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Heat Flow out of Regions in $\mathbb{R}^m$

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1. Introduction

Let D be an open set in $\mathbb{R}^m$ and let $H=H^*\geqq 0$ be the operator $-\Delta_D$ on $L^2(D)$ subject to Dirichlet boundary conditions, and defined by the method of quadratic forms [8, 9], so that $\mathrm{Quad}(H)=\mathrm{Dom}(H^{1/2})=W_0^{1,2}(D)$. The heat kernel $p_D(x, y; t)$ of e^{-Ht} is a positive C^∞ function on $(0, \infty)\times D\times D$. It is well-known that if $D\subset E$, E open, then

$$0\leqq p_D(x, y; t)\leqq p_E(x, y; t). \tag{1.1}$$

In particular putting $E=\mathbb{R}^m$ yields

$$0\leqq p_D(x, y; t)\leqq (4\pi t)^{-m/2} e^{-|x-y|^2/(4t)}. \tag{1.2}$$

In previous papers [1, 2, 7] we have been interested in finding necessary and sufficient conditions for

$$Z_D(t)=\mathrm{trace}(e^{-Ht})=\int_D dx\, p_D(x, x; t) \tag{1.3}$$

to be finite for some $t>0$. While $Z_D(t)$ is always finite for $t>0$ if the volume $|D|$ of D is finite this is not necessarily the case for regions with infinite volume. One reason for the importance of the function $Z_D(t)$ is the fact that if $Z_D(t)<\infty$ for $t>0$ and the asymptotic behaviour of $Z_D(t)$ as $t\downarrow 0$ is known, then the spectrum of H is discrete: $\lambda_1<\lambda_2\leqq\lambda_3\leqq\ldots\leqq\lambda_j\leqq\ldots$ and its asymptotic distribution ($j\uparrow\infty$) can be obtained via Karamata's tauberian theorem. See [1, 2, 13, 17, 18, 21, 22, 23].

In this paper we will be interested in necessary and sufficient conditions on D for a closely related function $Q_D(t)$ to be finite for some $t\geqq 0$. Here $Q_D(t)$ is defined by

$$Q_D(t)=\int_D dx \int_D dy\, p_D(x, y; t). \tag{1.4}$$

For regions D with finite volume one has in view of (1.2)

$$Q_D(t) \leqq \int_D dx \int_{\mathbb{R}^m} dy\, p_D(x, y; t) \leqq |D|, \qquad t \geqq 0. \tag{1.5}$$

$Q_D(t)$ represents the amount of heat contained in D at time t when D has temperature 1 at $t=0$ and the boundary ∂D of D is kept at temperature 0 for all $t>0$.

This paper is organized as follows. In Sect. 2 we will prove some elementary estimates for $Q_D(t)$ and $Z_D(t)$. In Sect. 3 and 4 we prove supercontractive estimates and gaussian upper bounds for the heat kernel $p_D(x, y; t)$ under conditions on D which imply compactness of the resolvent of H. These bounds on $p_D(x, y; t)$ imply bounds on $Q_D(t)$. All the results which we obtain support the following.

Conjecture. If $0 \leqq t < \infty$ and D is open in $\mathbb{R}^m$ then the following are equivalent.
(i) $Q_D(s) < \infty$ for all $s > t$,
(ii) $Z_D(s) < \infty$ for all $s > t$.

If D is a horn-shaped region in $\mathbb{R}^2$ we prove that the conjecture holds (Theorem 5.11) and that one cannot sharpen it to the case where $s=t$ (Theorem 5.5). Finally in Sect. 6 we obtain the first two terms in the asymptotic expansion of $Q_D(t)$ as $t \downarrow 0$ for bounded regions D in $\mathbb{R}^m$ with a smooth boundary. (See [3, 5] for the asymptotic behaviour in some special cases).

