Calculus 1 Workshops

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These workshops were designed to be done by the students in class. So one day each week, I use one lecture period to have them work on a workshop; I allow them to work alone or in small groups. My job is to answer their questions (and then grade their work). I have used the workshops both after delivering an introductory lecture on the topic of the workshop, and as an introduction to the topic. Both methods worked well.

Workshop 1

1. Let a and b denote fixed positive numbers. We want to deduce the Pythagorean Theorem, which states that $a^2 + b^2 = c^2$ where a, b are the lengths of the two legs of a right triangle and c is the length of the hypotenuse.
   (a) Draw (x, y)-axes, and draw the square with vertices $(0, 0)$, $(a + b, 0)$, $(0, a + b)$, and $(a + b, a + b)$. Compute the area of this square.
   (b) In your drawing, draw the line segments connecting $(0, b)$ to $(a, 0)$, $(a, 0)$ to $(a + b, a)$, $(a + b, a)$ to $(b, a + b)$, and $(b, a + b)$ to $(0, b)$. Compute the lengths and the slopes of these line segments. Now explain why these segments form a square. (Why are the interior angles of this polygon **right** angles? Recall that the interior angles of a triangle sum to 180 degrees. Which angles in your picture are identical?)
   (c) Let c denote the length of each side of this new square. Compute the area of the large square by adding together the area of the small square and the areas of the surrounding triangles.
   (d) Briefly explain why this proves the Pythagorean Theorem.

Recall: Given a right triangle with another interior angle of α, we define $\cos(\alpha)$ to be the length of the leg adjacent to angle α divided by the length of the hypotenuse, and $\sin(\alpha)$ the length of the leg opposite angle α divided by the length of the hypotenuse.

2. Draw a right triangle with hypotenuse of length 1 and one interior angle of $45°$.
   (a) What are the measurements of all the interior angles of this triangle? How are the lengths of the legs of this triangle related?
   (b) According to the Pythagorean Theorem, how are the three sides of the triangle related?
   (c) Determine the lengths of the legs of the triangle, and then determine the values of $\cos(45°)$ and $\sin(45°)$.

3. (a) In the (x, y)-plane, draw the triangle with vertices $(0, 0)$, $(c, 0)$, $(c, 1/2)$. Suppose the hypotenuse of this triangle is 1. According to the Pythagorean Theorem, what is c?
   (b) Now include in your picture a reflection of this triangle across the x-axis. What are the vertices of this reflection?
   (c) The two triangles together form one larger triangle. What are the lengths of the sides of this triangle? What are the interior angles of this triangle?
   (d) Use (c) to determine the interior angles of the triangle you drew in (a).
   (e) Evaluate $\sin(30°)$, $\cos(30°)$, $\sin(60°)$, and $\cos(60°)$.

Workshop 2

The prototypical example of perpendicular lines are the x- and y-axes. The x-axis has slope 0, and the y-axis has undefined (or, according to some, infinite) slope. We now consider other situations. We want to **deduce** that a line of slope m is perpendicular to a line of slope $-1/m$, so do not **assume** this fact.

1. Let L_1 denote the line of slope 5 through the point $(0,0)$. Let L_2 denote the line of slope $-\dfrac{1}{5}$ through the point $(0,0)$.
   (a) Present formulas for the lines L_1 and L_2.
   (b) First draw the x- and y-axes, then carefully sketch in the lines L_1 and L_2. (So you should obtain one picture containing all these things.) **Label things!**
   (c) Choose two points A and B on the line L_1 that are equidistant from $(0,0)$. (**Suggestion:** Consider $x = 1$ and $x = -1$.) **Verify** that these two points are equidistant from $(0,0)$ (that is, verify that the distance from $(0,0)$ to A is the same as the distance from $(0,0)$ to B).
   (d) Choose a point C on the line L_2 so that $C \neq (0,0)$. (**Suggestion:** To ease the upcoming algebra, you may want to let C have an x-coordinate that is an integer multiple of 5, like 5 or 10 or 15 *etc.*) Compute the distance from C to A, and compute the distance from C to B.
   (e) Argue that the triangle with vertices $(0,0)$, A and C is identical to the triangle with vertices $(0,0)$, B and C.
   (f) Argue that the lines L_1 and L_2 are perpendicular. (Look at your triangles! Which angles are identical to each other? Which angles together form a straight line?)

2. Let L_1 denote the line of slope m ($m \neq 0$) through the point $(0,0)$. Let L_2 denote the line of slope $-\dfrac{1}{m}$ through the point $(0,0)$. **Note:** This part of the Workshop is to generalize the first part, replacing the slope 5 by any slope m ($m \neq 0$). So do **not** replace m by a specific number.
   (a) Present formulas for the lines L_1 and L_2.
   (b) First draw the x- and y-axes, then sketch in the lines L_1 and L_2. (So you should obtain one picture containing all these things.) **Label things!**
   (c) Choose two points A and B on the line L_1 that are equidistant from $(0,0)$. (**Suggestion:** Consider $x = 1$ and $x = -1$.) **Verify** that these two points are equidistant from $(0,0)$ (that is, verify that the distance from $(0,0)$ to A is the same as the distance from $(0,0)$ to B).

