LIMIT THEOREMS FOR STOCHASTIC PROCESSES ASSOCIATED WITH A BOSON GAS

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§1 INTRODUCTION

In this lecture, we discuss the density of particles having energy less than λ in a boson system as a stochastic process indexed by λ. The notation is that of [1] in this volume. Recall that the hamiltonian for the free boson gas is given by

$$H_\ell(\omega) = \sum_{j\geq 1} \lambda_\ell(j)\sigma_j(\omega), \tag{1.1}$$

where $0 = \lambda_\ell(1) \leq \lambda_\ell(2) \leq \ldots$. For a system in a region of volume V_ℓ, the grand canonical pressure $p_\ell(\mu)$ is defined for $\mu < 0$ by

$$p_\ell(\mu) = \frac{1}{\beta V_\ell} \ln\left\{\sum_{\omega\in\Omega} e^{\beta(\mu N(\omega) - H_\ell(\omega))}\right\}. \tag{1.2}$$

In [2] in this volume, we recalled results (proved in [3]) on the existence of the pressure in the thermodynamic limit:

$$p(\mu) = \lim_{\ell\to\infty} p_\ell(\mu). \tag{1.3}$$

In order to discuss the phenomenon of boson condensation, we introduced in [3] the family of random variables $\{X_\ell(\cdot;\lambda):\lambda \geq 0\}$ defined by

$$X_\ell(\omega;\lambda) = \frac{1}{V_\ell} \sum_{\{j:\lambda_\ell(j)\leq\lambda\}} \sigma_j(\omega). \tag{1.4}$$

For the free boson gas, we have the following result:

THEOREM 1

Suppose that (S1) *and* (S2) *hold; then, for* ρ_c *finite,*

$$\lim_{\lambda\downarrow 0} \lim_{\ell\to\infty} \mathbb{E}_\ell^\rho[X_\ell(\lambda)] = (\rho - \rho_c)^+. \tag{1.5}$$

[Conditions (S1) and (S2) and the critical density ρ_c are defined in §2 and §3 of [2] in this volume. Here $\mathbb{E}_\ell^\rho[\cdot]$ denotes the expectation taken with respect to the grand canonical probability measure $\mathbb{P}_\ell^\mu[\cdot]$ with $\mu = \mu_\ell(\rho)$, defined in §3 of

[2]; it is the expectation at fixed mean density ρ.]

Proof:

From the definition of $X_\ell(\lambda)$, we have

$$\mathbb{E}^\rho_\ell[X_\ell(\lambda)] = \int_{[0,\lambda)} p'(\mu_\ell(\rho)|\lambda)dF_\ell(\lambda) = \rho - \int_{[\lambda,\infty)} p'(\mu_\ell(\rho)|\lambda)dF_\ell(\lambda). \tag{1.6}$$

But, for $\mu < \lambda$, the sequence

$$\left\{ \int_{[\lambda,\infty)} p'(\mu|\lambda)dF_\ell(\lambda) : \ell = 1,2,\ldots \right\} \tag{1.7}$$

converges uniformly in μ on compacts to

$$\int_{[\lambda,\infty)} p'(\mu|\lambda)dF(\lambda). \tag{1.8}$$

Hence, by Proposition 2 of [2], we have for $\lambda > 0$:

$$\lim_{\ell\to\infty} \mathbb{E}^\rho_\ell[X_\ell(\lambda)] = \rho - \int_{[\lambda,\infty)} p(\mu(\lambda)|\lambda)dF(\lambda). \tag{1.9}$$

But, by hypothesis, ρ_c is finite so that we may invoke the dominated convergence principle to conclude that

$$\lim_{\lambda\downarrow 0} \int_{[\lambda,\infty)} p'(\mu(\rho)|\lambda)dF(\lambda) = \int_{[0,\infty)} p'(\mu(\rho)|\lambda)dF(\lambda) = \begin{cases} \rho\ , & \rho < \rho_c, \\ \rho_c\ , & \rho \geqslant \rho_c. \end{cases}$$