The techniques rely on the representation (see [19]) of $\int_D dy\, p_D(x, y; t)$ as a Wiener probability $P_x[T_D > t]$ that a brownian motion $x(\cdot)$ with $x(0)=x$ does not leave D until t:

$$P_x[T_D > t] = \int_D dy\, p_D(x, y; t). \tag{1.6}$$

A crucial ingredient of our calculations is the quadratic form inequality

$$m \int_D \frac{f^2(x)}{4m^2(x)}\, dx \leqq \int_D |\nabla f(x)|^2\, dx, \tag{1.7}$$

for all $f \in C_c^\infty(D)$. Here the mean distance function $m(x)$ is defined on D by

$$\frac{1}{m^2(x)} = \int_{\|u\|=1} \frac{dS(u)}{d_u^2(x)}, \tag{1.8}$$

where dS is the normalized surface measure on the unit sphere of $\mathbb{R}^m$ and

$$d_u(x) = \min\{|t| : t \in \mathbb{R},\ x + tu \notin D\}, \tag{1.9}$$

with $d_u(x) = +\infty$ if the set is empty. The bound (1.7) may be found in [6, 7, 9] where it is also shown that $m(x) \geqq d(x) = \min\{|x-y| : y \notin D\}$. We say D is regular if there exists a constant $c \geqq 1$ such that $d(x) \leqq m(x) \leqq c\,d(x)$ for all $x \in D$, and refer to [6, 7, 9] for conditions on ∂D which imply regularity. We also note that if D is regular, then H has compact resolvent if and only if $d(x) \to 0$

as $|x|\to\infty$. (H has compact resolvent if and only if the embedding of $W_0^{1,2}(D)$ into $L^2(D)$ is compact. More complicated necessary and sufficient conditions for this, which do not require D to be regular may be found in [12, 15]).

2. Elementary Estimates

Lemma 2.1. *For D open in $\mathbb{R}^m$*

$$p_D(x, x; t) \geqq (4\pi t)^{-m/2} e^{-m^2\pi^2 t/(4d^2(x))}. \tag{2.1}$$

Proof. A cube C_x with centre x and edge length $2d(x)/m^{1/2}$ is contained in D. Since the heat kernel for a cube is the product of m one-dimensional heat kernels we obtain (2.1) by using Lemma 8 of [4].

Lemma 2.2. *For D open in $\mathbb{R}^m$*

$$\int_D dy\, p_D(x, y; t) \geqq 2^{-m} e^{-m^2\pi^2 t/(4d^2(x))}. \tag{2.2}$$

Proof. By positivity and monotonicity of heat kernels

$$\int_D dy\, p_D(x, y; t) \geqq \int_{C_x} dy\, p_D(x, y; t) \geqq \int_{C_x} dy\, p_{C_x}(x, y; t).$$

By the eigenfunction expansion of $p_{C_x}(x, y; t)$:

$$\begin{aligned} \int_{C_x} dy\, p_{C_x}(x, y; t) &= \left\{ \sum_{j=0}^{\infty} e^{-mt\pi^2(2j+1)^2/(4d^2(x))} \frac{4}{\pi(2j+1)} (-1)^j \right\}^m \\ &\geqq \left\{ e^{-mt\pi^2/(4d^2(x))} \frac{4}{\pi} - e^{-mt\pi^2 9/(4d^2(x))} \frac{4}{3\pi} \right\}^m \\ &\geqq 2^{-m} e^{-m^2 t\pi^2/(4d^2(x))}. \end{aligned} \tag{2.3}$$

By combining the above lemmas we obtain the following.

Corollary 2.3. *If either $Z_D(t) < \infty$ or $Q_D(t) < \infty$ then*

$$\int_D dx\, e^{-m^2\pi^2 t/(4d^2(x))} < \infty. \tag{2.4}$$

The following lemma gives an upper bound on $Q_D(t)$ in terms of $p_D(x, x; t)$ and the normalized eigenfunction in $L^2(D)$ corresponding to λ_1.