(d) Choose a point C on the line L_2 so that $C \neq (0,0)$. (**Suggestion:** To ease the upcoming algebra, you may want to let C have an x-coordinate that is an integer multiple of m, like m or $2m$ or $3m$ *etc.*) Compute the distance from C to A, and compute the distance from C to B.

(e) Argue that the triangle with vertices $(0,0)$, A and C is identical to the triangle with vertices $(0,0)$, B and C.

(f) Argue that the lines L_1 and L_2 are perpendicular. (Look at your triangles! Which angles are identical to each other? Which angles together form a straight line?)

Workshop 3

1. (a) What is $\lim_{x\to\infty} 2^x$? What is $\lim_{x\to-\infty} 2^x$?
   (**Note:** $\lim_{x\to-\infty} 2^x = \lim_{x\to\infty} 2^{-x} = \frac{1}{\lim_{x\to\infty} 2^x}$.)

   (b) Use the definition of the derivative and rules of exponents to show $\frac{d}{dx}2^x$, the derivative of 2^x, is

   $$2^x \cdot \lim_{h\to 0} \frac{2^h - 1}{h}.$$

   (c) What is 2^0? When $h > 0$, is 2^h larger or smaller than 1? When $h < 0$, is 2^h larger or smaller than 1? Is 2^x ever negative? (Write your answers in complete sentences.)

   (d) **Fact:** $\lim_{h\to 0} \frac{2^h-1}{h} > 0$. Use this fact and (b) and (c) to briefly explain why $\frac{d}{dx}2^x$ is always positive.

   (e) Assuming that $y = 2^x$ is a continuous function, use the information from (a)–(d) to sketch a rough graph of $y = 2^x$.

Notation: Recall that f' denotes the derivative of f; we also write $\frac{d}{dx}f$ for f'.

Throughout, let f and g be functions. Recall that when $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist,

$$\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) \ + \ \lim_{x\to a} g(x),$$

$$\lim_{x\to a}(f-g)(x) = \lim_{x\to a} f(x) \ - \ \lim_{x\to a} g(x),$$

$$\lim_{x\to a}(f\cdot g)(x) = \left(\lim_{x\to a} f(x)\right)\cdot\left(\lim_{x\to a} g(x)\right),$$

and provided $\lim_{x\to a} g(x) \neq 0$, $\lim_{x\to a}\left(\frac{f}{g}\right)(x) = (\lim_{x\to a} f(x)) / (\lim_{x\to a} g(x))$. Also, when c is a constant, $\lim_{x\to a}(cf)(x) = c\cdot\lim_{x\to a} f(x)$. (Here cf denotes the function given by $(cf)(x) = c\cdot f(x)$.)

2. Recall that we define the function $f+g$ by $(f+g)(x) = f(x)+g(x)$.

   (a) Using the definition of derivative, explain why

   $$(f+g)'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}.$$

   (b) Use the definition of $f'(x)$ and $g'(x)$ together with (a) to explain why $(f+g)'(x) = f'(x) + g'(x)$.

3. Recall that $\left(\frac{f}{g}\right)(x) = f(x)/g(x)$.

(a) Use the definition of the derivative, the definition of $(f/g)(x)$, and algebra to show

$$\left(\frac{f}{g}\right)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)h}.$$

(b) Now explain why

$$\left(\frac{f}{g}\right)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x)}{g(x)g(x+h)h} - \lim_{h\to 0} \frac{f(x)g(x+h) - f(x)g(x)}{g(x)g(x+h)h}.$$

(c) Now use (c) to explain why

$$\left(\frac{f}{g}\right)'(x) = f'(x) \cdot \lim_{h\to 0} \frac{g(x)}{g(x)g(x+h)} - g'(x) \cdot \lim_{h\to 0} \frac{f(x)}{g(x)g(x+h)}.$$

(d) Assuming that $\lim_{h\to 0} g(x+h) = g(x)$, use (c) to conclude that

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

Note: This called the Quotient Rule.

Workshop 4

1. Our friend Pat is driving a car along a straight flat road. Sensing danger ahead, Pat applies the brake to stop. From the instant the brake is applied (at time $t = 0$) until the instant the car stops, the car's position is given by

$$s(t) = 10t - t^2.$$

(Distance is measured from the spot where Pat applies the brake, and distance is measured in feet. Time is measured in seconds.)

