UNCOUNTABILITY

Recall: we say that a set X is **countable** if there is a bijective function $f : \mathbb{N} \to X$.

We say that a set X is **UNcountable** if there is an injective function $f : \mathbb{N} \to X$, but there is no **bijective** function $f : \mathbb{N} \to X$.

We have seen that $\mathbb{N} \times \mathbb{N}$ is countable.

Are all infinite sets countable, meaning the same "size" as $\mathbb{N}$?
No!

We can show that there are more **real** numbers in the interval between 0 and 1 than there are positive integers.

To do this, we argue by **contradiction**: We assume there **are** as many positive integers as there are real numbers between 0 and 1, and we then derive a **contradiction**.

Technical fact: Each real number has a **unique** decimal expansion, **provided** we not use any decimal expansion that ends in all 9's.

Using geometric progressions, we can show that $0.99999\cdots = 1$, *so for instance,* $0.356799999\ldots = 0.35680000\ldots$

Suppose that the real numbers between 0 and 1 are countable, meaning that they are in one-to-one correspondence with the positive integers. Then we can enumerate these real numbers as

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \ldots$$

Each of these has a decimal expansion. For instance,

$$\alpha_1 = 0.a_{1,1} a_{1,2} a_{1,3} a_{1,4} a_{1,5} \cdots$$

where each $a_{1,j}$ is a digit in the decimal expansion of α_1.

(So $a_{1,j} = 0, 1, 2, 3, 4, 5, 6, 7, 8,$ or 9.)

We create a grid of these numbers with their decimal expansions:

$$
\begin{aligned}
\alpha_1 &= 0.a_{1,1}a_{1,2}a_{1,3}a_{1,4}a_{1,5}\cdots \\
\alpha_2 &= 0.a_{2,1}a_{2,2}a_{2,3}a_{2,4}a_{2,5}\cdots \\
\alpha_3 &= 0.a_{3,1}a_{3,2}a_{3,3}a_{3,4}a_{3,5}\cdots \\
\alpha_4 &= 0.a_{4,1}a_{4,2}a_{4,3}a_{4,4}a_{4,5}\cdots \\
\alpha_5 &= 0.a_{5,1}a_{5,2}a_{5,3}a_{5,4}a_{5,5}\cdots \\
&\vdots
\end{aligned}
$$

Now we construct another real number β between 0 and 1 that is **not** in the list $\alpha_1, \alpha_2, \alpha_3, \cdots$:

We look at the "diagonal" of our grid of decimal expansions:

$$\alpha_1 = 0.a_{1,1}a_{1,2}a_{1,3}a_{1,4}a_{1,5}\cdots$$
$$\alpha_2 = 0.a_{2,1}a_{2,2}a_{2,3}a_{2,4}a_{2,5}\cdots$$
$$\alpha_3 = 0.a_{3,1}a_{3,2}a_{3,3}a_{3,4}a_{3,5}\cdots$$
$$\alpha_4 = 0.a_{4,1}a_{4,2}a_{4,3}a_{4,4}a_{4,5}\cdots$$
$$\alpha_5 = 0.a_{5,1}a_{5,2}a_{5,3}a_{5,4}a_{5,5}\cdots$$
$$\vdots$$

We set $\beta = 0.b_1b_2b_3b_4b_5\cdots$ where

$$b_i = \begin{cases} 1 & \text{if } a_{i,i} \neq 1, \\ 2 & \text{if } a_{i,i} = 1. \end{cases}$$

We know that β is not equal to any α_i since the ith digit in the decimal expansion of β is not equal to the ith digit in the decimal expansion of α_i.

This is our **contradiction**:

We *assumed* that the set of real numbers between 0 and 1 was countable, meaning that we could enumerate all of them as $\alpha_1, \alpha_2, \alpha_3, \cdots$, and then we *constructed* a real number β between 0 and 1 that is not in this list.

Hence it **cannot** be the case that the set of real numbers between 0 and 1 is countable.
Thus the magnitude of the set of real numbers between 0 and 1 is **larger** than that of the positive integers.

Mathematicians have proved that there is a one-to-one correspondence between **all** real numbers, and those between 0 and 1 (we will prove this). (So in some sense, the magnitude of the unit interval is the same as the magnitude of the real line.)

Mathematicians have also proved that there are magnitudes of infinity **larger** than that of the real line (we will prove this).

Unknown: Is there a magnitude of infinity **between** that of the positive integers and that of the real numbers?