q}} \theta^{(n)}(K_{j-q}; \tau) \right)$$

where K_{j-q} runs over all p^{j-q}-neighbors of L (as defined in Proposition 6.3).

Proof. Given the results of Propositions 6.1 and 6.3, we need to show that for any even integral $\Omega \subseteq \frac{1}{p} L$ with formal rank n,

$$\sum_{q=0}^{j} u_q(j) \widetilde{c}_{j-q}(\Omega) = \sum_{q=0}^{j} v_q(j) b_{j-q}(\Omega).$$

So we fix $\Omega = \frac{1}{p}\Omega_0 \oplus \Omega_1 \oplus p\Omega_2$ with $\Omega_i \in L$, $\Omega_0 \oplus \Omega_1$ primitive in L modulo p.

First suppose that $\chi'(p) = 1$. Then with $\overline{\Omega}_1 = \Omega_1 / p\Omega_1$,

$$\begin{aligned} \sum_{q=0}^{j} u_q(j) \widetilde{c}_{j-q}(\Omega) = \sum_{\ell, t} & p^{E(\ell,t,\Omega)} R^*(\overline{\Omega_1} \perp \langle 2 \rangle, \langle 0 \rangle^{\ell}) \delta(k-r_0-\ell, t) \beta(r_2, t) \\ & \cdot \sum_{q=0}^{j} u_q(j) \beta(n-r_0-\ell-t, n-j+q). \end{aligned}$$

Note that

$$\begin{aligned}\beta(m+r,r+q)\beta(r+q,q) &= \frac{\mu(m+r,r)\mu(m,q)\mu(r+q,q)}{\mu(r+q,q)\mu(r,r)\mu(q,q)}\\ &= \beta(m+r,r)\beta(m,q).\end{aligned}$$

Using that $\beta(m,q) = p^q\beta(m-1,q) + \beta(m-1,q-1)$, we find that when $m \geq 1$, $\sum_{q=0}^{m}(-1)^q p^{q(q-1)/2}\beta(m,q) = 0$. Therefore, with $m = j - r_0 - \ell - t$ and $r = n - j$,,

$$\begin{aligned}\sum_q u_q(j)\widetilde{c}_{j-q}(\Omega) = \sum_\ell & p^{E(\ell,j-r_0-\ell,\Omega)}R^*(\overline{\Omega_1} \perp \langle 2\rangle, \langle 0\rangle^\ell)\\ &\cdot \delta(k-r_0-\ell, j-r_0-\ell)\beta(r_2, j-r_0-\ell).\end{aligned}$$

Also, $E(\ell, j-r_0-\ell,\Omega) = (j-r_0)(k-n) + \ell(n-r_1+\ell-j) + (j-r_0)(j-r_0-1)/2$.

On the other hand,

$$\begin{aligned}\sum_q v_q(j)b_{j-q}(\Omega) = p^{(j-r_0)(j-r_0-1)/2}&\sum_\ell p^{\ell(k-r_1+\ell-j)}R^*(\overline{\Omega_1} \perp \langle 2\rangle, \langle 0\rangle^\ell\\ &\cdot \frac{\delta(k-r_0-\ell, j-r_0-\ell)}{\mu(j-r_0-\ell, j-r_0-\ell)}S(j-r_0-\ell)\end{aligned}$$

where

$$S(m) = \sum_{q=0}^{m}(-1)^q p^{q(q+1)/2-qm}\mu(k-r_0-r_1, m-q)\mu(k-n-1+q,q)\beta(m,q).$$

Using that $n = r_0 + r_1 + r_2$ and that $\beta(m,q) = \beta(m-1,q) + p^{m-q}\beta(m-1,q-1)$, we find that

$$S(m) = p^{k-n}(p^{r_2-m+1}-1)S(m-1) = p^{m(k-n)}\mu(r_2,m).$$

Thus

$$\sum_q v_q(j)b_{j-q}(\Omega) = \sum_q u_q(j)\widetilde{c}_{j-q}(\Omega).$$

The case when $\chi'(p) = -1$ is virtually the same, and so the details are left to the reader. □

In the next corollary, we average across the generalised Eichler Commutation Relation to show that $\theta^{(n)}(\mathrm{gen}L)$ is a Hecke eigenform for primes $p \nmid N$, where

$$\theta^{(n)}(L) = \sum_{\mathrm{cls}K\in\mathrm{gen}L}\frac{1}{o(K)}\theta^{(n)}(K)$$

(here cls$K$ varies over all isometry classes within the genus of L).

Corollary 6.5. *With p a prime, $p \nmid N$, and $1 \le j \le n$ so that also $j \le k$,*

$$\theta^{(n)}(genL)|T'_j(p^2) = \lambda_j(p^2)\theta^{(n)}(genL)$$

where

$$\lambda_j(p^2) = \begin{cases} p^{j(j-1)/2+j(k-n)}\beta(n,j)\delta(k,j) & \text{if } \chi'(p) = 1, \\ p^{j(j-1)/2+j(k-n)}\beta(n,j)\mu(k,j) & \text{if } \chi'(p) = -1. \end{cases}$$

Proof. First note that for $K \in \text{gen}L$, K is a p^m-neighbor of L if and only if $pL \subseteq K \subseteq \frac{1}{p}L$, and either $\text{mult}_{\{L:K\}}(p) = m$ or $\text{mult}_{\{L:K\}}(1/p) = m$. Classifying the p^m-neighbors into isometry classes, we see that the number of p^m-neighbors of L in $\text{cls}K \in \text{gen}L$ is

$$\frac{\#\{\text{isometries } \sigma : \ pL \subseteq \sigma K \subseteq \frac{1}{p}L, \ \text{mult}_{\{L:\sigma K\}}(p) = m \ \}}{o(K)}$$