   (a) Explain why $s'(t)$ describes the car's velocity. (Remember: $s'(t)$ measures for us the instantaneous rate of change of $s(t)$ relative to change in t.)

   (b) How long will it take the car to stop completely? (What is the velocity when the car stops? For which value of t does this occur?)

   (c) The danger that Pat sensed is a brick wall at the end of the road. When Pat applied the brake, the front of Pat's car was 28 feet from the wall. Will Pat's car stop before it hits the wall? (How long until Pat's car stops? What is the position of Pat's car at that time?)

2. A ball is thrown upward at time $t = 0$. Its height at time t (measured in seconds) is given by

$$f(t) = -16t^2 + 160t$$

(where distance is measured in feet). When $f'(t) > 0$, is the ball moving upward or downward? When $f'(t) < 0$, is the ball moving upward or downward? When does the ball reach its maximum height? Knowing how long it takes for the ball to reach its maximum height, compute its maximum height. (Remember what $f(t)$ tells you!)

3. The position function $x = f(t)$ of a particle moving in a horizontal straight line is given by

$$x = 100 - 20t - 5t^2.$$

Find its location x when its velocity is zero.

Workshop 5

proportional: having the same or a constant ratio

Throughout, $0 < \alpha < \pi/2$.

1. Consider the circle with center $(0,0)$ and radius 1. Let (u,v) be a point on the circle with $u > 0$ and $v > 0$. (So (u,v) is in the first quadrant.) Let α be the angle intercepting the arc from $(1,0)$ (counterclockwise) to (u,v) (where angles are measured in radians).
   (a) Draw a picture of this situation. Label things!
   (b) Suppose $\alpha = \pi/4$ radians. What are u and v? What proportion of the entire circle is the sector with angle α? What is the area of the sector?
   (c) Suppose $\alpha = \pi/6$ radians. What are u and v? What proportion of the entire circle is the sector with angle α? What is the area of the sector?
   (d) Whatever the value of α, how is α related to u and v? (Label u and v as such in your picture.) What proportion of the entire circle is the sector with angle α? (How many radians are in the circle? How many radians are in this sector?) What is the area of the sector? (What is the area of the entire circle? What proportion of this area is within the sector?) **For later convenience, let A_1 denote the area of this sector with radius 1 and angle α.** (So presumably you have just worked out a formula for A_1 in terms of α.)
2. Now consider the triangle with vertices $(0,0)$, $(u,0)$ and (u,v).
   (a) Supplement your drawing in 1(a) to include this triangle.
   (b) Say $\alpha = \pi/4$. What is the area of the triangle?
   (c) Say $\alpha = \pi/6$. What is the area of the triangle?
   (d) Say α is some angle between 0 and $\pi/2$ radians. In terms of α, what is the area of the triangle? (In terms of α, what are u and v?) **For later convenience, let A_2 denote the area of this right triangle with hypotenuse 1 and interior angle α.**
3. Now consider the circle with center $(0,0)$ and radius u.
   (a) Supplement your drawing in 1(a) to include this circle.
   (b) Suppose $\alpha = \pi/4$ radians. What proportion of the entire circle of radius u is the sector with angle α? What is the area of the sector? (Recall that in 1(b) you computed the value of u. What is the area of the circle with radius u? What proportion of this area is in the sector?)
   (c) Suppose $\alpha = \pi/6$ radians. What proportion of the entire circle is the sector with angle α? What is the area of the sector?

(d) Whatever the value of α, what proportion of the entire circle is the sector with angle α? What is the area of the sector? **For later convenience, let A_3 denote the area of this sector with radius u and angle α.**

4. Explain why $A_3 \leq A_2 \leq A_1$.

5. Multiply your inequalities from #4 by $\dfrac{2}{\alpha \cos(\alpha)}$ to obtain

$$\cos(\alpha) \leq \frac{\sin(\alpha)}{\alpha} \leq \frac{1}{\cos(\alpha)}.$$

(Is $\cos(\alpha)$ positive? How do you know?)

6. (a) Evaluate

$$\lim_{\alpha \to 0^+} \cos(\alpha) \quad \text{and} \quad \lim_{\alpha \to 0^+} \frac{1}{\cos(\alpha)}.$$

(b) Use the inequality from #5 and the limits from #6 (a) to explain why we must have

$$\lim_{\alpha \to 0^+} \frac{\sin(\alpha)}{\alpha} = 1.$$

Workshop 6

Note: We only identify f as continuous or discontinuous at those x in domain f; so f can be continuous even when its domain consists of several connected components.

1. Define the function f by

$$f(x) = \frac{x^2+1}{x^2-5}.$$

Notice that f is continuous (that is, f is continuous at every x-value in its domain). Through the following steps we create the graph of f.

(a) Determine the domain of f and briefly explain why f is continuous. Draw the x- and y-axes.