Thus we have

$$\lim_{\ell\to\infty} \mathbb{E}^\rho_\ell[X_\ell(\lambda)] = (\rho - \rho_c)^+ \quad \blacksquare$$

In the free boson gas there is a second effect, discovered by M. Kac in 1971. We saw in §3 of [2] that the free-energy has a first-order phase-transition segment $[\rho_c,\infty)$; it follows that for $\rho > \rho_c$ there is no guarantee that the weak law of large numbers will hold for the distribution $\mathbb{K}^\rho_\ell = \mathbb{P}^\rho_\ell \circ X_\ell^{-1}$ of the number density $X_\ell = N/V_\ell$. In fact, there is no guarantee that for, $\rho > \rho_c$, the sequence $\{\mathbb{K}^\rho_\ell : \ell = 1,2,\ldots\}$ will converge; nevertheless, by the Helly Selection Principle, a subsequence will converge, but the limit distribution will depend on the detailed behaviour of the corresponding subsequence of the sequence $\{\lambda_\ell(\cdot) : \ell = 1,2,\ldots\}$. In other words, it is possible to have two sequences, $\{\lambda_\ell(\cdot) : \ell = 1,2,\ldots\}$ and $\{\hat{\lambda}_\ell(\cdot) : \ell = 1,2,\ldots\}$, each satisfying (S1) and (S2) and having the same integrated density of states $F(\cdot)$ but having limit distributions $\mathbb{K}^\rho$ and $\hat{\mathbb{K}}^\rho$ which are

distinct for $\rho > \rho_c$. (For $\rho < \rho_c$, they must both be equal to δ_ρ, the degenerate distribution concentrated at ρ, by Theorem 1 of [1].) For example, Kac showed that in the standard example (described in §3 of [2]) the limit distribution is the exponential distribution supported on $[\rho_c, \infty)$ with mean ρ, for $\rho > \rho_c$; other examples are investigated in detail in [3]. We shall see in the next section that, in the mean-field model, this phenomenon disappears: there is no first-order phase-transition segment, the grand canonical pressure exists for all values of μ and is a differentiable function; the weak law of large numbers holds for X_ℓ for all values of the mean density ρ; nevertheless, condensation persists. In these circumstances it becomes interesting to regard $\lambda \longrightarrow X_\ell(\cdot;\lambda)$ as a stochastic process and to enquire about the convergence in distribution of a re-scaled, centred version of it. This we do in §3.

§2 THE MEAN FIELD-MODEL

To describe the mean-field model, we define a sequence of hamiltonians $\{\tilde{H}_\ell : \ell = 1,2,\ldots\}$ by

$$\tilde{H}_\ell(\omega) = H_\ell(\omega) + \frac{a}{2V_\ell} N^2(\omega) \tag{2.1}$$

with $a > 0$. The term $\frac{a}{2V_\ell} N^2$, which provides a crude caricature of the interaction, can be understood classically: it arises in an "index of refraction" approximation in which we imagine each particle to move through the system as if it were moving in a uniform optical medium and so receiving an increment of energy proportional to the density $X_\ell = N/V_\ell$; since a is positive the interaction is repulsive.

First, we compute the pressure $\tilde{p}_\ell(\mu)$, as explained in §4 of [1]: writing $u(x) = (\mu - \alpha)x - \frac{a}{2}x^2$, a straight-forward manipulation gives

$$\tilde{p}_\ell(\mu) = p_\ell(\alpha) + \frac{1}{\beta V_\ell} \ell n \, \mathbb{E}_\ell^\alpha[e^{\beta V_\ell u(X_\ell)}] = p_\ell(\alpha) + \frac{1}{\beta V_\ell} \ell n \int_{[0,\infty)} e^{\beta V_\ell u(x)} \mathbb{K}_\ell^\alpha[dx] \tag{2.2}$$