(since $\sigma K = \sigma' K$ if and only if $\sigma^{-1}\sigma' \in O(K)$). Also, using Proposition 6.3 (a),

$$\begin{aligned} &\sum_{\text{cls}L' \in \text{gen}L} \frac{\#\{\text{isometries } \sigma : \ pL' \subseteq \sigma K \subseteq \frac{1}{p}L', \ \text{mult}_{\{L':\sigma K\}}(p) = m \ \}}{o(L')o(K)} \\ &\quad = \frac{1}{o(K)} \sum_{\text{cls}L'} \frac{\#\{\text{isometries } \sigma : \ pK \subseteq \sigma L' \subseteq \frac{1}{p}K, \ \text{mult}_{\{K:\sigma L'\}}(p) = m \ \}}{o(L')} \\ &\quad = \frac{1}{o(K)} \#\{p^m\text{-neighbors of } K\} \\ &\quad = \frac{1}{o(K)} p^{m(m-1)/2}\beta\delta(k,m). \end{aligned}$$

Thus

$$\theta^{(n)}(\text{gen}L)|T'_j(p^2) = \lambda_j(p^2)\theta^{(n)}(\text{gen}L)$$

where

$$\lambda_j(p^2) = \sum_{q=0}^{j} v_q(j) p^{(j-q)(j-q-1)/2}\beta\delta(k, j-q)\frac{\mu(j,q)}{\mu(j,q)}.$$

Our standard technique for evaluating such sums yields the result. $\square$

Theorem 6.6. *When $1 \le a \le n-k$ and p a prime not dividing N,*

$$\theta^{(n)}(L)|T'_{k+a}(p^2) = 0.$$

Proof. This is proved just as in the integral weight case (see §3 [12]); for completeness, we give a quick sketch.

First, we claim $\widetilde{c}_{k+a}(\Omega) = \sum_{q=0}^{k} w_q(a)\widetilde{c}_{k-q}(\Omega)$, and hence

$$\theta^{(n)}(L)|\widetilde{T}_{k+a}(p^2) = \theta^{(n)}(L)|\sum_{q=0}^{k} w_q(a)\widetilde{T}_{k-q}(p^2)$$

where $w_q(a) = (-1)^q p^{q(q+1)/2}\beta(a+q-1,q)\beta(n-k+q,a+q)$. To verify this, notice that in Proposition 6.1, the only term in our formula for $\widetilde{c}_j(\Omega)$ that is dependent on j is $\beta(n-k+x,n-j) = \beta(n-k+x,x-k+j)$ where $x = k-r_0-\ell-t$. Then, since

$$\begin{aligned}\beta(a+q-1,q)&\beta(n-k+q,a+q)\beta(n-k+x,x-q)\\ &= \frac{\beta(n-k+x,x+a)}{\mu(x,x)}\mu(a+q-1,q)\mu(x+a,x-q)\beta(x,q),\end{aligned}$$

we need to verify that $S_a(x,1) = \mu(x,x)$ where

$$S_a(x,y) = \sum_{q=0}^{x}(-1)^q p^{q(q-1)/2+qy}\mu(a+q-1,q)\mu(x+a-1+y,x-q)\beta(x,q).$$

Using our standard technique, we see $S_a(x,y) = (p^y-1)S_a(x-1,y+1)$, and so $S_a(x,1) = \mu(x,x)S_a(0,x+1) = \mu(x,x)$.

Next, we claim that

$$\sum_{q=0}^{r}\beta(n-q,r-q)T'_q(p^2) = \widetilde{T}_r(p^2).$$

To see this, we begin with the definition of $T'_q(p^2)$, then we make some changes of variables:

$$\begin{aligned}\sum_{q=0}^{r}&\beta(n-q,r-q)T'_q(p^2)\\ &= \sum_{q=0}^{r}\sum_{i=0}^{q}(-1)^{q-i}p^{(q-i)(q-i-1)/2}\beta(n-i,q-i)\beta(n-q,r-q)\widetilde{T}_i(p^2)\\ &= \sum_{i=0}^{r}\beta(n-i,r-i)\widetilde{T}_i(p^2)\sum_{q=0}^{r-i}(-1)^q p^{q(q-1)/2}\beta(r-i,q).\end{aligned}$$

When $r > i$, our standard technique shows this last sum is 0.

Hence, with $a \geq 1$,

$$\begin{aligned}
&\theta^{(n)}(L)|\sum_{q=0}^{k+a}\beta(n-q,k+a-q)T'_q(p^2)\\
&\quad=\theta^{(n)}(L)|\widetilde{T}_{k+a}(p^2)\\
&\quad=\theta^{(n)}(L)|\sum_{\ell=0}^{k}w_\ell(a)\widetilde{T}_{k-\ell}(p^2)\\
&\quad=\theta^{(n)}(L)|\sum_{q=0}^{k}\sum_{\ell=0}^{k-q}w_\ell(a)\beta(n-q,k-\ell-q)T'_q(p^2).
\end{aligned}$$

As shown at the beginning of this proof (replacing x by $k-q$),

$$\sum_{\ell=0}^{k-q}w_\ell(a)\beta(n-q,k-q-\ell)=\beta(n-q,k+a-q).$$

Hence induction on a shows $\theta^{(n)}(L)|T'_{k+a}(p^2)=0$ for all $a\geq 1$. $\square$

§7. Unimodular lattices over $\mathbb{Z}_p$

Recall that a regular (or nondegenerate) lattice L over $\mathbb{Z}_p$ is unimodular if $L^\# = L$, where $L^\# = \{v \in \mathbb{Q}_pL : \ B_Q(v,L) \subseteq \mathbb{Z}_p\ \}$, B_Q the symmetric bilinear form associated to the quadratic form Q on L. Here we give a brief accounting of the facts we use about unimodular lattices over $\mathbb{Z}_p$.