(**Suggestion:** In your drawing, somehow distinguish those x not in domain f. For instance, if domain $f = (-\infty, 1) \cup (1, \infty)$, then you could draw a dotted line where $x = 1$.)

(b) If 0 is in domain f, plot $(0, f(0))$. If possible, determine when $f(x) = 0$ and plot the corresponding points $(x, f(x)) = (x, 0)$.

(c) Compute all relevant limits. (For instance, if domain $f = (-\infty, 1) \cup (1, \infty)$, compute $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$, $\lim_{x\to 1^+} f(x)$, $\lim_{x\to 1^-} f(x)$.)

(**Suggestion:** Somehow indicate the limits in your graph, perhaps using arrows.)

(d) Find $f'(x)$. Determine where $f'(x) = 0$ or where $f'(x)$ is undefined (with x in domain f). Plot the corresponding points $(x, f(x))$.

(**Suggestion:** At the points where $f'(x) = 0$, draw a small horizontal dash to remind yourself the "slope" of the curve at this point is 0.)

(e) Using the above information, complete the sketch of the graph. (Remember, since f is continuous, the number of continuous components in the graph is the number of connected components of domain f.)

2. Define the function f piecewise by the formula

$$f(x) = \begin{cases} x^2 - 9 & \text{if } x \geq 1, \\ \frac{1}{x^2+1} & \text{if } x < 1. \end{cases}$$

(a) Determine the domain of f and whether f is continuous. (Since $x^2 - 9$ and $\frac{1}{x^2+1}$ are algebraic functions, they are continuous; so to determine whether f is continuous, you need to determine whether f is continuous at $x = 1$, i.e. whether $\lim_{x\to 1} f(x) = f(1)$. **Caution:** Does $\lim_{x\to 1^+} f(x)$ equal $\lim_{x\to 1^-} f(x)$?)

(b) If 0 is in domain f, plot $(0, f(0))$. If possible, determine when $f(x) = 0$ and plot the corresponding points $(x, f(x)) = (x, 0)$.

(c) Compute all relevant limits that you have not already computed.

(d) Find $f'(x)$. Determine where $f'(x) = 0$ or where $f'(x)$ is undefined (with x in domain f). Plot the corresponding points $(x, f(x))$.

(e) Using the above information, complete the sketch of the graph.

Workshop 7

1. Set $y = \sqrt{x} = x^{\frac{1}{2}}$.
   (a) For which values of x is y defined?
   (b) In terms of x, what is y^2? State this as an equality.
   (c) Apply $\frac{d}{dx}$ to both sides of your equality in (b). (That is, differentiate each side of the equality with respect to x. Remember the chain rule!)
   (d) Solve the equation you obtained in (c) for $\frac{dy}{dx}$. Use the definition of y to express $\frac{dy}{dx}$ in terms of x, and use exponential notation.

2. Set $y = \sqrt[3]{x} = x^{\frac{1}{3}}$.
   (a) For which values of x is y defined?
   (b) In terms of x, what is y^3? State this as an equality.
   (c) Apply $\frac{d}{dx}$ to both sides of your equality in (b).
   (d) Solve the equation you obtained in (c) for $\frac{dy}{dx}$. Use the definition of y to express $\frac{dy}{dx}$ in terms of x, and use exponential notation.

3. Set $y = \sqrt[n]{x} = x^{\frac{1}{n}}$ where n is a positive integer (so $n = 1$ or 2 or 3 or 4 *etc.*).
   (a) For which values of x is y defined?
   (b) In terms of x, what is y^n? State this as an equality.
   (c) Apply $\frac{d}{dx}$ to both sides of your equality in (b).
   (d) Solve the equation you obtained in (c) for $\frac{dy}{dx}$. Use the definition of y to express $\frac{dy}{dx}$ in terms of x, and use exponential notation.

4. Set $y = \frac{1}{x} = x^{-1}$.
   (a) For which values of x is y defined?
   (b) What is $y \cdot x$? State this as an equality.
   (c) Apply $\frac{d}{dx}$ to both sides of your equality in (b).
   (d) Solve the equation you obtained in (c) for $\frac{dy}{dx}$. Use the definition of y to express $\frac{dy}{dx}$ in terms of x, and use exponential notation.

5. Set $y = \frac{1}{x^m} = x^{-m}$ where m is a positive integer.
   (a) For which values of x is y defined?
   (b) What is $y \cdot x^m$? State this as an equality.
   (c) Apply $\frac{d}{dx}$ to both sides of your equality in (b).
   (d) Solve the equation you obtained in (c) for $\frac{dy}{dx}$. Use the definition of y to express $\frac{dy}{dx}$ in terms of x, and use exponential notation.