for each $\alpha < 0$, where $\mathbb{K}_\ell^\alpha = \mathbb{P}_\ell^\alpha \circ X_\ell^{-1}$. But $x \longrightarrow u(x)$ is continuous and bounded above and $\{\mathbb{K}_\ell^\alpha : \ell = 1,2,\ldots\}$ satisfies the Large Deviation Principle with rate-function $I^\alpha(x) = p(\alpha) + f(x) - \alpha x$, by Theorem 1 of [2]. Hence, by Varadhan's First Theorem, $\tilde{p}(\mu) = \lim_{\ell \to \infty} \tilde{p}_\ell(\mu)$ exists and is given by

$$\tilde{p}(\mu) = p(\alpha) + \sup_x \{u(x) - I^\alpha(x)\} = \sup_x \{\mu x - \tilde{f}(x)\}, \tag{2.3}$$

where the mean-field free-energy $\tilde{f}(\cdot)$ is given by

$$\tilde{f}(x) = f(x) + \frac{a}{2}x^2. \tag{2.4}$$

Thus we have proved:

THEOREM 2

Suppose that (S1) *and* (S2) *hold; then the mean-field pressure exists for all real* μ *and is given by*

$$\tilde{p}(\mu) = \sup_x\{\mu x - \tilde{f}(x)\}, \tag{2.5}$$

where $x \longrightarrow \tilde{f}(x)$ *is the mean-field free energy, given by* $\tilde{f}(x) = f(x) + \frac{a}{2}x^2$.

Next, we introduce the mean-field expectation functional $\tilde{\mathbb{E}}^{\mu}_{\ell}[\cdot]$ defined by

$$\tilde{\mathbb{E}}^{\mu}_{\ell}[\cdot] = \mathbb{E}^{\alpha}_{\ell}[\cdot e^{\beta M_{\ell}}]/\mathbb{E}^{\alpha}_{\ell}[e^{\beta M_{\ell}}], \tag{2.6}$$

and the associated probability measure $\tilde{\mathbb{P}}^{\alpha}_{\ell}[\cdot]$, where

$$M_{\ell} = V_{\ell}u(X_{\ell}). \tag{2.7}$$

COROLLARY

The mean-field pressure $\mu \longrightarrow \tilde{p}(\mu)$ *is differentiable for all values of* μ. *The sequence of distribution functions* $\{\tilde{\mathbb{K}}^{\mu}_{\ell} = \tilde{\mathbb{P}}^{\mu}_{\ell} \circ X_{\ell}^{-1}\}$ *converges weakly to the degenerate distribution* δ_{ρ} *concentrated at* $\rho = \tilde{p}'(\mu)$ *and satisfies the Large Deviation Principle with constants* $\{V_{\ell}\}$ *and rate-function* $\tilde{I}^{\mu}(x) = \tilde{p}(\mu) + \tilde{f}(x) - \mu x$.

Proof:

Since $x \longrightarrow f(x)$ is strictly convex for $0 \leq x < \rho_c$ and constant for $\rho_c \leq x < \infty$ and $x \longrightarrow \frac{a}{2}x^2$ is strictly convex for $0 \leq x < \infty$, the function $x \longrightarrow \tilde{f}(x) = f(x) + \frac{a}{2}x^2$ is strictly convex for $0 \leq x < \infty$; hence there is no first-order phase-transition segment; equivalently, $\mu \longrightarrow \tilde{p}(\mu)$, the Legendre transform of $x \longrightarrow \tilde{f}(x)$, is differentiable for $\mu < \infty$. It follows from Theorem 1 of [1] that $\tilde{\mathbb{K}}^{\mu}_{\ell} \to \delta_{\rho}$, where $\rho = \tilde{p}'(\mu)$, and from Theorem 4 of [1] that $\{\tilde{\mathbb{K}}^{\mu}_{\ell}: \ell = 1,2,\ldots\}$ satisfies the Large Deviation Principle with constants $\{V_{\ell}\}$ and rate-function $\tilde{I}^{\mu}(\cdot)$ ■

Although the first-order phase-transition segment, which was present in the free energy function of the free-gas, has disappeared, the phenomenon of condensation persists:

THEOREM 3

Suppose that (S1) *and* (S2) *hold; then, for* ρ_c *finite, we have*

$$\lim_{\lambda \downarrow 0} \lim_{\Lambda \to \infty} \tilde{\mathbb{E}}_\Lambda^\mu [X_\Lambda(\lambda)] = (\rho - \rho_c)^+ , \tag{2.8}$$

where $\tilde{\mathbb{E}}_\Lambda^\mu[\cdot]$ *is the mean-field expectation functional and* $\rho = \tilde{p}'(\mu)$.