Proposition 7.1. *Say L is a rank m unimodular lattice over $\mathbb{Z}_p$, $p \neq 2$.*
(a) For any $\mu_1,\dots,\mu_m \in \mathbb{Z}_p^\times$ so that $discL = \mu_1\cdots\mu_m$, we have

$$L \simeq \langle \mu_1,\dots,\mu_m\rangle.$$

(b) The matrices $\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\langle 1,-1\rangle$ represent the same (isotropic) lattice. For $\omega\in\mathbb{Z}_p^\times$ with $\left(\frac{\omega}{p}\right)=-1$, the lattice represented by $\langle 1,-\omega\rangle$ is anisotropic.

(c) If $m=2k+1$, then $L\simeq \begin{pmatrix}0&1\\1&0\end{pmatrix}\perp\cdots\perp\begin{pmatrix}0&1\\1&0\end{pmatrix}\perp\langle\varepsilon\rangle$ where $(-1)^k\varepsilon =$ $discL$. If $m=2k$ then

$$L\simeq\begin{cases}\begin{pmatrix}0&1\\1&0\end{pmatrix}\perp\cdots\perp\begin{pmatrix}0&1\\1&0\end{pmatrix} & \textit{if } (-1)^k = discL,\\ \begin{pmatrix}0&1\\1&0\end{pmatrix}\perp\cdots\perp\begin{pmatrix}0&1\\1&0\end{pmatrix}\perp\langle 1,-\omega\rangle & \textit{otherwise.}\end{cases}$$

Proof. (a) As discussed at the beginning of §92 [8], L has an orthogonal base. Hence for some $\varepsilon_i \in \mathbb{Z}_p^\times$, $L \simeq \langle \varepsilon_1, \dots, \varepsilon_m \rangle$. By 92:1 [8], $L \simeq \langle 1, \dots, 1, \varepsilon \rangle$ where $\varepsilon = \varepsilon_1 \cdots \varepsilon_m$. So ε represents $\operatorname{disc} L$. If $\mu_1, \dots, \mu_m \in \mathbb{Z}_p^\times$ so that $\mu = \mu_1 \cdots \mu_m$ represents $\operatorname{disc} L$, then $L \simeq \langle 1, \dots, 1, \mu \rangle$ and again by 92:1 [8], $\langle 1, \dots, 1, \mu \rangle \simeq \langle \mu_1, \dots, \mu_m \rangle$.

(b) Since $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is unimodular with determinant -1, by (a) it must represent the same lattice as $\langle 1, -1 \rangle$. On the other hand, since ω is not a square in $\mathbb{Z}_p^\times$, $\alpha^2 - \omega\beta^2 = 0$ is not soluble, $\langle 1, -\omega \rangle$ is anisotropic.

(c) This follows from repeated use of the fact that, by (a) and (b), $\langle 1, 1, \varepsilon \rangle$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \langle -\varepsilon \rangle$ represent the same lattice. □

Similar to the discussion in §93B [8], we have the following.

Lemma 7.2. *Say $J = \mathbb{Z}_2 x \oplus \mathbb{Z}_2 y$ is unimodular. Then either $J \simeq \langle \mu_1, \mu_2 \rangle$ for some $\mu_1, \mu_2 \in \mathbb{Z}_2^\times$, or $J \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, or $J \simeq \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Also, $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ is anisotropic, and hence $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \not\simeq \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. So when J is even (meaning $Q(J) \subseteq 2\mathbb{Z}_2$), $J \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ if $\operatorname{disc} J = -1$, and $J \simeq \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ if $\operatorname{disc} J = 3$.*

Proof. Relative to the basis x, y, $J \simeq \begin{pmatrix} \nu & \varepsilon \\ \varepsilon & \eta \end{pmatrix}$ for some $\nu, \varepsilon, \eta \in \mathbb{Z}_2$. If either ν or η is a unit, then J can be diagonalized (by an appropriate change of basis).

So suppose $J \simeq \begin{pmatrix} 2\nu & \varepsilon \\ \varepsilon & 2\eta \end{pmatrix}$; since J is unimodular, $4\nu\eta - \varepsilon^2$ is a unit and hence ε is a unit. Thus by scaling one basis vector by ε^{-1}, we can assume $J \simeq \begin{pmatrix} 2\nu & 1 \\ 1 & 2\eta \end{pmatrix}$.

First, when $\nu = 2\mu$ for $\mu \in \mathbb{Z}_2$, we claim $J \simeq \mathbb{H}$. To see this, suppose $\mathbb{Z}_2 u \oplus \mathbb{Z}_2 w \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since 1 is a solution to $X^2 - X + 2\mu\eta \equiv 0 \pmod 2$, Hensel's lemma allows us to lift 1 to a solution $\alpha \in \mathbb{Z}_2^\times$ to the equation $X^2 - X + 2\mu\eta = 0$. Then with $\eta' = \eta\alpha^{-1}$,

$$\mathbb{Z}_2 u \oplus \mathbb{Z}_2 w = \mathbb{Z}_2(u + 2\mu w) \oplus \mathbb{Z}_2(\eta' u + \alpha w) \simeq \begin{pmatrix} 4\mu & 1 \\ 1 & 2\eta \end{pmatrix}.$$