6. Set $y = \sqrt[n]{x^m} = x^{m/n}$ (where m, n are nonzero integers),
   (a) In terms of x, what is y^n? State this as an equality.
   (b) Apply $\frac{d}{dx}$ to both sides of your equality in (a).
   (c) Solve the equation you obtained in (b) for $\frac{dy}{dx}$. Use the definition of y to express $\frac{dy}{dx}$ in terms of x, and use exponential notation.

Workshop 8

1. We want to choose 2 nonnegative numbers whose sum is 1, and we want to know when the sum of their squares is minimal and when the sum of their squares is maximal.
   (a) Since we don't yet know the values of these numbers, let's give them names. Let's call them x and z. Given the above constraints, what is the smallest x could be? What is the largest x could be? So what values of x are we considering? What is the smallest z could be? What is the largest z could be? What values of z are we considering?
   (b) Given the above constraints on x and z, find a formula for z in terms of x.
   (c) Find a formula for the sum of the squares of x and z; use (b) to express this formula in terms of x only. (So your final formula should not invlove z.)
   (d) Remembering the constraints on x (see (a)), graph the function you obtained in (c). (Be careful what domain you use – see (a)!)
   (e) What is the lowest point on this graph? What is x at that point? What is the correspoinding value of z? (See (b)!) When is the sum of the squares of x and z minimal? **Briefly** explain your reasoning.
   (f) What is the highest point of this graph? What is x at that point? What is the corresponding value of z? (See (b)!) When is the sum of the squares of x and z maximal? **Briefly** explain your reasoning.

   proportional: having the same or a constant ratio

2. The strength of a rectangular beam is jointly proportional to its width and the cube of its height. Given a cylindrical log, we want to find the dimensions of the strongest rectangular beam that can be cut from the log.
   (a) Draw a picture of a circle, the cross-section of the cylindrical log. Inscribe in the circle a rectangle, the cross-section of the beam. Choose letters to label the width and the height of the rectangle. Choose a letter to represent the radius (or diameter) of the circle. **Note:** The radius and diameter of the log are contant.
   (b) Find a formula relating the width and height of the beam to the radius (or the diameter) of the log. Which of these values are constant, and which are variable? What are the physical constraints on your variables?
   (c) Find a formula for the strength of the beam. Use (b) to rewrite this either in terms of the width of the beam and the radius (or diameter) of the log, or in terms of the height of the beam and the radius (or diameter) of the log.
   (d) Use calculus to find the maximum value of the strength function, keeping in mind the physical constraints on the variables.

Workshop 9

1. A spherical balloon is being inflated at a rate of 1 cubic centimeter per second. Our goal is to find how the radius of the balloon is increasing.
   (a) Draw a picture of the balloon, marking its radius.
   (b) Translate the first sentence of this problem into an equation for $\frac{dV}{dt}$ where V is the volume of the balloon (measured in cubic centimeters) and t is time (measured in seconds).
   (c) We have a hypothesis on $\frac{dV}{dt}$, and we want to use this to find $\frac{dr}{dt}$. To be able to relate $\frac{dr}{dt}$ to $\frac{dV}{dt}$, we first relate V and r: so write an equation that relates V and r.
   (d) Differentiate your formula in (b) with respect to t; that is, apply $\frac{d}{dt}$ to both sides of your equation in (b). (This requires implicit differentiation, reflecting the fact that the volume V and the radius r of the balloon are functions of time t. You should get an equation involving $\frac{dV}{dt}$ and $\frac{dr}{dt}$.)
   (e) Now substitute from (a) for $\frac{dV}{dt}$ and solve for $\frac{dr}{dt}$.

2. Sand is being emptied from a hopper at a rate of 10 cubic feet per second. The sand forms a conical pile whose height is always twice its radius. Our goal is to find the rate at which the radius is increasing at the instant the height is 5 feet.
   (a) Draw a picture of the sand pile, marking its height and the radius of its base.
   (b) Translate the first sentence of this problem into an equation.
   (c) The volume V of a cone of radius r and height h is

$$V = \frac{\pi}{3} r^2 h.$$

   Use the second sentence of the problem to write V in terms of r only (i.e. not involving h).
   (d) Differentiate your formula for V in (b) with respect to t where t represents time (measured in seconds). Use this together with (a) to find an equation for $\frac{dr}{dt}$.
   (e) What is the value of $\frac{dr}{dt}$ when $h = 5$? (When $h = 5$, what is r? Remember how h and r are related.)

3. Say the radius of a spherical balloon is increasing at a rate of 1 centimeter per second. At what rate is the volume increasing? At what rate is the volume increasing at the instant the radius is 3 centimeters? At what rate is the volume increasing at the instant the radius is 5 centimeters?

Workshop 10

Recall: Given a function f (whose derivative f' exists), we know that f is increasing when $f' > 0$, and f is decreasing when $f' < 0$. Similarly, the function f' is increasing when its derivative, f'', is positive, and f' is decreasing when f'' is negative. So the *slope* of f is increasing when $f'' > 0$, and the *slope* of f is decreasing when $f'' < 0$.