Lemma 2.4. *Suppose the spectrum of $-\Delta_D$ is discrete and ϕ_1 is the normalized eigenfunction in $L^2(D)$ corresponding to λ_1. Then*

$$Q_D(t) \leqq e^{t\lambda_1} \{ \int_D p_D(x, x; t) (\phi_1(x))^{-1} dx \}^2. \tag{2.5}$$

Proof. Since $p_D(x, y; t)$ is of positive type

$$p_D(x, y; t) \leqq (p_D(x, x; t)\, p_D(y, y; t))^{1/2}. \tag{2.6}$$

Hence

$$Q_D(t) \leqq \{\int_D dx (p_D(x, x; t))^{1/2}\}^2. \tag{2.7}$$

Using the eigenfunction expansion and the positivity of ϕ_1

$$\begin{aligned}(p_D(x, x; t))^{1/2} &= \left\{\sum_{j=1}^{\infty} e^{-t\lambda_j}(\phi_j(x))^2\right\}^{1/2} \\ &\geqq e^{-t\lambda_1/2}\phi_1(x),\end{aligned} \tag{2.8}$$

so that

$$Q_D(t) \leqq \{\int_D dx\, p_D(x, x; t)\, e^{t\lambda_1/2}(\phi_1(x))^{-1}\}^2. \tag{2.9}$$

The following proposition was proved in [7]. However, the first part follows more directly from Lemma 2.1.

Proposition 2.5. *If D is an arbitrary open set in $\mathbb{R}^m$ then the first of the following conditions implies the second. If D is regular then the two conditions are equivalent.*

(i) $Z_D(t) < \infty$ *for all* $t > 0$,

(ii) $\int_D e^{-t/d^2(x)}dx < \infty$ *for all* $t > 0$.

Much of our analysis is motivated by the attempt to find something close to a converse of the following result.

Lemma 2.6. *If D is open in $\mathbb{R}^m$ and $Q_D(t) < \infty$ for some $t > 0$, then $Z_D(s) < \infty$ for all $s > t$.*

Proof. If $\varepsilon > 0$ then

$$\begin{aligned}Z_D(t+\varepsilon) &= \int_D p_D(x, x; t+\varepsilon)\, dx \\ &= \int_D dx \int_D dy\, p_D(x, y; \varepsilon)\, p_D(y, x; t) \\ &\leqq (4\pi\varepsilon)^{-m/2} Q_D(t).\end{aligned} \tag{2.10}$$

3. Supercontractive Estimates

Throughout this section we shall take φ to be the function

$$\varphi(x) = (1 + x^2)^{-\alpha} \tag{3.1}$$

on an open set D in $\mathbb{R}^m$, where $\alpha > m/2$. Our key assumption in this section, that

$$m(x)^2 = o\left(\frac{1}{\log(1+x^2)}\right) \tag{3.2}$$

as $|x| \to \infty$, $x \in D$, should be compared with (2.4). Lemma 3.1 and Theorem 3.4 are fairly close to being converses to Corollary 2.3.

Lemma 3.1. *If D is open and regular and* (3.2) *holds then $Z_D(t) < \infty$ for all $t > 0$.*

Proof. Since $m(x)$ and $d(x)$ are of the same order of magnitude, (3.2) implies that for any $u > 0$ there exists $v > 0$ such that

$$\frac{1}{d^2(x)} \geqq u \log(1+x^2) - v, \qquad x \in D. \tag{3.3}$$

Therefore

$$\int_D e^{-t/d^2(x)} dx \leqq \int_D e^{-tu\log(1+x^2)+tv} dx \leqq e^{tv} \int_{\mathbb{R}^m} (1+x^2)^{-tu} dx < \infty \tag{3.4}$$

provided $u > m/(2t)$. The proof is completed by applying Proposition 2.5.

