Semiclassical approach to spectral correlation functions

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1 Introduction

One motivation for deriving semiclassical methods for chaotic systems like the Gutzwiller trace formula was to find a substitute for the EBK quantisation rules which provide semiclassical approximations for energy levels and wave functions in integrable systems [1]. Although the Gutzwiller trace formula allows to determine energy levels semiclassically from a knowledge of classical periodic orbits, in practise only a relatively small number of the lowest energies can be determined in this way due to the exponential proliferation of the number of long periodic orbits in chaotic systems.

However, the trace formula is very powerful in applications in which one is interested not in individual levels, but in statistical properties of the spectrum in the semiclassical regime. These spectral statistics are in the centre of interest in quantum chaos, because they are found to be universal and to provide a clear signature for chaos in the underlying classical system. All numerical evidence points to an agreement of spectral statistics in generic chaotic systems with those of random matrix theory (RMT) [2]. The aim of the semiclassical method is to find a theoretical explanation and, possibly, a proof for this connection between quantum chaos and random matrix theory. Similar as on the quantum side, what is needed for this task is not a knowledge of individual periodic orbits, but of statistical properties of long periodic orbits.

The first steps in this direction were made in the seminal works of Hannay and Ozorio de Almeida, and Berry. Using the mean distribution of long periodic orbits [3] Berry showed that two-point statistics agree with RMT in the regime of long-range correlations in the energy spectrum [4]. To go beyond Berry's so-called diagonal approximation requires the evaluation of correlations between periodic orbits. Vice versa, the conjectured connection between quantum chaos and random matrix theory can be used to predict the existence of universal correlations between very long periodic orbits [5, 6]. So in some sense quantum mechanics reveals fundamental universal properties of periodic orbits that were not known before.

In recent years there has been a rapid development of methods to derive these periodic orbit correlations on the classical side. Proofing that these correlations do indeed exist would show the validity of the random matrix hypothesis. The purpose of this short lecture series is to give an introduction into the evaluation of correlations between periodic orbits. This is done by discussing in detail the simplest example of periodic orbit correlations which involves orbits with two loops. This example reveals the basic mechanism that is also behind more complicated correlations.

In accordance with the topic of this school, the calculations are done for the motion on compact Riemann surfaces of constant negative curvature. For these systems the derivation is more transparent and can be done with more rigour than for general chaotic systems. Some of the special features that facilitate the derivation in these systems are the existence of the exact Selberg trace formula, the non-existence of conjugate points, the uniform hyperbolicity, and the tessellation properties of representations of the surfaces on the hyperbolic plane.

The course is structured as follows. In the second section the semiclassical approximation of the form factor is derived and its properties are discussed. The diagonal approximation is evaluated and the two-loop orbits which contribute to the leading off-diagonal approximation are introduced. The third section contains an introduction to phase space methods for Riemann surfaces and an evaluation of transition probabilities which are used to calculate the number of two-loop pairs. In section 4 the contribution of the two-loop orbit pairs to the form factor are evaluated. Finally, section 5 contains a discussion and an overview over further developments.

2 The spectral form factor

The spectral form factor is a two-point statistics that is convenient for semiclassical evaluations, because its argument is directly related to the time along orbits. In general chaotic systems the semiclassical approximation to the form factor is derived by applying Gutzwiller's trace formula. In the case of Riemann surfaces the existence of the exact Selberg trace formula gives the opportunity to understand the effect of the approximations involved in arriving at the semiclassical form factor.

The spectral form factor is defined as the Fourier transform of the two-point correlation function of the density of states

$$K(\tau) = \left\langle \int_{-\infty}^{\infty} \frac{d\eta}{\bar{d}(E)} \, \langle d_{\text{osc}}(E+\eta/2)\, d_{\text{osc}}(E-\eta/2) \rangle_E \; e^{2\pi\, i\, \eta\, \bar{d}(E)\, \tau} \right\rangle_\tau \, . \tag{1}$$

Here $\bar{d}(E)$ and $d_{\text{osc}}(E)$ are the mean and oscillatory part of the density of states, $d(E)$, respectively,

$$d(E) = \sum_n \delta(E - E_n) = \bar{d}(E) + d_{\text{osc}}(E) \, . \tag{2}$$

The mean density of states is asymptotically given by Weyl's law $\bar{d}(E) \sim A/(4\pi)$ as $E \to \infty$, where A is the area of the Riemann surface. The energy average in (1) is performed over an energy window ΔE which is classically small (i.e. small in comparison to E) and large in comparison to the mean spacing between levels $1/\bar{d}(E)$. In addition, a local average is performed over a small window $\Delta\tau$ in order to suppress strong fluctuations that would occur otherwise. The dependence of the form factor $K(\tau)$ on E (and the choice of the smoothing) is omitted in its argument.

The random matrix hypothesis predicts that the form factor of generic chaotic systems with time-reversal symmetry approaches the form factor of the Gaussian Orthogonal Ensemble (GOE) of random matrices in the semiclassical limit, $\lim_{E\to\infty} K(\tau) = K^{\mathrm{GOE}}(\tau)$, where [7]

$$K^{\mathrm{GOE}}(\tau) = \begin{cases} 2\tau - \tau \log(1+2\tau) & \text{if } \tau < 1 \\ 2 - \tau \log \frac{2\tau+1}{2\tau-1} & \text{if } \tau > 1 \, . \end{cases} \tag{3}$$

Requirements for this assumption to hold are that $\bar{d}(E)\, \Delta E \to \infty$, $\Delta E/E \to 0$, and $\Delta\tau \to 0$ in this limit.

One difficulty is to define what is meant by "generic". In the following we require that the multiplicity of lengths of periodic orbits is typically two. This means that there are typically only two periodic orbits whose lengths are identical, an orbit and its time-reverse. This condition excludes systems with symmetries, and it excludes arithmetic systems for which the mean multiplicity increases exponentially with their lengths. These arithmetic systems have spectral statistics that are described by a Poisson process, and not by one of the random matrix ensembles [8, 9, 10]. It is, in fact, not possible to require the multiplicity to be exactly two, because it is known that for Riemann surfaces with constant negative curvature the multiplicity of lengths is unbounded [11]. We assume that lengths with higher multiplicity than two are so sparse that they do not modify the calculations in the semiclassical limit.