Next, when $\nu, \eta \notin 2\mathbb{Z}_2$, we claim $J \simeq \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. To see this, suppose $\mathbb{Z}_2 u \oplus \mathbb{Z}_2 w \simeq \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Using Hensel's lemma, we can find $\alpha, \beta \in \mathbb{Z}_2$ so that

$$3\alpha^2 - 3\alpha + 1 = \nu,\ (4\nu^2 - \nu)\beta^2 + (1 - 4\nu)\beta + 1 = \eta.$$

Then with $u' = (1-2\alpha)u + \alpha w$, $w' = \beta u' + (1-2\beta\nu)w$, we have $\mathbb{Z}_2 u \oplus \mathbb{Z}_2 w = \mathbb{Z}_2 u' \oplus \mathbb{Z}_2 w = \mathbb{Z}_2 u' \oplus \mathbb{Z}_2 w'$, and

$$\mathbb{Z}_2 u' \oplus \mathbb{Z}_2 w \simeq \begin{pmatrix} 2\nu & 1 \\ 1 & 2 \end{pmatrix} \text{ and } \mathbb{Z}_2 u' \oplus \mathbb{Z}_2 w' \simeq \begin{pmatrix} 2\nu & 1 \\ 1 & 2\eta \end{pmatrix}.$$

Finally, we verify that $\mathbb{Z}_2 u \oplus \mathbb{Z}_2 w \simeq \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ is anisotropic. Say $\alpha, \beta \in \mathbb{Z}_2$, and without loss of generality, suppose $\mathrm{ord}_2\alpha \geq \mathrm{ord}_2\beta$. Then $Q(\alpha u + \beta w) = 2\alpha^2 + 2\alpha\beta + 2\beta^2 = 0$ if and only if $\gamma^2 + \gamma + 1 = 0$ where $\gamma = \alpha/\beta \in \mathbb{Z}_2$; however, $X^2 + X + 1 \equiv 0 \pmod 2$ has no solution. Thus $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ is anisotropic. □

Recall that for any prime p we use $\mathbb{H}$ to denote a binary lattice over $\mathbb{Z}_p$ with $\mathbb{H} \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and we call $\mathbb{H}$ a hyperbolic plane; we use $\mathbb{A}$ to denote an anisotropic binary lattice over $\mathbb{Z}_p$, and we call $\mathbb{A}$ an anisotropic plane.

Lemma 7.3. *Say L is a $\mathbb{Z}_2$-lattice with $L \simeq \mathbb{A} \perp \mathbb{A}$. Then $L \simeq \mathbb{H} \perp \mathbb{H}$.*

Proof. We first show that L is isotropic. Choosing $\alpha \in \mathbb{Z}_2^\times$ so that $5\alpha = -7$, we see that

$$\alpha \equiv -35 \equiv 1 \pmod 8.$$

Hence by the Local Square Theorem (63:1 [8]), $\alpha = \varepsilon^2$ for some $\varepsilon \in \mathbb{Z}_2^\times$. Then with x, y, x', y' a basis for L corresponding to the matrix representation $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, we see that with $w = x + 2y + \varepsilon x' - 4\varepsilon y'$, $Q(w) = 0$.

By 82:16 [8], w lies in a binary sublattice of L that splits L, meaning $L = J \perp J'$. Notice that since L is even, so are J and J'. Then since J is isotropic, $J \simeq \mathbb{H}$. Also, $J' \simeq \mathbb{H}$ or $\mathbb{A}$; since $\mathrm{disc} L = \mathrm{disc} J \cdot \mathrm{disc} J'$, by Lemma 7.2XS we have $J' \simeq \mathbb{H}$. □

Proposition 7.4. *Say L is a unimodular lattice over $\mathbb{Z}_2$. Then*

$$L \simeq \langle \mu_1, \ldots, \mu_t \rangle \perp \mathbb{H} \perp \cdots \mathbb{H} \perp A$$

where $\mu_i \in \mathbb{Z}_2^\times$ and $A = \mathbb{H}$ or $\mathbb{A}$. Hence if L is even unimodular over $\mathbb{Z}_2$ then rank$L$ is even.

Proof. By 93:15 [8] and Lemma 7.2,

$$L \simeq \langle \mu_1, \ldots, \mu_t \rangle \perp \begin{pmatrix} 2\nu_1 & 1 \\ 1 & 2\eta_1 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 2\nu_s & 1 \\ 1 & 2\eta_s \end{pmatrix}$$

where $\mu_i \in \mathbb{Z}_2^\times$, $\nu_j, \eta_j \in \mathbb{Z}_2$. The proposition now follows from Lemma 7.3. □

§8. Representation numbers of quadratic forms over $\mathbb{Z}/p\mathbb{Z}$, $p \neq 2$

Fix a prime $p \neq 2$, and set $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$. Recall that $J_s = I_{s-1} \perp \langle \omega \rangle$ where $\left(\frac{\omega}{p}\right) = -1$, and $r^*(Q,T)$ is the number of times the symmetric matrix Q primitively represents the symmetric matrix T. Also, $o(Q)$ is the order of the orthogonal group of Q.