Lemma 3.2. *The function φ defined in* (3.1) *satisfies*

$$\int_D \varphi(x)\, dx < \infty, \tag{3.5}$$

$$|\Delta \varphi| \leqq c_1 \varphi, \tag{3.6}$$

for some $c_1 < \infty$ and all $x \in D$. Furthermore for all $\varepsilon > 0$ there exists $\gamma > 0$ such that for all $x \in D$

$$-\log \varphi \leqq \frac{\varepsilon}{m^2(x)} + \gamma. \tag{3.7}$$

Proof. Inequality (3.5) follows from the assumption that $\alpha > m/2$. Inequality (3.6) is a direct computation using the formula

$$\Delta \varphi = \frac{d^2 \varphi}{dr^2} + \frac{m-1}{r} \frac{d\varphi}{dr}. \tag{3.8}$$

Finally (3.7) follows immediately from (3.1) and (3.2).

We now follow the standard procedure [9] of transferring the problem to the weighted space $L^2(D, \varphi^2 dx)$. If we put $V = \Delta\varphi/\varphi$ then V is bounded and $(-\Delta + V)\varphi = 0$. We define the unitary operator U from $L^2(D, \varphi^2 dx)$ to $L^2(D, dx)$ by $Uf = \varphi f$ and consider the quadratic form Q_φ defined on the subspace $C_c^\infty(D)$ of $L^2(D, \varphi^2 dx)$ by

$$Q_\varphi(f) = \int_D |\nabla f|^2 \varphi^2 dx. \tag{3.9}$$

This form is closable and its closure is associated with the self-adjoint operator

$$0 \leqq H_\varphi = U^{-1}(-\Delta + V)U \tag{3.10}$$

on $L^2(D, \varphi^2 dx)$. The condition $\varphi \in L^2(D, dx)$ often imposed in this argument is actually irrelevant.

Theorem 3.3. *The operator $e^{-H_\varphi t}$ is bounded from the weighted space $L^p(D, \varphi^2 dx)$ to the weighted space $L^q(D, \varphi^2 dx)$ for all $1 < p \leqq q < \infty$ and all $0 < t < \infty$.*

Proof. Inequality (3.7) implies that if $\varepsilon > 0$ then

$$-\log \varphi \leqq \varepsilon(-\Delta + V) + \mu \tag{3.11}$$

as a quadratic form inequality on $L^2(D, dx)$ for some $\mu > 0$. Rosen's lemma [9, 16] now establishes that for all $\varepsilon > 0$ there exists $\beta(\varepsilon) < \infty$ such that

$$\int_D (f^2 \log f)\varphi^2 dx \leqq \varepsilon Q_\varphi(f) + \beta(\varepsilon)\|f\|_2^2 + \|f\|_2^2 \log \|f\|_2 \tag{3.12}$$

for all $0 \leqq f \in \mathrm{Quad}(H_\varphi) \cap L^1 \cap L^\infty$. It follows by [16] or by a simplified version of Theorem 2.2.7 of [9] that $e^{-H_\varphi t}$ is bounded from $L^2(D, \varphi^2 dx)$ to $L^q(D, \varphi^2 dx)$ for all $2 \leqq q < \infty$ and $t > 0$. The result now follows by duality and the fact that $e^{-H_\varphi t}$ is a self-adjoint semigroup.

Theorem 3.4. *For D open in $\mathbb{R}^m$ satisfying (3.2) one has $Q_D(t) < \infty$ for all $t > 0$.*

Proof. If p' is the integral kernel of $e^{-(-\Delta+V)t}$ and p_φ is the integral kernel of $e^{-H_\varphi t}$ then

$$\begin{aligned} 0 \leqq p_D(x, y; t) &\leqq e^{t\|V\|_\infty} p'(x, y; t) \\ &= e^{t\|V\|_\infty} p_\varphi(x, y; t)\varphi(x)\varphi(y). \end{aligned} \tag{3.13}$$