In the following we use the Selberg trace formula to derive the semiclassical expression for the form factor. The energy spectrum for Riemann surfaces is the spectrum of minus the Laplace-Beltrami operator $-\Delta\Psi = E\Psi$. For compact Riemann surfaces with constant negative curvature the trace formula is given by (see [12])

$$\sum_n h(p_n) = \frac{A}{4\pi} \int_{-\infty}^{\infty} dp' \; h(p')\, p' \, \tanh(\pi p') + \sum_\gamma A_\gamma \, g(L_\gamma) \, , \tag{4}$$

where p_n are momentum eigenvalues which are related to the energies by $E_n = p_n^2 + 1/4$, and

$$A_\gamma = \frac{L_\gamma}{2R_\gamma \, \sinh L_\gamma/2} \, , \qquad g(x) = \frac{1}{\pi} \int_0^\infty dp \; h(p) \, \cos px \, . \tag{5}$$

The sum on the right-hand side of (4) runs over all periodic orbits γ with length L_γ and repetition number R_γ. (Repetitions of a primitive periodic orbit count as distinct orbits.) The function $h(p)$ has to satisfy certain conditions (for details see [12]). It has to be an even function in p. It has to be analytic in a strip $|\operatorname{Im} p| < 1/2 + \eta$ for some $\eta > 0$, which guarantees the convergence of the periodic orbit sum. Finally, it has to fall off sufficiently fast for large p, $|h(p)| = \mathcal{O}(|p|^{-2-\delta})$ as $|p| \to \infty$, for some $\delta > 0$. This guarantees the convergence of the sum over energies.

One example of an allowed function is

$$h_\varepsilon(p') = \frac{1}{\sqrt{2\pi\varepsilon^2}} \left(\exp\left\{ -\frac{(p-p')^2}{2\varepsilon^2} \right\} + \exp\left\{ -\frac{(p+p')^2}{2\varepsilon^2} \right\} \right) \tag{6}$$

in terms of which one can define a smoothed density of states $d_\varepsilon(E) := (2|p|)^{-1} \sum_n h_\varepsilon(p_n)$, where $E = p^2 + 1/4$.

The smoothed density of states has the property that $\lim_{\varepsilon\to 0} d_\varepsilon(E) = d(E)$. Clearly the function $h_\varepsilon(p')$ that is obtained in the limit $\varepsilon \to 0$, $h_0(p') = \delta(p'-p) + \delta(p'+p)$ is not an allowed function. Hence we cannot use the Selberg trace formula directly to evaluate (1). Let us look instead at the quantity $K_\varepsilon(\tau)$ which is obtained by replacing the spectral density in (1) by the smoothed density $d_\varepsilon(\tau)$. We then can express $K_\varepsilon(\tau)$ in terms of classical periodic orbits by using Selberg's trace formula, and evaluate the integral in (1) in leading semiclassical order. Then we take the limit $\varepsilon \to 0$. This leads to a well-defined, absolutely convergent sum over periodic orbits for the particular choice of averaging that we will use in the following. Since these steps involve a semiclassical approximation and the interchange of limit and integral we will look carefully at the final result in order to see the effect of these steps.

The semiclassical form factor that is obtained in this way is

$$K_{\rm sc}(\tau) = \frac{1}{L_H} \left\langle \sum_{\gamma,\gamma'} A_\gamma \, A_{\gamma'} \, e^{i\,p\,(L_\gamma - L_{\gamma'})} \, \delta\left(L - \frac{L_\gamma + L_{\gamma'}}{2} \right) \right\rangle_{E,\tau} , \tag{7}$$

where $\tau = L/L_H$, and $L_H = p\,A$ is the "Heisenberg length". We choose to perform the energy average by a Gaussian in the variable p with width Δp, and the τ-average by a Gaussian in the variable L with width ΔL. In more detail,

$$\langle f(x) \rangle_x = \int_{-\infty}^{\infty} dx' \, \frac{1}{\sqrt{2\pi}\,\Delta x} \, \exp\left\{ -\frac{(x-x')^2}{2\,\Delta x^2} \right\} f(x') , \tag{8}$$

where x stands for p or L. This has the additional advantage that for the particular choice $\Delta p \, \Delta L = 1/2$ the form factor can be written as absolute square of a single sum [13]

$$K_{\rm sc}(\tau) = \sqrt{\frac{2}{\pi}} \, \frac{\Delta p}{L_H} \left| \sum_\gamma A_\gamma \, \exp\left\{ i\,p\,L_\gamma - \Delta p^2 \, (L - L_\gamma)^2 \right\} \right|^2 . \tag{9}$$

The conditions for the semiclassical limit after (3) then translate into $p \to \infty$ with $\Delta p/p \to 0$ and $p\Delta p \to \infty$.

Let us consider expression (9) in more detail. The Selberg trace formula is exact, but we made an approximation by evaluating the integral in (1) asymptotically for large p and, after introducing a smoothing parameter ε, interchanged limit $\varepsilon \to 0$ and integral. The main question is whether (9) is a good approximation to (1) and has the same semiclassical limit.

A numerical evaluation of (9) in [14] showed, somewhat surprisingly, that $K_{\rm sc}(\tau)$ is an exponentially increasing function in τ. This exponential increase can be understood, however, by using the trace formula again in order to express (9) in terms of the spectrum. One finds that the semiclassical form factor is exactly identical to the form factor of the *momentum spectrum*. Hence the effect of the approximations involved is to replace the energy spectrum by the momentum spectrum. One main difference is that there is at least one imaginary momentum eigenvalue which is $p_0 = i/2$, corresponding to the ground state $E_0 = 0$. There might be finitely many other imaginary eigenvalues. These imaginary momentum eigenvalues are the reason for the exponential increase of $K_{\rm sc}(\tau)$. To suppress them requires a stronger condition for performing the semiclassical limit $\Delta p/\sqrt{p} \to 0$ as $p \to \infty$ instead of $\Delta p/p \to 0$ [14]. This is a necessary condition for $\lim_{p\to\infty} K_{\rm sc}(\tau) = K_{\rm GOE}(\tau)$.