1. Sketch a graph of a continuous function f with the following properties:
   the domain of f is $[-3, 5]$;
   $f(-3) = 1$, $f(5) = 4$;
   $f'(x) > 0$ for $-3 < x < 1$, $f'(x) < 0$ for $1 < x < 5$;
   $f''(x) < 0$ for $-3 < x < 5$.
   (Mild suggestion: Invent values for $f'(x)$ at, say, $x = -2, 0, 2$, and 4 so that $f'(x) > 0$ for $-3 < x < 1$, $f'(x) < 0$ for $1 < x < 5$, and so that $f'(x)$ decreases as x varies from -3 up to 5.)

2. Sketch a graph of a continuous function f with the following properties:
   the domain of f is $[-3, 5]$;
   $f(-3) = 1$, $f(5) = 4$;
   $f'(x) > 0$ for $-3 < x < 1$, $f'(x) < 0$ for $1 < x < 5$;
   $f''(x) > 0$ for $-3 < x < 1$ and for $1 < x < 5$.

3. Sketch a graph of a continuous function f with the following properties:
   the domain of f is $[-\infty, \infty]$;
   $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = -\infty$;
   $f'(x) > 0$ for $x < -2$ and $x > 4$, $f'(x) < 0$ for $-2 < x < 4$;
   $f''(x) < 0$ for $x < 1$, $f''(x) > 0$ for $x > 1$.

4. Sketch a graph of a continuous function f with the following properties:
   the domain of f is $[-\infty, \infty]$;
   $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = -\infty$;
   $f'(x) > 0$ for $x < -2$ and $x > 4$, $f'(x) < 0$ for $-2 < x < 4$;
   $f''(x) > 0$ for $x < -2$ and for $-2 < x < 4$, $f''(x) < 0$ for $x > 4$.

5. Let $f(x) = x^3 - 9x^2 + 5x$. (So f is continuous.) Find the domain of f, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, where $f(x) = 0$ (if possible), where $f'(x) > 0$ and where $f'(x) < 0$, where $f''(x) > 0$ and where $f''(x) < 0$. Now sketch a graph of f reflecting all the information computed above.

Workshop 11

1. (a) Suppose f is a continuous function defined on some interval with $f'(x) = 0$. Explain why this means the graph of f is a horizontal line.

   (b) What is an equation for a horizontal line of height 15? What is an equation for a horizontal line of height c? Using this and (a), briefly explain why $f(x) = c$ if $f'(x) = 0$.

   (c) Suppose f and g are continuous functions defined on some interval with $f'(x) = g'(x)$. Explain why this means $(f - g)(x) = c$ for some constant c. Then express $f(x)$ in terms of $g(x)$ and c.

   (d) Describe (i.e. give a formula for) all continuous functions f so that

$$f'(x) = 5x^2 - 7x + 3.$$

2. Dustball is strolling along a straight and narrow path, and Dustball's position on the path at time t is given by

$$f(t) = 8t^2 - 5t + 7.$$

   (Here the position is measured in feet from a fixed point of reference on the path, and time is measured in minutes.)

   (a) What is Dustball's position at time $t = 0$?

   (b) What is Dustball's velocity at time $t = 0$?

   (c) Recall that Dustball's acceleration is the rate of change of Dustball's velocity. What is Dustball's acceleration at time $t = 0$?

3. Dustball is now lounging on the side of the path and Tabasco is cruising along the path, accelerating at a rate of 2 ft/min per minute, or equivalently, 2 ft/(min)^2. (So each minute Tabasco's speed increases by 2 ft/min.)

   (a) Given that Tabasco's velocity at time $t = 0$ is 1 ft/min, find a formula for Tabasco's velocity (as a function of t).

   (b) Given that Tabasco's position at time $t = 0$ is 5 ft from our fixed point of reference, find a formula for Tabasco's position (as a function of t).

4. Now both Dustball and Tabasco are off the path, and Coyope is running on the path. Coyope's acceleration at time t is $5\cos(t)$ (meaning $5\cos(t$ radians)). Also, Coyope's velocity is 0 when $t = 0$, and Coyope's position is -2 when $t = 0$.

   (a) Find a formula for Coyope's position (as a function of t).

   (b) What are Coyope's position and velocity when $t = 4\pi$?

   (c) What are Coyope's position and velocity when $t = \frac{9\pi}{2}$?

   (d) What are Coyope's position and velocity when $t = 5\pi$?

Workshop 12

1. Let $f(t) = \dfrac{1}{t}$ with domain $(0, \infty)$.
   (a) Graph f. (Include concavity computations.) **Note:** Your horizontal axis is the t-axis.