Lemma 8.1. *Working over $\mathbb{F}$, here we present formulas for primitive representation numbers of a regular quadratic form Q, and for $o(Q)$. Since $R^*(Q,T) = r^*(Q,T)/o(T)$, this gives us formulas for $R^*(Q,T)$.*
(a) We have:

$$\begin{aligned} r^*(I_{2s},1) &= p^{s-1}(p^s - \varepsilon^s) = r^*(I_{2s},\omega),\\ r^*(J_{2s},1) &= p^{s-1}(p^s + \varepsilon^s) = r^*(J_{2s},\omega),\\ r^*(I_{2s+1},1) &= p^s(p^s + \varepsilon^s) = r^*(J_{2s+1},\omega),\\ r^*(J_{2s},1) &= p^s(p^s - \varepsilon^s) = r^*(I_{2s},\omega),\\ r^*(I_{2s},0) &= (p^s - \varepsilon^s)(p^{s-1} + \varepsilon^{s-1}),\\ r^*(J_{2s},0) &= (p^s + \varepsilon^s)(p^{s-1} - \varepsilon^{s-1}),\\ r^*(I_{2s+1},0) &= (p^s + \varepsilon^s)(p^s - \varepsilon^s) = r^*(J_{2s+1},0). \end{aligned}$$

(b) With Q,T symmetric matrices and $\det Q \neq 0$*,*

$$\begin{aligned} r^*(Q \perp \langle 1 \rangle, T \perp \langle 1 \rangle) &= r^*(Q \perp \langle 1 \rangle, 1) r^*(Q,T),\\ r^*(Q \perp \langle 1,-1 \rangle, T \perp \langle 0 \rangle) &= p^{\dim T} r^*(Q \perp \langle 1,-1 \rangle, 0) r^*(Q,T). \end{aligned}$$

(c) With $c \leq s$,

$$\begin{aligned} r^*(I_{2s+1},I_{2c}) &= r^*(I_{2s+1},J_{2c}) = r^*(J_{2s+1},I_{2c}) = r^*(J_{2s+1},J_{2c})\\ &= p^{2sc-c^2}\mu\delta(s,c),\\ o(I_{2s}) &= 2p^{s(s-1)}(p^s - \varepsilon^s)\mu\delta(s-1,s-1),\\ o(J_{2s}) &= 2p^{s(s-1)}(p^s + \varepsilon^s)\mu\delta(s-1,s-1),\\ o(I_{2s+1}) &= 2p^{s^2}\mu\delta(s,s) = o(J_{2s+1}), \quad o(I_s \perp \langle 0 \rangle^a) = p^{as}\eta(a,a)o(I_s). \end{aligned}$$

Proof. (a) These formulas are well-known, and fairly easily derived. For instance, in Theorems 2.59 and 2.60 of [5], Gerstein derives formulas for all representations of $1, \omega$, and 0 by a regular quadratic form over $\mathbb{F}$. Note that for Q such a regular quadratic form, we have $r(Q,1) = r^*(Q,1)$, $r(Q,\omega) = r^*(Q,\omega)$, and $r(Q,0) = r^*(Q,0) + 1$.

(b) Let V' be a vector space over $\mathbb{F}$ with quadratic form $Q' = Q \perp \langle 1 \rangle$. Choose $u \in V'$ so that $Q'(u) = 1$. Then $\mathbb{F}u$ is a regular subspace of V', and hence splits V' as $V \perp \mathbb{F}u$ where $V \simeq Q$. Hence

$$r^*(Q', T \perp \langle 1 \rangle) = r^*(Q',1) r^*(Q,T).$$

Now let V' be the vector space with quadratic form $Q' = Q \perp \langle 1, -1 \rangle$; since we have assumed $\det Q \neq 0$, V' is a regular space. Let $u \in V'$ so that $u \neq 0$ and $Q'(u) = 0$. Then, since V' is regular, u lies in a hyperbolic plane $\mathbb{F}u \oplus \mathbb{F}v$, which splits V' as $(\mathbb{F}u \oplus \mathbb{F}v) \perp V$ where $V \simeq Q$. Also, for $w \in V$, $Q'(w + \alpha u) = Q'(w)$ for any $\alpha \in \mathbb{F}$. Consequently

$$r^*(Q', T \perp \langle 0 \rangle) = p^{\dim T} r^*(Q', 0) r^*(Q, T).$$

(c) All but the final formula follow easily from (a) and (b). For the final formula, notice that the orthogonal group of $I_s \perp \langle 0 \rangle^a$ is

$$O(I_s \perp \langle 0 \rangle^a) = \left\{ \begin{pmatrix} G_1 & 0 \\ G_3 & G_4 \end{pmatrix} : \ G_1 \in O(I_s), \ G_3 \in \mathbb{F}^{a,s}, \ G_4 \in GL_a(\mathbb{F}) \right\}. \qquad \square$$

Recall that $r(Q, T)$ denotes the total number of times Q represents T, regardless of whether the representation is primitive.

Lemma 8.2. *For $s \geq 1$,*

$$r(\mathbb{H}^s, \langle 0 \rangle^{2s}) = \sum_{d=0}^{s} (-1)^d p^{d(d-1)+2s(s-d)} \beta(s, s-d) \delta(s-1, s-d),$$
$$r(\mathbb{H}^{s-1} \perp \mathbb{A}, \langle 0 \rangle^{2s}) = -\sum_{d=0}^{s} (-1)^d p^{d(d-1)+2s(s-d)} \beta(s-1, s-d) \delta(s, s-d).$$

Thus

$$(p^s + \varepsilon^s) r(I_{2s}, \langle 0 \rangle^{2s}) = \varepsilon^s \sum_{d=0}^{s} (-1)^d p^{d(d-1)+2s(s-d)} \beta\delta(s, s-d)(p^d + \varepsilon^d),$$
$$(p^s - \varepsilon^s) r(J_{2s}, \langle 0 \rangle^{2s}) = -\varepsilon^s \sum_{d=0}^{s} (-1)^d p^{d(d-1)+2s(s-d)} \beta\delta(s, s-d)(p^d - \varepsilon^d).$$

Proof. Take a vector space V over $\mathbb{F}$ so that $V \simeq \mathbb{H}^s$ or $V \simeq \mathbb{H}^{s-1} \perp \mathbb{A}$. Note that a totally isotropic subspace of V has dimension at most s.