2.1 The diagonal approximation

The contributions of pairs of periodic orbits in (7) and (9) are suppressed by the averaging, and also by cancellations of oscillatory contributions from different pairs of periodic orbits which are uncorrelated. The main contribution comes from a relatively small number of pairs of orbits which are correlated. Berry concluded that the most important contribution for small τ comes from pairs of orbits with identical lengths [4]. This corresponds to the diagonal approximation in which only pairs of orbits are included which are either identical or related by time inversion. The diagonal approximation for (9) has the form

$$K_{\rm sc}^{(1)}(\tau) = \sqrt{\frac{2}{\pi}}\,\frac{\Delta p}{L_H}\,2\sum_\gamma A_\gamma^2\,\exp\{-2\,\Delta p^2\,(L-L_\gamma)^2\}\;. \tag{10}$$

If the semiclassical limit is performed with τ fixed and $p \to \infty$ it follows that $L \to \infty$. Hence the contributions come from very long orbits and we can replace the amplitude in (5) by

$$A_\gamma^2 \to \frac{L_\gamma^2}{e^{L_\gamma}}\;. \tag{11}$$

The sum in (10) is evaluated asymptotically for large length by using the prime geodesic theorem

$$N(L) = \#\{L_\gamma \le L\} \sim \frac{e^L}{L} \qquad \text{as} \qquad L \to \infty\;. \tag{12}$$

The mean density of periodic orbits in a length interval is then obtained taking a derivative

$$\bar{\rho}(L) \sim \frac{e^L}{L} \qquad \text{as} \qquad L \to \infty\,, \tag{13}$$

and the sum over periodic orbits is replaced by an integral

$$\sum_\gamma \to \int dL' \; \bar{\rho}(L') \; . \tag{14}$$

Hence one finds

$$\begin{aligned} K_{\text{sc}}^{(1)}(\tau) &\sim \sqrt{\frac{2}{\pi}} \, \frac{2\,\Delta p}{L_H} \int_{-\infty}^{\infty} dL' \, L' \, \exp\{-2\,\Delta p^2\,(L-L')^2\} \\ &\sim \sqrt{\frac{2}{\pi}} \, \frac{2\,\Delta p\,L}{L_H} \sqrt{\frac{\pi}{2}} \, \frac{1}{\Delta p} \\ &\sim 2\tau \end{aligned} \tag{15}$$

as $L \to \infty$. The calculations in (15) involve an application of the method of steepest descent with the condition that $L\,\Delta p \propto p\,\Delta p \to \infty$ as $L \to \infty$, in accordance with our previous conditions for performing the semiclassical limit. The result is indeed the first term in the Taylor expansion of the spectral form factor for the GOE ensemble $K_{\text{GOE}}(\tau) = 2\tau - 2\tau^2 + \ldots$.

2.2 Off-diagonal terms

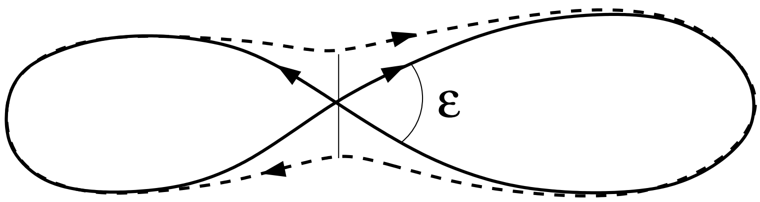

Fig. 1. A pair of two-loop orbits.

The assumption behind the evaluation of off-diagonal terms is that correlated orbits follow each other very closely almost everywhere in position space. They consist of different orbit segments, or loops, along which both orbits are almost identical. These segments might be traversed in opposite directions in systems with time-reversal symmetry. The orbits differ in the way in which the segments are connected.

The simplest example is that of a pair of orbits with two long loops as shown schematically in figure 1. The two loops are connected by two long orbit stretches which are almost parallel. (The schematic picture 1 does not really show this. Imagine ε being small.) This makes it possible that the two loops are connected in different ways for the two orbits [15, 16].

Consider now one of the two orbits. We look more closely at the encounter region where the two segments of the orbit which connect the two loops are almost parallel. (More precisely, they are anti-parallel if the sense of traversal is taken into account, but in this section we disregard the direction in which they are traversed.) We investigate how the two segments approach each other and separate again in the encounter region. For this purpose we consider the relative motion along one segment by linearising the motion around the other segment.

In the vicinity of a geodesic on a surface of constant curvature K the linearised motion is described by the Jacobi equation [1]. It has the form

$$\frac{d^2\rho}{dl^2} + K\,\rho = 0 \,. \tag{16}$$

Here l is the distance along the trajectory and ρ measures the distance perpendicular to the trajectory. The solution for $K = 1$ and initial conditions $\rho(0) = \rho_0$ and $\rho'(0) = {\rho'}_0$ is

$$\rho = \rho_0\,\cosh l + {\rho'}_0\,\sinh l \,. \tag{17}$$

Introducing a conjugate momentum variable $\sigma = \rho'$ leads to

$$\begin{pmatrix}\rho\\ \sigma\end{pmatrix} = M\begin{pmatrix}\rho_0\\ \sigma_0\end{pmatrix} \,, \quad \text{where} \quad M = \begin{pmatrix}\cosh l & \sinh l\\ \sinh l & \cosh l\end{pmatrix} \,. \tag{18}$$

Eigenvalues and eigenvectors are

$$\lambda_u = e^l \,, \quad \mathbf{e}_u = \begin{pmatrix}1\\ 1\end{pmatrix} \,, \quad \lambda_s = e^{-l} \,, \quad \mathbf{e}_s = \begin{pmatrix}1\\ -1\end{pmatrix} \,. \tag{19}$$

It is convenient to express the distance vector (in phase space) in terms of the eigenvectors

$$\begin{pmatrix}\rho\\ \sigma\end{pmatrix} = s\,\mathbf{e}_s + u\,\mathbf{e}_u \,, \quad \text{where} \quad s = s_0\,e^{-l} \,, \quad u = u_0\,e^l \,. \tag{20}$$

Here u and s are coordinates along the unstable and stable directions, respectively, and u_0 and s_0 are their initial values.

One sees that, in general, a neighbouring trajectory separates exponentially in positive and negative l-direction from the central trajectory. In between there is a closest encounter where the distance in phase space is minimal. If we define the distance as the square root of $\rho^2 + \sigma^2 = 2(s^2 + u^2)$ the closest encounter occurs when

$$0 = \frac{d}{dl}(s^2 + u^2) = -2s^2 + 2u^2 \,. \tag{21}$$

Solutions are either $u = s$, corresponding to $\rho = 2u$ and $\sigma = 0$, or $u = -s$, corresponding to $\rho = 0$ and $\sigma = -2u$. The orbits in figure 1 provide examples for these two cases. For the inner orbit the two segments become closest at the crossing where $\rho = 0$ and $\sigma = \varepsilon$. For the outer orbit the two segments become closest when they are parallel, i.e. $\sigma = 0$ and $\rho \neq 0$.

The encounter region in figure 1 is thus characterised by the fact that one orbit has an intersection with a small crossing angle ε, and the other orbit does not self-intersect. In the following we count the number of two-loop orbit pairs by counting the number of self-intersections with small crossing angles ε along periodic orbits. We thus assume that for every self-intersection with small ε there exists a partner periodic orbit without self-intersection. This is true in the linearised approximation [15], and is evident in symbolic dynamics [17]. The fact that orbits self-intersect either once or do not self-intersect in the encounter region is a special property of the uniformly hyperbolic dynamics. In general chaotic systems the two almost parallel segments of an orbit can, for example, cross several times before they separate again and the number of self-intersections does not agree with the number of orbit pairs [18, 19, 20].