We define a new function F by setting $F(x)$ equal to the area of the region above the t-axis and below the graph of f, and with $1 \le t \le x$.

   (b) Quickly sketch another graph of f, and in this graph shade the region above the t-axis and below the graph of f, and with $1 \le t \le 3$.
   (c) Use two rectangles to **approximate** the area of the region shaded in (b).
   (d) Using two rectangles, what dimension rectangles could you use to over-estimate the area of this region, yet still get a fairly reasonable approximation? Using two rectangles, what dimension rectangles could you use to under-estimate the area of this region?
   (e) Use four rectangles to over-estimate the area shaded in (b). Then use four rectangles to under-estimate the area of this region. Are these approximations better or worse than the approximations in (d)? Briefly explain your reasoning.
   (f) Quickly sketch another graph of f (with the horizontal axis the t-axis). Say x is a fixed number greater than 1, and h is a fixed, positive number. (Later we will let h approach 0, so think of h as rather small.) Mark x and $x + h$ on the t-axis, then shade the region whose area is $F(x)$. Also, hatch or cross-hatch the region whose area is $F(x + h)$. Describe what region's area is $F(x + h) - F(x)$.
   (g) Now use one rectangle to over-estimate the area $F(x + h) - F(x)$, and use one rectangle to under-estimate this area. (What is the height of the curve $y = f(t)$ when $t = x$? What is the height of the curve when $t = x + h$?) Use inequalities to compare these estimates to $F(x + h) - F(x)$.
   Recall: By definition,
$$F'(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h}.$$
   (h) Assuming $h > 0$, Use your inequalities from (g) to give upper and lower bounds for the quantity
$$\frac{F(x + h) - F(x)}{h}.$$
   Now use your inequalities to evaluate
$$\lim_{h \to 0^+} \frac{F(x + h) - F(x)}{h}.$$

(i) Repeat (f) and (g) with h negative. Then use this to conclude

$$\lim_{h\to 0^-} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0^+} \frac{F(x+h)-F(x)}{h}.$$

(j) Use (h) and (i) to evaluate

$$F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h}.$$

Workshop 13

Recall: We defined $\int_a^b f(x)\,dx$ to be the "area" beneath the curve $y = f(x)$ (i.e. between $y = f(x)$ and the x-axis), with x running from a to b. (We define area precisely using Riemann sums.)

Note: These areas are oriented. That is, regions below the x-axis have negative orientation and thus negative area, and reversing $\int_a^b$ to $\int_b^a$ reverses the orientation.

When measuring areas of regions between curves, we measure all regions with positive area (i.e. the areas are not oriented).

1. Let $f(x) = x + 1$ and $g(x) = -3$. We want to find the area of the region between $y = f(x)$ and $y = g(x)$ and with $-2 \leq x \leq 2$.
   (a) Where do the curves $y = f(x)$ and $y = g(x)$ intersect? That is, when is $f(x) = g(x)$?
   (b) Sketch rough graphs of f and g for $-2 \leq x \leq 2$. Clearly indicate which curve is on top.
   (c) Compute the area of the region between these curves using the Fundamental Theorem of Calculus; so compute the area beneath the top curve and subtract from that the area beneath the bottom curve.
   (d) Use geometry to check your area computation. (Suggestion: Partition this region into geometric shapes whose areas you can easily compute.)

2. Let $f(x) = 2x$ and $g(x) = 8 - x^2$. Find all points where the graphs of these functions intersect, then find the area between the curves.

3. Let $f(x) = x^3 - 3x + 2 = (x + 2)(x - 1)^2$ and $g(x) = -\frac{1}{2}x^2 + 2$. Find all points where the graphs of these functions intersect, then find the area between the curves.

Workshop 14

Recall: $\int f(u)du \;=\; \int f(u)\frac{du}{dx}\, dx.$

Throughout this workshop, we set

$$F(x) = \int_1^x \frac{1}{t}\, dt.$$

(So $F(x)$ is the area beneath the curve $y = \frac{1}{t}$ with t running from 1 to x.)

1. Sketch a graph of $y = \dfrac{1}{t}$ for $t > 0$. With $F(x)$ as above and a, b positive numbers, explain why

$$F(ab) = \int_1^a \frac{1}{t}\, dt \;+\; \int_a^{ab} \frac{1}{t}\, dt.$$

Use the substitution $u = \dfrac{t}{a}$ to show

$$\int_a^{ab} \frac{1}{t}\, dt \;=\; \int_1^b \frac{1}{u}\, du.$$

(As t varies from a to ab, what does u do?) Conclude that

$$F(ab) = F(a) + F(b).$$

2. Set $u = t^{1/n}$. First find $\dfrac{du}{dt}$, and use this to show $\dfrac{1}{t} = \dfrac{n}{u}\dfrac{du}{dt}$. Now show

$$\int_1^{a^n} \frac{1}{t}\, dt \;=\; n \cdot \int_1^a \frac{1}{u}\, du.$$

Explain why this implies $F(a^n) = n \cdot F(a)$. (**Note:** Problems 1 and 2 show that the function $F(x)$ behaves like a logarithm, and in fact, it is. $F(x) = \ln(x)$ where ln, the "natural" log, has base e where e is a particular number between 2 and 3. Also recall that $\log_a(b) = c$ means $a^c = b$. Here a is called the base of the logarithm.)