Set

$$\psi(V) = \sum_{d=0}^{s} \alpha_d p^{2s(s-d)} R^*(V, \langle 0 \rangle^{s-d})$$

where

$$\alpha_d = \begin{cases} (-1)^d p^{d(d-1)} & \text{if } V \simeq \mathbb{H}^s, \\ (-1)^{d+1} p^{d(d-1)} & \text{if } V \simeq \mathbb{H}^{s-1} \perp \mathbb{A}. \end{cases}$$

Since there are $p^{2s(s-d)}$ ways to choose $x_1,\dots,x_t\in U$ where $\dim U = s-d$,

$$\psi(V) = \sum_{s=0}^{d} \sum_{\substack{U\subseteq V\\ U\simeq\langle 0\rangle^{s-d}}} \alpha_d\cdot\#\{x_1,\dots,x_{2s}\in U\}$$
$$= \sum_{\substack{x_1,\dots,x_{2s}\in V\\ B_Q(x_i,x_j)=0}} \sum_{d=0}^{s}\alpha_d\cdot\#\{U\subseteq V:\ U\simeq\langle 0\rangle^{s-d},\ x_1,\dots,x_{2s}\in U\ \}.$$

Take $x_1,\dots,x_{2s}\in V$ so that $B_Q(x_i,x_j)=0$; let W be the subspace generated by $x_1,\dots,x_{2s}$, and ℓ the dimension of W. So there exists a subspace $W'\subseteq V$ so that $W\oplus W'\simeq\mathbb{H}^\ell$ and $V=(W\oplus W')\perp V'$ where

$$V'\simeq\begin{cases}\mathbb{H}^{s-\ell} & \text{if } V\simeq\mathbb{H}^s,\\ \mathbb{H}^{s-\ell-1}\perp\mathbb{A} & \text{if } V\simeq\mathbb{H}^{s-1}\perp\mathbb{A}.\end{cases}$$

Also, with $x_1,\dots,x_{2s}$ as above, and using Lemma 8.1,

$$\begin{aligned}&\#\{U\subseteq V:\ U\simeq\langle 0\rangle^{s-d},\ x_1,\dots,x_{2s}\in U\ \}\\ &\quad= R^*(V',\langle 0\rangle^{s-d-\ell})\\ &\quad=\begin{cases}\beta(s-\ell,s-\ell-d)\delta(s-\ell-1,s-\ell-d) & \text{if } V\simeq\mathbb{H}^s,\\ \beta(s-\ell-1,s-\ell-d)\delta(s-\ell,s-\ell-d) & \text{if } V\simeq\mathbb{H}^{s-1}\perp\mathbb{A}.\end{cases}\end{aligned}$$

Lemma 4.2 (b) tells us that

$$\sum_{d=0}^{s}\alpha_d R^*(V',\langle 0\rangle^{s-\ell-d}) = \sum_{d=0}^{s-\ell}\alpha_d R^*(V',\langle 0\rangle^{s-\ell-d}) = 1.$$

Therefore

$$\psi(V) = \#\{x_1,\dots,x_{2s}\in V:\ B_Q(x_i,x_j)=0\ \} = r(V,\langle 0\rangle^{t}).$$

The other statements of the lemma follow immediately from the observation that $I_{2s}\simeq\mathbb{H}^s$ if $\varepsilon^s=1$ and $I_{2s}\simeq\mathbb{H}^{s-1}\perp\mathbb{A}$ otherwise, and $J_{2s}\simeq\mathbb{H}^s$ if $\varepsilon^s=-1$ and $J_{2s}\simeq\mathbb{H}^{s-1}\perp\mathbb{A}$ otherwise. □

Remark. The technique used to prove Lemma 8.2 can also be used to show that with $\nu\neq 0$,

$$r(\mathbb{H}^s\perp\langle\nu\rangle,\langle 0\rangle^{2s+1}) = \sum_{d=0}^{s}(-1)^d p^{d^2+(2s+1)(s-d)}\beta\delta(s,s-d).$$

§9. Lemmas on symmetric matrices

Lemma 9.1. *Say p is a prime and $U,V \in \mathbb{Z}^{n,n}$, $V = diag\{p^{a_1}I_{r_1},\dots,p^{a_\ell}I_{r_\ell}\}$, $a_i < a_{i+1}$, and $({}^tU,V)$ is a coprime symmetric pair. There there is a matrix $Y \in SL_n(\mathbb{Z})$ so that $VY = Y'V$, $Y' \in SL_n(\mathbb{Z})$, and*

$$YUV^{-1}\,{}^tY \equiv diag\{p^{-a_1}U'_1,\dots,p^{-a_\ell}U'_\ell\}\,(mod\ \mathbb{Z})$$

where $U'_i \in \mathbb{Z}^{r_i,r_i}$ is symmetric with $p \nmid \det U'_i$ unless $i=1$ and $a_i = 0$. Further, there is a matrix $Y_p \in SL_n(\mathbb{Z}_p)$ so that $VY_p = Y'_pV$, $Y'_p \in SL_n(\mathbb{Z}_p)$, and

$$Y_pUV^{-1}\,{}^tY_p = diag\{p^{-a_1}U''_1,\dots,p^{-a_\ell}U''_\ell\}$$

where $U''_i \in \mathbb{Z}_p^{r_i,r_i}$ is symmetric and $U'_i \equiv U''_i$ $(mod\ p^{a_i}\mathbb{Z}_p)$.