In the remaining part of this lecture course we calculate the contribution of the two-loop orbit pairs to the form factor. This requires two main ingredients. One is the length difference of the two orbits that form a pair, and the other is the number of pairs. The latter is calculated in the next section.

The length difference can be evaluated by determining the length difference in the linearised approximation [15]. For Riemann surfaces there is a simpler way that uses hyperbolic geometry [17]. Consider a hyperbolic triangle whose three sides A, B and C are formed by geodesics. The sine-law in hyperbolic geometry has the form

$$\frac{\sin\alpha}{\sinh A} = \frac{\sin\beta}{\sinh B} = \frac{\sin\gamma}{\sinh C} \tag{22}$$

In figure 1 one can recognise four (approximate) hyperbolic triangles that are formed by the two orbits and the thin line that connects them at the crossing. Applying the sine-law with angles $\alpha = \pi/2$ $\beta = (\pi - \varepsilon)/2$, $\gamma = 0$ results in $\cos\varepsilon/2 = \sinh B/\sinh A \approx \exp(B - A)$. This leads to the following formula for the length difference that is valid for all angles ε

$$\Delta L \approx -4 \ln\cos\frac{\varepsilon}{2} \tag{23}$$

The error is exponentially small in the lengths of the loops.

3 The number of self-intersections

The aim of section 3 is to calculate the average number of self-intersections along arbitrary (non-periodic) trajectories of length L. The calculation can be done rigorously by using specific properties of Riemannian geometry. The intention is to transfer this result in section 4 to periodic orbits by using the uniform distribution of periodic orbits in phase space. The final result of this section can be obtained more quickly by using heuristic arguments such as in subsection 3.3 (see also e.g. [21]), but we want to avoid making any assumptions in this section.

We start by listing some properties of hyperbolic geometry that will be needed in the following. For details see the lectures of Buser [22].

We consider the upper-half plane model of the hyperbolic plane $\mathbb{H} = \{z = x + i\,y \,|\, y > 0\}$. In this model the line element and the volume element are given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2}\,, \qquad d\tilde{\mu} = \frac{dx\,dy}{y^2}\,, \tag{24}$$

where a tilde is used to distinguish $d\tilde{\mu}$ from the phase space measure that is introduced later.

Of particular importance are the orientation preserving isometries of the hyperbolic plane. In the upper half plane model they take the form of fractional linear transformations.

$$z \longmapsto \frac{a\,z + b}{c\,z + d}\,, \tag{25}$$

where the associated matrices are elements of the group

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \text{ real; } \det(g) = 1 \right\}\,. \tag{26}$$

Matrices which differ by an overall sign correspond to the same fractional linear transformation. Hence the group of orientation-preserving isometries can be identified with the group

$$\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm\mathbb{I}\}\,, \tag{27}$$

where $\mathbb{I}$ is the identity matrix.

Phase space (resp. a surface of constant energy in phase space) is represented by the unit tangent bundle of $\mathbb{H}$. It is parametrised by the coordinate $z = x + i\,y$ and an angle θ which describes the direction of motion at the point z.

The transformation of θ under the fractional linear transformation (25) is found by considering the transformation of an infinitesimally modified point $z + dz$. This leads to

$$\theta \longmapsto \theta - 2\arg(c\,z + d)\,. \tag{28}$$

The invariant measure on the unit tangent bundle is

$$d\mu = d\tilde{\mu}\,d\theta = \frac{dx\,dy\,d\theta}{y^2}\,. \tag{29}$$

3.1 Parametrisation of phase space

In the following a particular parametrisation of points on the unit tangent bundle is introduced which greatly simplifies the evaluation of coordinate transformations and time evolution. It exploits a one-to-one relationship that exists between elements of $\mathrm{PSL}(2,\mathbb{R})$ and points on the unit tangent bundle. This leads to a very convenient method to evaluate transition probabilities in phase space which will be used to find the number of self-intersections.

Consider a matrix $g \in \mathrm{SL}(2,\mathbb{R})$ with elements g_{ij}. It can be uniquely written in the form

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix} , \tag{30}$$

where $z = x + i\,y \in \mathbb{H}$, and $\theta \in [0, 4\pi)$. The relation between x, y, θ and the matrix elements of g is given by

$$x = \frac{g_{11}\,g_{21} + g_{12}\,g_{22}}{g_{21}^2 + g_{22}^2} , \quad y = \frac{1}{g_{21}^2 + g_{22}^2} , \quad \theta = -2\arg(g_{22} + i\,g_{21}) . \tag{31}$$

The matrix g changes by an overall sign if θ changes by 2π. Hence by identifying θ-values which differ by 2π we obtain a one-to-one relationship between elements of $\mathrm{PSL}(2,\mathbb{R})$ and points on the unit tangent bundle.

The identification of a point (z, θ) with an element in $\mathrm{PSL}(2,\mathbb{R})$ has further advantages. Consider the following matrix product

$$g' := \begin{pmatrix} a & b \\ c & d \end{pmatrix} g = \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} {y'}^{1/2} & 0 \\ 0 & {y'}^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta'/2 & \sin\theta'/2 \\ -\sin\theta'/2 & \cos\theta'/2 \end{pmatrix} . \tag{32}$$

A short calculation shows that

$$z' = x' + iy' = \frac{a\,z + b}{c\,z + d} , \quad \theta' = \theta - 2\arg(c\,z + d) . \tag{33}$$

These transformation rules are identical to those in equations (25) and (28). This means that the action of an isometry on a point in the unit tangent bundle is simply given by a matrix multiplication in this representation. Note that the point $(z, \theta) = (i, 0)$ is represented by the identity matrix I. Hence a general point (z, θ) is represented by the matrix g that corresponds to the isometry which maps $(i, 0)$ onto (z, θ).

As a consequence the time evolution also simplifies to a matrix multiplication. The point $(i, 0)$ evolves in time t (with unit speed) to

$$(i, 0) \quad \longmapsto \quad (i\,e^t, 0) \quad \hat{=} \quad \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} . \tag{34}$$

The time evolution of a general point is obtained by applying the isometry g

$$g \longmapsto g\,\Phi^t \ , \quad \text{where} \quad \Phi^t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \ . \tag{35}$$

We will avoid using the time variable t in the following, because it is convenient to express all quantities in terms of the lengths of trajectories. We will represent time t by the length l that is covered during the time evolution.