Proof:

First, we remark that an elementary exercise yields the following alternative formula for the mean-field pressure $\tilde{p}(\mu)$:

$$\tilde{p}(\mu) = \inf_{\alpha<0} \left\{ \frac{(\mu-\alpha)^2}{2a} + p(\alpha) \right\}, \tag{2.9}$$

where $p(\alpha)$ is the free-gas pressure. The idea of the proof of (2.8) is that we compute the cumulant generating function of $X_\Lambda(\lambda)$; since

$$V_\Lambda X_\Lambda(\lambda) = V_\Lambda X_\Lambda - \sum_{\{j:\lambda_\Lambda(j)>\lambda\}} \sigma_j \tag{2.10}$$

we get

$$\tilde{\mathbb{E}}_\Lambda^\mu \left[e^{\beta s V_\Lambda X_\Lambda(\lambda)} \right] = \tilde{\mathbb{E}}^{(s,\lambda),\mu}_\Lambda \left[e^{\beta s V_\Lambda X_\Lambda} \right],$$

where $\tilde{\mathbb{E}}^{(s,\lambda),\mu}_\Lambda[\cdot]$ is the mean-field expectation functional for which the free-gas hamiltonian has been modified by the addition of the term $\sum_{\{j:\lambda_\Lambda(j)>\lambda\}} s\sigma_j$. These considerations yield the formula

$$\lim_{\Lambda \to \infty} \tilde{\mathbb{E}}_\Lambda^\mu [X_\Lambda(\lambda)] = \frac{\partial}{\partial s} \tilde{p}(\mu + s; s, \lambda) \Big|_{s=0}, \tag{2.11}$$

where

$$\tilde{p}(\mu+s;s,\lambda) = \inf_{\alpha<0} \left\{ \frac{(\mu+s-\alpha)^2}{2a} + p(\alpha;s,\lambda) \right\} ,$$

and

$$p(\alpha;s,\lambda) = \int_{[0,\lambda)} p(\alpha|\lambda) dF(\lambda) + \int_{[\lambda,\infty)} p(\alpha|s+\lambda) dF(\lambda).$$

A standard argument, using Griffith's Lemma, yields the result. ■

§3 FLUCTUATIONS IN THE MEAN-FIELD MODEL

Fluctuations in $X_\Lambda = N/V_\Lambda$ in the mean-field model in the thermodynamic limit

were studied for the standard example, described in §3 of [2] in this volume, by Davies [4], Wreszinski [5], Fannes and Verbeure [6] and Buffet and Pule [7]. The mean-field model in the general situation, where the only assumptions about the single-particle spectrum are that (S1) and (S2) hold, was investigated in [8]; we have summarized the results of [8] in §2 and now go on to investigate the fluctuations in X_ℓ. In fact, we do rather more; we regard $\lambda \longrightarrow X_\ell(\lambda)$ as a stochastic process and prove a central limit theorem:

THEOREM 4

Let $Z_\ell(\lambda) = V_\ell^{1/2}\{X_\ell(\lambda) - \tilde{\mathbb{E}}_\ell^\mu[X_\ell(\lambda)]\}$; *then, for* $\mu < a\rho_C$, $Z_\ell(\lambda) \xrightarrow{(d)} Z(\lambda)$, *where* $Z(\lambda)$ *is gaussian with mean zero and covariance* $\Gamma(\lambda_1,\lambda_2)$ *given by*

$$\Gamma(\lambda_1,\lambda_2) = J^\mu_{\lambda_1\wedge\lambda_2} - \frac{aJ^\mu_{\lambda_1}J^\mu_{\lambda_2}}{1 + aJ^\mu_\infty} \quad , \tag{3.1}$$

where

$$J^\mu_\lambda = \int_{[0,\lambda)} p''(\alpha(\mu)|\lambda)dF(\lambda) \quad , \tag{3.2}$$

and $\alpha(\mu)$ *is the value of* α *at which* $\inf_{\alpha<0}\left\{\frac{(\mu-\alpha)^2}{2a} + p(\alpha)\right\}$ *is attained.*