Notation: Let $\ln(x)$ denote the function we called $F(x)$ above. So

$$\ln(x) = \int_1^x \frac{q}{t}\, dt$$

and

$$\ln(ab) = \ln(a) + \ln(b), \ \ln(a^n) = n\ln(a), \text{ and } \frac{d}{dx}\ln(x) = \frac{1}{x}.$$

3. Suppose that y is a function of x so that $\ln(y) = x$. (**Note:** We assume $y > 0$.) Differentiate both sides of this equation with respect to x (remember that y is a function of x). Then solve for $\frac{dy}{dx}$ in terms of x and y.

4. Let $y = \dfrac{\sqrt{x^2+1}}{\sqrt[3]{x^3+5}}$.
   (a) Use the identities from #1 and #2 to rewrite $\ln(y)$.
   (b) Differentiate both sides of your equation in (a) with respect to x.
   (c) Solve for $\frac{dy}{dx}$. (This method of finding $\frac{dy}{dx}$ is called logarithmic differentiation.)

Workshop 15

Recall: To find the volume of a sphere of radius r, we rotated the region bounded by $y = \sqrt{r^2 - x^2}$ about the x-axis. We noted that cross-sections perpendicular to the x-axis are disks; the disk at a given x-coordinate has radius $\sqrt{r^2 - x^2}$. Integrating the cross-sectional areas gave us the exact volume:

$$\int_{-r}^{r} \pi(\text{radius})^2 \, dx = \int_{-r}^{r} \pi(r^2 - x^2) \, dx.$$

We use a similar procedure to find the surface area of this sphere. Instead of rotating the entire region bounded by $y = \sqrt{r^2 - x^2}$ and the x-axis, we rotate the arc $y = \sqrt{r^2 - x^2}$ about the x-axis.

(a) Draw the x-axis and the y-axis, then sketch a reasonably large graph of $f(x) = \sqrt{r^2 - x^2}$. (What is the domain of this function?)

(b) Let x_0 be a generic value in the domain of this function; mark and label this on the x-axis.

(c) Let $y_0 = f(x_0)$. Mark the corresponding point $(x_0, y_0) = (x_0, \sqrt{r^2 - x_0^2})$ on the curve, and sketch the line tangent to the curve at this point, i.e. the line through the point (x_0, y_0) with slope $m = f'(x_0)$. (What is the value of m in terms of x_0?)

(d) Let h be a small positive value; mark and label $x_0 + h$ on the x-axis.

(e) On the tangent line, mark the point with x-coordinate $x_0 + h$; call this point P. In terms of x_0, y_0, h and $f'(x_0)$, describe the y-coordinate of this point P. (Remember the point-slope formula for a line!)

(f) In terms of x_0, y_0, h and $f'(x_0)$, describe the distance from (x_0, y_0) to the point P.

(g) Imagine taking the line segment from (x_0, y_0) to P and rotating this about the x-axis. In this way you obtain something that is almost a cylinder; call this a skewed cylinder. What is the "radius" of this skewed cylinder? Consider the distance from (x_0, y_0) to P to be the "height" of this skewed cylinder. In terms of x_0, y_0, h and $f'(x_0)$, approximate the surface area of this skewed cylinder; call this approximated surface area A.

(h) The curve $y = \sqrt{r^2 - x^2}$ can be approximated by many line segments, like the one from (x_0, y_0) to P. Draw a picture in which $y = \sqrt{r^2 - x^2}$ is approximated by 6 line segments.

(i) When we rotate $y = \sqrt{r^2 - x^2}$ about the x-axis, we obtain the surface of the sphere. So to approximate this surface area, we can add together the surface areas of the skewed cylinders obtained by rotating the line segments approximating the curve $y = \sqrt{r^2 - x^2}$. To make this approximation exact, we take the limit as $h \to 0$. This yields the integral of $\frac{A}{h}$ as x_0 varies through the domain of $y = \sqrt{r^2 - x^2}$. To evaluate this:

(1) Write y_0 in terms of x_0, then compute $f'(x_0)$.

(2) Now write $\frac{A}{h}$ only in terms of r and x_0.

(3) Finally, integrate $\frac{A}{h}$ over the domain of the curve $y = \sqrt{r^2 - x^2}$.