Finally, we need delta-functions on the unit-tangent bundle. They are introduced by

$$\int \delta_0(g', g)\, f(g')\, d\mu(g') = f(g) \ , \qquad d\mu(g') = \frac{dx'\, dy'\, d\theta'}{{y'}^2} \ . \tag{36}$$

where $f(g)$ is here a continuous function of g. The delta-function is symmetric and invariant under the application of isometries

$$\delta_0(g, g') = \delta_0(g', g) = \delta_0(h\, g', h\, g) = \delta_0(g'\, h, g\, h) \ . \tag{37}$$

It can be factorised in the form

$$\delta_0(g', g) = y^2\, \delta(x - x')\; \delta(y - y')\; \delta_p(\theta - \theta') \tag{38}$$

where δ_p denotes the 2π-periodic delta-function.

So far we have considered the motion on the full hyperbolic plane, but we are interested in the motion on compact smooth Riemann surfaces. These surfaces are represented by a compact area on the hyperbolic plane with the property that one can tessellate the full hyperbolic plane with copies of it. They are associated with a discrete subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$ whose elements map one copy onto all other copies. For a compact smooth Riemann surface this subgroup contains only hyperbolic elements (with the exception of the identity). The hyperbolic elements correspond to boosts which are analogs of translations in the Euclidean plane and are characterised by $|\operatorname{Tr} g| > 2$. The boosts in Γ cannot be arbitrarily small, in mathematical terms Γ is a strictly hyperbolic Fuchsian group [22]. A delta-function on a Riemann surface is then defined by periodising the original delta-function by summing over all copies

$$\delta(g, g') = \sum_{\gamma \in \Gamma} \delta_0(\gamma\, g, g') \ . \tag{39}$$

3.2 The average number of self-intersections

Consider an arbitrary trajectory of length L. To each self-intersection of this trajectory there is a corresponding loop which starts and ends at the self-intersection. In the following l denotes the length of the loop, ε its opening angle, and l' the distance along the trajectory up to the starting point of the loop.

The average number of self-intersections with intersection angle in an interval $[\varepsilon, \varepsilon + d\varepsilon]$ along trajectories of length L is then given by $P(\varepsilon, L)\, d\varepsilon$ where the density of self-intersections $P(\varepsilon, L)$ is

$$P(\varepsilon, L) = \frac{1}{\Sigma} \int d\mu(g) \int_0^L dl \int_0^{L-l} dl' \; \delta(g\Phi^{l'} g_\varepsilon, g\Phi^{l'+l}) \, |\sin\varepsilon| \; . \tag{40}$$

Here the average is taken over all initial conditions on the unit tangent bundle, and $\Sigma = \int d\mu(g) = 2\pi A$. The matrix g_ε rotates the θ-coordinate by $\varepsilon - \pi$.

$$g_\varepsilon = \begin{pmatrix} \cos\frac{\varepsilon-\pi}{2} & \sin\frac{\varepsilon-\pi}{2} \\ -\sin\frac{\varepsilon-\pi}{2} & \cos\frac{\varepsilon-\pi}{2} \end{pmatrix} = \begin{pmatrix} \sin\frac{\varepsilon}{2} & -\cos\frac{\varepsilon}{2} \\ \cos\frac{\varepsilon}{2} & \sin\frac{\varepsilon}{2} \end{pmatrix} \; . \tag{41}$$

Equation (40) can be understood in the following way: $g\Phi^{l'}$ is the starting point of a loop, and the integral gives a contribution if after an evolution of length l the trajectory arrives at a point in phase space that is obtained from the starting point by a rotation by $\varepsilon - \pi$. The $|\sin\varepsilon|$-term in (40) arises from a Jacobian that is necessary in order that the l- and l'- integrals correctly count the number of self-intersections. More accurately, with an additional integral over some ε-interval $\Delta\varepsilon$, the l- and l'-integrals yield a one for any self-intersection with intersection angle in $\Delta\varepsilon$. The origin of the $|\sin\varepsilon|$-term is most easily seen by considering the transformation from the arguments of the delta-function in x- and y- coordinates, $x(l+l') - x(l')$ and $y(l+l') - y(l')$ to the l- and l'- coordinates with the Jacobian

$$|J| = \left| \frac{dx(l+l')}{dl} \frac{dy(l')}{dl'} - \frac{dy(l+l')}{dl} \frac{dx(l')]}{dl'} \right| \; . \tag{42}$$

This can be recognised as cross-product of two direction-vectors with length y, hence $|J| = y^2 |\sin\varepsilon|$. (The y^2 term of J is obtained by the factorisation of the delta-function.)

After interchanging integrals and changing the integration variable $g\Phi^{l'} \mapsto g$ one obtains

$$P(\varepsilon, L) = \int_0^L dl \; (L - l) \, p(\varepsilon, l) \, |\sin\varepsilon| \; , \tag{43}$$

where $p(\varepsilon, l)$ is defined as

$$p(\varepsilon, l) = \frac{1}{\Sigma} \int d\mu(g) \; \delta(g \, g_\varepsilon, g \, \Phi^l) \; . \tag{44}$$

It can be interpreted as the probability density to form a loop with opening angle ε and length l. It is different from zero only if is there is a trajectory of length l which forms a loop with opening angle ε.

In the following it will be shown that $p(\varepsilon, l)$ and hence $P(\varepsilon, L)$ can be expressed in terms of the periodic orbits of the Riemann surface. A substitution $g \mapsto g \, g_\varepsilon^{-1}$ leads to

$$p(\varepsilon, l) = \frac{1}{\Sigma} \int d\mu(g) \; \delta(g, g \, g_\varepsilon^{-1} \, \Phi^l) \, , \tag{45}$$

where

$$g_\varepsilon^{-1}\,\Phi^l = \begin{pmatrix} e^{l/2}\sin\frac{\varepsilon}{2} & e^{-l/2}\cos\frac{\varepsilon}{2} \\ -e^{l/2}\cos\frac{\varepsilon}{2} & e^{-l/2}\sin\frac{\varepsilon}{2} \end{pmatrix} . \tag{46}$$

The eigenvalues of this matrix are real with modulus $|\lambda| = e^{\pm\tilde{L}/2}$ where

$$\cosh\frac{\tilde{L}}{2} = \frac{1}{2}\left|\mathrm{Tr}\, g_\varepsilon^{-1}\,\Phi^l\right| = \cosh\frac{l}{2}\,\sin\left|\frac{\varepsilon}{2}\right| , \qquad -\pi < \varepsilon \leq \pi . \tag{47}$$

The matrix $g_\varepsilon^{-1}\,\Phi^l$ can be diagonalised by a similarity transformation $g_\varepsilon^{-1}\,\Phi^l = h^{-1}\,\Phi^{\tilde{L}}\,h$ and we find

$$\begin{aligned} p(\varepsilon, l) &= \frac{1}{\Sigma}\int \delta(g, g\,h^{-1}\Phi^{\tilde{L}}h)\; d\mu(g) \\ &= \frac{1}{\Sigma}\int \delta(g\,h^{-1}, g\,h^{-1}\Phi^{\tilde{L}})\; d\mu(g) \\ &= \frac{1}{\Sigma}\int \delta(g, g\,\Phi^{\tilde{L}})\; d\mu(g)\,. \end{aligned} \tag{48}$$

The last expression is identical to $p(\pi, \tilde{L})$, the probability density to return in phase space. It is different from zero only if there exists a periodic orbit γ with length $L_\gamma = \tilde{L}$.