Proof. We know UV^{-1} is symmetric, so writing $U = (U_{ij})$ where U_{ij} is an $r_i \times r_j$ block, we have ${}^tU_{ij} = p^{a_j-a_i}U_{ji}$ when $i<j$. Also, we know that $\begin{pmatrix}U\\V\end{pmatrix}$ has rank n over $\mathbb{Z}$, hence over $\mathbb{Z}_p$, U_{ii} must have rank r_i for $1 < i \le \ell$, and if $a_1 > 0$, U_{11} has rank r_1. So for $i<j$, choose integral Y_{ij} so that ${}^tU_{ji} \equiv -Y_{ij}U_{jj}\,(\text{mod } p^{a_j-a_i})$. Let

$$Y = \begin{pmatrix} I_{r_1} & p^{a_2-a_1}Y_{12} & \cdots & p^{a_\ell-a_1}Y_{1\ell} \\ & I_{r_2} & \cdots & p^{a_\ell-a_2}Y_{2\ell} \\ & & \ddots & \cdots \\ & & & I_{r_\ell} \end{pmatrix}.$$

Then we have

$$YWUV^{-1}W\,{}^tY \equiv \begin{pmatrix} p^{-a_1}U'_{11} & & & \\ & p^{-a_2}U'_{22} & & \\ & & \ddots & \\ & & & p^{-a_\ell}U'_{\ell\ell} \end{pmatrix} (\text{mod } \mathbb{Z})$$

where U'_{ii} is symmetric and $U'_{ii} \equiv U_{ii}\,(\text{mod } p)$. Also, $VY = Y'V$ where

$$Y' = \begin{pmatrix} I_{r_1} & Y_{12} & \cdots & Y_{1\ell} \\ & I_{r_2} & \cdots & Y_{2\ell} \\ & & \ddots & \cdots \\ & & & I_{r_\ell} \end{pmatrix} \in GL_n(\mathbb{Z}).$$

To find Y_p, we simply modify the above construction to choose $Y_{ij} = -{}^tU_{ij}U_{jj}^{-1}$. $\square$

Lemma 9.2. *Suppose* $({}^tB, {}^tD)$ *is a symmetric coprime pair of* $n\times n$ *integral matrices with* $\det D \neq 0$. *Suppose also that* $Q \in \mathbb{Z}^{m,n}$ *is symmetric with* $\det Q \neq 0$ *and* $(\det Q, \det D) = 1$.

(a) If $G \in \mathbb{Z}^{m,n}$ *so that* $GB \in Q^{-1}\mathbb{Z}^{m,n}D$ *then* $G \in \mathbb{Z}^{m,n}\,{}^tD$.

(b) As G_0 *varies over* $\mathbb{Z}^{m,n}/\mathbb{Z}^{m,n}\,{}^tD$, QG_0B *varies over* $\mathbb{Z}^{m,n}/\mathbb{Z}^{m,n}D$*; also,* $Q\mathbb{Z}^{m,n}B + \mathbb{Z}^{m,n}D = \mathbb{Z}^{m,n}$.

Proof. (a) Say $G \in \mathbb{Z}^{m,n}$ with $GB \in Q^{-1}\mathbb{Z}^{m,n}D$; we show that $G \in \mathbb{Z}_p^{m,n}\,{}^tD$ for all primes p. If $p \nmid \det D$ then this is clear; so say $p|\det D$. Thus $p \nmid \det Q$ and so $GB \in \mathbb{Z}_p^{m,n}D$. If $p \nmid \det B$ then

$$G \in \mathbb{Z}_p^{m,n}DB^{-1} = \mathbb{Z}_p^{m,n}\,{}^tB^{-1}\,{}^tD = \mathbb{Z}_p^{m,n}\,{}^tD.$$

So suppose $p|\det B$. By the Elementary Divisor Theorem there are $E_1, E_2 \in GL_n(\mathbb{Z}_p)$ so that $V = {}^tE_2^{-1}DE_1 = \mathrm{diag}\{p^{a_1}I_{r_1}, \dots, p^{a_\ell}I_{r_\ell}\}$. Set $U = E_2BE_1^{-1}$. Then by the proof of Lemma 9.1 we know there is some $Y \in GL_n(\mathbb{Z}_p)$ so that

$$YE_2BD^{-1}\,{}^tE_2 = \begin{pmatrix} U_{11} & 0 & \cdots & 0 \\ U_{21} & U_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ U_{\ell 1} & U_{\ell 2} & \cdots & U_{\ell\ell} \end{pmatrix} V^{-1}$$

where $U_{ij} \in \mathbb{Z}_p^{r_i,r_j}$ and $p \nmid \det U_{ii}$ for $i > 1$. Write GE_2^{-1} as (G_{ij}) where G_{ij} is $r_i \times r_j$; set

$$X = \begin{pmatrix} I & {}^tU_{21} & \cdots & {}^tU_{\ell 1} \\ 0 & {}^tU_{22} & \cdots & {}^tU_{\ell 2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & {}^tU_{\ell\ell} \end{pmatrix}, \qquad G_1 = \begin{pmatrix} G_{11} & 0 & \cdots & 0 \\ G_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G_{\ell 1} & 0 & \cdots & 0 \end{pmatrix},$$

and $G_2 = GE_2^{-1} - G_1$. Thus, recalling that BD^{-1} is symmetric, we get

$$GBD^{-1}\,{}^tE_2\,{}^tY = G_1\begin{pmatrix} {}^tU_{11} & \\ & 0 \end{pmatrix} + G_2V^{-1}X.$$

We have $GBD^{-1}, G_1\begin{pmatrix} {}^tU_{11} & \\ & 0 \end{pmatrix} \in \mathbb{Z}_p^{m,n}$, so $G_2V^{-1}X \in \mathbb{Z}_p^{m,n}$. Since we chose $X \in GL_n(\mathbb{Z}_p)$, we have $G_2V^{-1} \in \mathbb{Z}_p^{m,n}$. Noting that $G_1V^{-1} = G_1$, we see $GE_2^{-1}V^{-1} = G_1V^{-1} + G_2V^{-1} \in \mathbb{Z}_p^{m,n}$. So $G\,{}^tD = GE_2^{-1}V^{-1}\,{}^tE_1 \in \mathbb{Z}_p^{m,n}$.