Sketch of proof:

The result follows from a routine, but somewhat tedious, calculation of

$$\lim_{\ell\to\infty}\tilde{\mathbb{E}}^\mu_\ell\left[e^{\beta(s_1Z_\ell(\lambda_1)+s_2Z_\ell(\lambda_2))}\right]$$

along the lines of the proof of Theorem 3. ∎

It is interesting to identify the process $Z(\cdot)$ in terms of a standard process.

THEOREM 5

Let $B(\cdot)$ *be a* BM(1), *a brownian motion in* $\mathbb{R}^1$ *starting at zero; then, for* $\mu < a\rho_C$,

$$Z(\lambda) \overset{(d)}{=} B(J^\mu_\lambda) - \frac{aJ^\mu_\lambda}{1 + aJ^\mu_\infty}B(J^\mu_\infty + 1/a). \tag{3.3}$$

Proof:

A routine computation shows that the mean of the right-hand side of (3.3) is zero and the covariance is the same as that of $Z(\cdot)$, given by (3.1). Hence the two gaussian processes are equal in distribution. ■

The process (3.3) is a modification of a time-changed brownian bridge; it never reaches the point at which it is tied-down but, as a increases, that point comes closer to J^{μ}_{∞}. This shows how, as the strength of the interaction increases, the fluctuations in $Z(\infty)$ are damped down.

It is a little more difficult to deal with the case $\mu > a\rho_c$; we introduce

$$W_{\ell}(\lambda) = Z_{\ell}(\infty) - Z_{\ell}(\lambda) \tag{3.4}$$

and prove in analagous fashion:

THEOREM 6

For $\mu > a\rho_c$, $W_{\ell}(\lambda) \xrightarrow{(d)} W(\lambda)$, *where* $W(\lambda)$ *is a gaussian process with mean zero and covariance* $\Gamma(\lambda_1,\lambda_2)$ *given by*

$$\Gamma(\lambda_1,\lambda_2) = K^{\mu}_{\lambda_1 \vee \lambda_2}, \tag{3.5}$$

where

$$K^{\mu}_{\lambda} = \int_{[\lambda,\infty)} p''(0|\lambda)dF(\lambda). \tag{3.6}$$

In this case,

$$W(\lambda) \overset{(d)}{=} K^{\mu}_{\lambda} B\left[\frac{1}{K^{\mu}_{\ell}}\right]. \tag{3.7}$$

The method by which we discovered the representations may be of some interest. The stochastic differential equation satisfied by a process $(X_t)_{t\geq 0}$ with filtration $(\mathcal{F}_t)$ is discussed by Nelson [9]; see also McGill [10].

Suppose that a process $(X_t,\mathcal{F}_t)$ *satisfies the stochastic differential equation*

$$X_t = X_s + \int_s^t \sigma(u,X_u)dB(u) + \int_s^t \tau(u,X_u)du; \tag{3.8}$$

then

$$\tau(s,X_s) = \lim_{t \downarrow s} \frac{1}{t-s} \mathbb{E}\,[X_t - X_s | \mathcal{F}_s] \tag{3.9}$$

and

$$\sigma^2(s,X_s) = \lim_{t \downarrow s} \frac{1}{t-s} \mathbb{E}\,[X_t - X_s)^2 | \mathcal{F}_s]. \tag{3.10}$$

Assuming that the processes $Z(\lambda), W(\lambda)$ satisfy stochastic differential equations, the corresponding coefficients σ and τ can be computed using (3.9) and (3.10); this is a routine exercise starting from the expressions (3.1) and (3.5) for the covariances since the processes are gaussian. Obvious time-changes then give the stochastic differential equations for a brownian bridge and a brownian motion respectively.

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