The above calculation shows that there is a unique relationship between a loop of length l and opening angle ε and a periodic orbit of length $\tilde{L}$ given by (47). Geometrically this signifies that any loop can be continuously deformed into a periodic orbit [16]. This relationship is not one-to-one, but there is a one-parameter family of loops with the same length l and angle ε whose initial points lie on a closed curve which are related by (47) to a periodic orbit..

The relationship between a transition probability density in phase space and periodic orbits is not restricted to loops, but holds more general. In order to see this it is convenient to introduce a second parametrisation of $\mathrm{PSL}(2,\mathbb{R})$. A matrix g can be uniquely decomposed into

$$g = \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \begin{pmatrix} e^{r/2}\,(1+u\,s) & e^{r/2}\,s \\ e^{-r/2}\,u & e^{-r/2} \end{pmatrix} . \tag{49}$$

Using (31) one finds

$$y = \frac{e^r}{u^2+1}\ , \quad x = e^r\left(\frac{u}{u^2+1} + s\right)\ , \quad \theta = -2\arg(1+i\,u)\ . \tag{50}$$

Let us denote the particular form of the matrix on the right-hand side of equation (49) by g_{rsu}. The interpretation of the coordinates r, s and u can be understood by letting the time-evolution matrix act on it

$$g_{rsu}\begin{pmatrix} e^{l/2} & 0 \\ 0 & e^{-l/2} \end{pmatrix} = \begin{pmatrix} e^{(r+l)/2} & 0 \\ 0 & e^{-(r+l)/2} \end{pmatrix} \begin{pmatrix} 1 & s\,e^{-l} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u\,e^{l} & 1 \end{pmatrix} . \tag{51}$$

Hence r is a coordinate along a trajectory, and s and u are coordinates on the stable and unstable manifolds, respectively. For infinitesimal distances s and u agree with the previously introduced coordinates in section 2.2. The phase space measure is given by

$$\frac{dx\, dy\, d\theta}{y^2} = 2\, dr\, ds\, du\ . \tag{52}$$

Analogous to the calculation for $p(\varepsilon, l)$ one can express also the probability density for an arbitrary transition in phase space in terms of periodic orbits.

$$\hat{p}(r,s,u,l) = \int \delta(g\, g_{rsu}, g\Phi^l)\, d\mu(g) = \int \delta(g, g\,\Phi^{\tilde{L}})\, d\mu(g) = p(\pi, \tilde{L})\,, \tag{53}$$

where

$$g_{rsu}^{-1}\,\Phi^l = \begin{pmatrix} e^{-(r-l)/2} & -s\, e^{(r-l)/2} \\ -u\, e^{(-r-l)/2} & (1+u\,s)\, e^{(r-l)/2} \end{pmatrix} = h^{-1}\,\Phi^{\tilde{L}}\, h\,. \tag{54}$$

Now

$$\cosh\frac{\tilde{L}}{2} = \left|\cosh\frac{r-l}{2} + \frac{u\,v}{2}\exp\frac{r-l}{2}\right|\ . \tag{55}$$

This means that the transition probability density $\hat{p}(r,s,u,l)$ can be expressed in terms of the probability density to return $p(\pi, \tilde{L})$ and hence in terms of periodic orbits.

The remaining step for the evaluation of $p(\varepsilon, l)$ consists of an evaluation of the probability density to return

$$p(\pi, \tilde{L}) = \frac{1}{\Sigma}\int d\mu(g)\ \delta(g, g\Phi^{\tilde{L}}) = \frac{1}{\Sigma}\int d\mu(g) \sum_{\gamma\in\Gamma} \delta_0(\gamma\, g, g\Phi^{\tilde{L}})\,. \tag{56}$$

The steps in this calculation follow very closely those in the derivation of the trace formula [12] and we give here only the final result. The sum over the elements in γ is split into a sum over all conjugacy classes and a sum over all elements within each conjugacy class. The sum over the elements within a conjugacy class is used to replace the integral over the area of the surface by an integral over a strip in the full hyperbolic plane. The final integral over the delta- function is conveniently done in local coordinates r, s and u. The result is

$$p(\pi, \tilde{L}) = \frac{1}{\Sigma}\sum_\gamma \frac{L_\gamma\, \delta(\tilde{L} - L_\gamma)}{R_\gamma\,(2\sinh L_\gamma/2)^2}\,, \tag{57}$$

where the sum runs over all periodic orbits γ with length L_γ and repetition number R_γ.

Finally, by using the relation between loop length and periodic orbit length

$$\cosh\frac{l_\gamma(\varepsilon)}{2}\ \sin\left|\frac{\varepsilon}{2}\right| = \cosh\frac{L_\gamma}{2}\ , \tag{58}$$

we arrive at the following expression for $p(\varepsilon, l) = p(\pi, \tilde{L})$

$$p(\varepsilon, l) = \frac{1}{\Sigma} \sum_{\gamma} B_\gamma(\varepsilon)\, \delta(l - l_\gamma(\varepsilon))\,, \tag{59}$$

where

$$B_\gamma(\varepsilon) = \frac{L_\gamma}{4R_\gamma\, \sinh L_\gamma/2\, \sqrt{\sinh^2 L_\gamma/2 + \cos^2 \varepsilon/2}}\,. \tag{60}$$

The integration in (43) yields the result

$$P(\varepsilon, L) = \frac{|\sin\varepsilon|}{2\pi A} \sum_{l_\gamma(\varepsilon) < L} (L - l_\gamma(\varepsilon))\; B_\gamma(\varepsilon) \tag{61}$$

for the average density of self-intersections with angle ε along trajectories of length L.