(b) We show that as G_0 varies over $\mathbb{Z}^{m,n}/\mathbb{Z}^{m,n}\,{}^tD$, the product QG_0B varies over $\mathbb{Z}^{m,n}/\mathbb{Z}^{m,n}D$, and thus for any $G \in \mathbb{Z}^{m,n}$, there exist $G_0, G_1 \in \mathbb{Z}^{m,n}$ so that $G = QG_0B + G_1D$.

If $G_0 \in \mathbb{Z}^{m,n}\,{}^tD$, then $G_0 = G + 0'\,{}^tD$ and $QG_0B = QG_0'\,{}^tBD \in \mathbb{Z}^{m,n}D$. So suppose $G_0 \in \mathbb{Z}^{m,n}$ and $QG_0B \in \mathbb{Z}_{m,n}D$. Then $G_0B \in Q^{-1}\mathbb{Z}^{m,n}D$ and so by (a), $G_0 \in \mathbb{Z}^{m,n}\,{}^tD$. Thus for $G_0 \in \mathbb{Z}^{m,n}$, $G_0 \in \mathbb{Z}^{m,n}\,{}^tD$ if and only if $QG_0B \in \mathbb{Z}^{m,n}D$. □

Lemma 9.3. *Suppose $Q \in \mathbb{Z}^{m,m}$ is symmetric and even, $V = diag\{d_1, \ldots, d_n\}$ ($d_i \in \mathbb{Z}$, $d_i \neq 0$), and suppose $U \in \mathbb{Z}^{n,n}$ so that VU is symmetric. For each prime p dividing $\det V$, let $W_p = diag\{d_1 p^{-e_1}, \ldots, d_n p^{-e_n}\}$ where $e_i = ord_p(d_i)$. Then with $p_1, \ldots, p_s$ the primes dividing $\det V$,*

$$\mathbb{Z}^{m,n}W_{p_1}/\mathbb{Z}^{m,n}V \times \cdots \times \mathbb{Z}^{m,n}W_{p_s}/\mathbb{Z}^{m,n}V \approx \mathbb{Z}^{m,n}/\mathbb{Z}^{m,n}V$$

and

$$\sum_{G \in \mathbb{Z}^{m,n}/\mathbb{Z}^{m,n}V} e\{Q[G]UV^{-1}\} = \prod_{i=1}^{s} \left(\sum_{G \in \mathbb{Z}^{m,n}W_{p_i}/\mathbb{Z}^{m,n}V} e\{Q[G]UV^{-1}\} \right).$$

Proof. Map $(G_1W_{p_1}, \ldots, G_sW_{p_s})$ to $(\sum_i G_iW_{p_i}) + \mathbb{Z}^{m,n}V$. One easily verifies this is a homomorphism with kernel $(\mathbb{Z}^{m,n}V, \ldots, \mathbb{Z}^{m,n}V)$; a cardinality argument establishes the map is surjective.

Note that since Q is even, $Q[G] \in 2\mathbb{Z}^{n,n}$ for any $G \in \mathbb{Z}^{m,n}$. Then using that $\mathrm{e}\{{}^tM\} = \mathrm{e}\{M\}$ and $\mathrm{e}\{MM'\} = \mathrm{e}\{M'M\}$, we find that

$$\begin{aligned} &\mathrm{e}\{Q[G_1W_{p_1} + \cdots + G_sW_{p_s}]UV^{-1}\} \\ &\quad = \left(\prod_i \mathrm{e}\{Q[G_iW_{p_i}]UV^{-1}\} \right) \left(\prod_{i \neq j} \mathrm{e}\{{}^tG_iQG_jW_{p_j}UV^{-1}W_{p_i}\} \right), \end{aligned}$$

and $V^{-1}W_{p_i}$ is integral modulo $\mathbb{Z}_{(q)}$ for all primes $q \neq p_i$. But we know UV^{-1} is symmetric, so

$$\mathrm{e}\{{}^tG_iQG_jW_{p_j}UV^{-1}W_{p_i}\} = \mathrm{e}\{{}^tG_iQG_jW_{p_j}V^{-1}\,{}^tUW_{p_i}\},$$

and $W_{p_j}V^{-1}$ is integral modulo $\mathbb{Z}_{(p_i)}$. Thus $W_{p_j}UV^{-1}W_{p_i}$ is integral. Also,

$$\begin{aligned} \mathrm{e}\{{}^tG_iQG_jW_{p_j}UV^{-1}W_{p_i}\} &= \mathrm{e}\{{}^tW_{p_i}V^{-1}\,{}^tUW_{p_j}\,{}^tG_jQG_i\} \\ &= \mathrm{e}\{{}^tG_jQG_iW_{p_i}UV^{-1}W_{p_j}\}. \end{aligned}$$

Hence

$$\prod_{i \neq j} \mathrm{e}\{{}^tG_iQG_jW_{p_j}UV^{-1}W_{p_i}\} = \prod_{i<j} \mathrm{e}\{2\,{}^tG_iQG_jW_{p_j}UV^{-1}W_{p_i}\} = 1,$$

proving the lemma. □

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