3.3 Asymptotic expansion of the density of self-intersections

In this section the asymptotic behaviour of $P(\varepsilon, L)$ for large L is determined. But before we do this let us look at the relation between loop length and periodic orbit length in (58) in more detail. For large L_γ this simplifies to

$$l_\gamma(\varepsilon) \sim L_\gamma - 2 \log \sin \frac{|\varepsilon|}{2}\,, \tag{62}$$

and for small angles

$$l_\gamma(\varepsilon) \sim L_\gamma - 2 \log \frac{|\varepsilon|}{2}\,. \tag{63}$$

The logarithmic divergence in (63) has a simple physical interpretation. Consider a loop with a very small angle ε, and consider the two legs of the loop as two trajectories with initial conditions that differ by a small ε. The separation of the two legs during time evolution is governed by the unit Lyapunov exponent, and it has to be at least of order one for the two legs to be able to meet and form a closed loop. This leads to an estimate for the minimum length that a loop with angle ε must have

$$|\varepsilon|\, \exp(l_{min}/2) = \mathcal{O}(1)\,, \tag{64}$$

and $l_{min} \sim -2 \log c\varepsilon$ for some constant c in agreement with (63). Hence the logarithmic divergence of the loop length is due to the existence of a minimal loop length for small ε.

Now we go back to $P(\varepsilon, L)$ and use (61) to determine its asymptotic behaviour for large L. We will need not only the leading order term, but the next-to-leading order term as well. For this purpose the sum over periodic orbits is split into a sum $L_\gamma < L^*$ and a sum $L_\gamma \geq L^*$. The first sum is evaluated exactly, whereas the second sum is evaluated asymptotically by using the prime geodesic theorem (12).

$$\begin{aligned} P(\varepsilon, L) &= \frac{\sin|\varepsilon|}{2\pi A} \sum_{l_\gamma(\varepsilon)<L} B_\gamma(\varepsilon)\,(L - l_\gamma(\varepsilon)) \\ &\sim \frac{\sin|\varepsilon|}{2\pi A} \sum_{L_\gamma<L_*} B_\gamma(\varepsilon)\,(L - l_\gamma(\varepsilon)) \\ &\quad + \frac{\sin|\varepsilon|}{2\pi A} \int_{L_*}^{L+2\log\sin|\varepsilon|/2} dL' \left(L - L' + 2\log\sin\frac{|\varepsilon|}{2}\right) \\ &\sim \frac{\sin|\varepsilon|}{2\pi A} \sum_{L_\gamma<L_*} B_\gamma(\varepsilon)\,(L - l_\gamma(\varepsilon)) \\ &\quad + \frac{L^2}{4\pi A} \sin|\varepsilon| \left(1 - \frac{2L^*}{L} + \frac{4}{L}\log\sin\frac{|\varepsilon|}{2}\right) + \mathcal{O}(1) \,. \end{aligned} \tag{65}$$

Note that the correction to the prime geodesic theorem is exponentially small as $L \to \infty$ and can be neglected. One can let L^* go to infinity as well as $L \to \infty$ and obtains the following two terms in the asymptotic expansion of $P(\varepsilon, L)$ for $L \to \infty$

$$P(\varepsilon, L) \sim \frac{L^2}{4\pi A} \sin|\varepsilon| \left(1 + \frac{2C(\varepsilon)}{L} + \frac{4}{L}\log\sin\frac{|\varepsilon|}{2}\right) \,, \tag{66}$$

where

$$C(\varepsilon) = \lim_{L_*\to\infty} \left(\sum_{L_\gamma<L_*} B_\gamma(\varepsilon) - L_* \right) \,. \tag{67}$$

The limit in (67) converges uniformly in ε.

As will become clear later we actually need the asymptotic behaviour in the joint limit $L \to \infty$ and $\varepsilon \propto L^{-1/2} \to 0$. This can be obtained from (66), further corrections do not contribute

$$P(\varepsilon, L) \sim \frac{L^2}{4\pi A} \sin|\varepsilon| \left(1 + \frac{4}{L}\log c|\varepsilon|\right) \,, \tag{68}$$

where $c = \frac{1}{2}\exp(C(0)/2)$.

4 Contribution to the form factor

In the following we use the information about $P(\varepsilon, L)$ from the last section in order to evaluate the contribution of the pairs of two-loop orbits to the form factor in (9). Let us look at the leading order term as $L \propto p \to \infty$. The orbits have a length difference ΔL given by (23). As a consequence the amplitudes of the orbits differ slightly too, however this difference contributes to the next-to-leading order (see below). The number of orbit pairs are counted by counting the self-intersections of periodic orbits. As will be seen below, only self-intersections with small ε contribute. We are led to

$$K_{\mathrm{sc}}^{(2)}(\tau) \sim \sqrt{\frac{2}{\pi}} \frac{4\,\Delta p}{L_H} \operatorname{Re} \int d\varepsilon \sum_{\gamma} A_\gamma^2 e^{ip\Delta L(\varepsilon)}\, e^{-2\Delta p^2 (L-L_\gamma)^2}\, P_\gamma(\varepsilon) \qquad (69)$$

where $P_\gamma(\varepsilon)\, d\varepsilon$ is the number of self-intersections along the orbit γ with angle in $[\varepsilon, \varepsilon + d\varepsilon]$.

Long periodic orbits are uniformly distributed on the unit tangent bundle. As a consequence, averages over periodic orbits can be replaced by averages over the unit tangent bundle in the asymptotic limit of long orbit lengths. This property is applied in order to effectively replace the average number of self-intersections along periodic orbits by the average number of self-intersections along non-periodic trajectories within the integral in (69)

$$\sum_{\gamma} A_\gamma^2\, P_\gamma(\varepsilon) \to \int dL'\; L'\; P(L', \varepsilon) \qquad (70)$$

Doing this we arrive at

$$\begin{aligned} K_{\mathrm{sc}}^{(2)}(\tau) &\sim \sqrt{\frac{2}{\pi}} \frac{4\,\Delta p}{L_H} \operatorname{Re} \int d\varepsilon \int dL'\, e^{ip\Delta L(\varepsilon)}\, e^{-2\Delta p^2 (L-L')^2}\, L'\, P(L', \varepsilon) \\ &\sim \frac{2L^3}{\pi p A^2} \operatorname{Re} \int_0^\infty d\varepsilon\, e^{ip\varepsilon^2/2}\, \sin\varepsilon \\ &\sim \frac{2\tau^3}{\pi} pA \operatorname{Re} i \end{aligned}$$

where we evaluated the integral over L' by the method of steepest decent, using the leading order approximation of $P(L, \varepsilon)$ in (66), and the integral over ε in stationary phase approximation. In this step $\sin\varepsilon$ was replaced by ε. First we notice that the important contribution to the ε-interval comes from an interval of order $1/\sqrt{p}$. This means that we have to consider the joint limit $L \propto p \propto \varepsilon^{-2} \to \infty$ instead of a limit where ε is considered constant.

Second, assuming that (70) is still valid under the new limit, we see that the leading order contribution vanishes. Hence one has to consider the next-to-leading order contribution to $K^{(2)}(\tau)$ in the semiclassical limit. This arises from several corrections: the next-to-leading order term for $P(L, \varepsilon)$, the difference in the amplitudes of the orbits, the next-to-leading order correction to $\Delta L(\varepsilon)$, and the next term in the expansion of $\sin\varepsilon$ for small ε. Looking at the L-dependence of the contributions one sees that only the first contribution contributes to a τ^2 term whereas the other three contribute to a τ^3 term of the form factor.

Let us consider the first correction. We would like to use the uniform distribution of periodic orbits in order to conclude that the number of self-intersections along periodic orbits has the same asymptotic behaviour (68) as that along non-periodic trajectories. However, the argument of the uniform distribution can only applied to obtain the leading term in a large L expansion, and not the next-to-leading order term for a joint large L small ε expansion. We assume here that periodic and non-periodic trajectories have a similar asymptotic behaviour in the considered limit, and that we can effectively make the replacement

$$\sum_\gamma A_\gamma^2 \, P_\gamma(\varepsilon) \to \int dL' \; L' \, \frac{L'^2}{4\pi A} \sin|\varepsilon| \, \left(1 + \frac{4}{L'} \log \tilde{c}|\varepsilon|\right) \tag{71}$$

where the constant $\tilde{c}$ may possibly be different from the constant in the case of non-periodic trajectories. A justification is that the $\log|\varepsilon|$ term originates from the minimum loop length that exists for both, periodic and non-periodic trajectories.

Inserting (71) into (69) and following the same steps as before we arrive at

$$K_{\mathrm{sc}}^{(2)}(\tau) \sim \frac{8L^2}{\pi p A^2} \, \mathrm{Re} \int_0^\infty d\varepsilon \, e^{ip\varepsilon^2/2} \, \varepsilon \, \log(\tilde{c}\varepsilon) \tag{72}$$

The integral can be evaluated by substituting $\varepsilon = e^{i\pi/4} \, \varepsilon'$ and rotating the integration contour by $\pi/4$. This leads to

$$K_{\mathrm{sc}}^{(2)}(\tau) \sim -\frac{8L^2}{\pi p A^2} \, \mathrm{Im} \int_0^\infty d\varepsilon' \, e^{-p\varepsilon'^2/2} \, \varepsilon' \, \log(\tilde{c}\varepsilon' e^{i\pi/4}) = -2\tau^2 \tag{73}$$

This result agrees with the second term in the Taylor expansion of the GOE form factor (3) at $\tau = 0$.

Finally let us look at the other three corrections that contribute to a τ^3-term. Using the definition of the amplitude (5), the next-to-leading term of the length difference (23) for small ε, and the next-to-leading term in the Taylor expansion of $\sin\varepsilon$ we obtain

$$\begin{aligned} &\frac{2L^3}{\pi p A^2} \, \mathrm{Re} \int_0^\infty d\varepsilon \, e^{ip(\varepsilon^2/2+\varepsilon^4/48)} \, (1 + \frac{1}{4}\varepsilon^2)(\varepsilon - \frac{1}{6}\varepsilon^3) \\ \approx &\frac{2L^3}{\pi p A^2} \, \mathrm{Re} \int_0^\infty d\varepsilon' \, e^{ip\varepsilon'^2/2} \, \varepsilon'(1 + \mathcal{O}(\varepsilon'^4)) \end{aligned} \tag{74}$$

where a substitution $\varepsilon' = \varepsilon + \varepsilon^3/48$ has been made. The different corrections of relative order ε'^2 cancel each other and hence the term (74) vanishes in the semiclassical limit [23]. So although each of the three corrections contributes to a τ^3 term their joint contribution vanishes. In summary, we find that the two-loop orbit pairs contribute a τ^2 term to the spectral form factor that is in agreement with random matrix theory.

5 Discussion

In these notes we have discussed the simplest form of correlations between different periodic orbits on compact Riemann surfaces. It was shown that these orbit pairs are responsible for a τ^2-term of the spectral form factor which is in agreement with the τ^2-term of the GOE form factor of random matrix theory. Specific properties of Riemann surfaces were used in order to arrive at this result. Many of the steps in the derivation can be done quicker by using heuristic arguments, however, a main emphasis was on using rigorous methods. One remaining assumption is that the large L small ε asymptotics of the number of self-intersections has the same form for general trajectories and periodic orbits (see equation (71)).

During recent years there have been a number of developments (some of them after this lecture series was given) which extend the present results in several directions.

— For quantum graphs the τ^2 and τ^3 terms of the form factor were derived semiclassically and shown to be in agreement with random matrix theory [24, 25, 26]. Recently it was proved by a different method that uses a supersymmetric σ-model that the full form factor of individual quantum graphs does indeed agree with random matrix theory [27, 28].
— The transport through an open chaotic cavity was investigated in [29]. Here one has to consider correlations between trajectories that go from an entry lead to an exit lead of the cavity. It was shown that the leading-order off-diagonal terms give a weak-localisation correction to the conductance which is in quantitative agreement with results from random matrix theory. This result was generalised to all higher-order terms in [30], and to the treatment of shot noise in [31].
— Generalisations of periodic-orbit correlations to non-uniformly hyperbolic systems were considered in [18, 19, 20]. One main difference to the uniformly hyperbolic case is that self-intersections are not appropriate for counting the number of orbit pairs, because there is no one-to-one correspondence between self-intersections with small ε and pairs or periodic orbits. Instead one considers encounter regions with a finite length.
— Universality classes for systems with spin were treated in [32, 33, 34], and transitions between universality classes in [18, 35, 36].
— Higher dimensions and fluctuations of matrix elements were considered in [37].
— Higher order terms in the Taylor expansion of the form factor for general chaotic systems were calculated semiclassically in the following articles: the τ^3 term was obtained in [21] and, finally, all higher orders of τ in [38, 39].

One main remaining problem is to show that other pairs of periodic orbits, which have been omitted in the evaluation of off-diagonal contributions, do indeed not contribute to the form factor in the semiclassical limit.

Another mayor problem is to obtain the form factor semiclassically in the regime beyond its singularity, which in the GOE-case occurs at $\tau = 1$. In this region the form factor has a different functional form, see (3). Possibly this requires to identify a further kind of periodic orbit correlations. A first step to evaluate the form factor in this regime was taken in [40].

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