Therefore

$$\begin{aligned} Q_D(t) &\leqq e^{t\|V\|_\infty} \int_D dx \int_D dy\, p_\varphi(x, y; t)(\varphi(x))^{-1}(\varphi(y))^{-1}(\varphi(x))^2(\varphi(y))^2 \\ &= e^{t\|V\|_\infty}\langle e^{-H_\varphi t}\varphi^{-1}, \varphi^{-1}\rangle \\ &\leqq e^{t\|V\|_\infty}\|e^{-H_\varphi t}\|_{L^p \to L^q} \cdot \|\varphi^{-1}\|_{L^p}^2 \end{aligned} \tag{3.14}$$

provided $1 < p < 2$ and $p^{-1} + q^{-1} = 1$. But

$$\begin{aligned} \|\varphi^{-1}\|_p^p &= \int_D \varphi^{-p}\varphi^2 dx \\ &\leqq \int_D (1 + x^2)^{-\alpha(2-p)} dx < \infty, \end{aligned} \tag{3.15}$$

provided $p - 1 > 0$ is small enough.

It is clear that if we replace (3.2) by a stronger hypotheses, it will be possible to obtain sharper information about the heat kernel p_D and in particular to estimate the rate at which $Q_D(t)$ diverges as $t\downarrow 0$. One such specialization of the above calculations is given in Sect. 4, but we emphasize that many intermediate situations may also be considered.

4. Gaussian Upper Bounds on the Heat Kernel

In this section we obtain further upper bounds on the heat kernel under stronger hypotheses. Throughout this section we assume that

$$m(x)^2 \leqq c(1+x^2)^{-\alpha}, \tag{4.1}$$

for some $c>0$, $\alpha>0$ and all x in an open set D. We also put

$$\varphi(x)=(1+x^2)^{-\beta}, \tag{4.2}$$

for some $\beta>0$.

Lemma 4.1. *For D open in $\mathbb{R}^m$ and assuming hypotheses* (4.1), (4.2), *we have*

$$0 \leqq p_D(x, y; t) \leqq c\,\varphi(x)\,\varphi(y)\,t^{-\gamma/2}, \tag{4.3}$$

for all $0<t\leqq 1$ and $x, y\in D$, where

$$\gamma=m+4\beta/\alpha. \tag{4.4}$$

Proof. Direct calculations lead to the bounds

$$|\Delta\,\varphi(x)| \leqq c_3\,\varphi(x), \tag{4.5}$$

$$-\log\varphi(x) \leqq \frac{\varepsilon}{m(x)^2}+c_4-\frac{\beta}{\alpha}\log\varepsilon \tag{4.6}$$

for all $x\in D$ and $0<\varepsilon<\infty$. Applying Rosen's lemma as in the proof of Theorem 3.3 we see that (3.12) holds for all $0<\varepsilon\leqq 1$, with

$$\beta(\varepsilon)=a_2-\frac{\gamma}{4}\log\varepsilon. \tag{4.7}$$

By [9, 11] we deduce that

$$0 \leqq p_\varphi(x, y; t) \leqq a_3\,t^{-\gamma/2}, \tag{4.8}$$

for all $0<t\leqq 1$. This implies the claimed result as in the proof of Theorem 3.3.

Theorem 4.2. *Let D be open and* (4.1), (4.2) *hold, let $E\geqq 0$ be the bottom of the spectrum of H acting on $L^2(D)$ and let $0<\delta<1$. Then*

$$0<p_D(x, y; t) \leqq c_\delta\,t^{-\gamma/2}\,e^{(\delta-E)t-d^2(x,y)/(4t+4\delta t)}\,\varphi(x)\,\varphi(y), \tag{4.9}$$

for all $x \in D$, $y \in D$, $0 < t < \infty$ *where*

$$d(x, y) = \sup\{|\psi(x) - \psi(y)| : \psi \in E\}, \tag{4.10}$$

and

$$E = \{\psi : D \to \mathbb{R} \quad \text{such that} \ \|\nabla \psi\|_\infty \leqq 1\}. \tag{4.11}$$

Proof. This is a straightforward application of Theorem 4 of [10]. The replacement of the dimension of D in that paper by γ causes no difficulties. The quantity d in that paper is called the riemannian distance between x and y in D, but (4.10) is the more precise definition.

For gaussian lower bounds on the heat kernel we refer to Theorem 3 of [4].

It is always the case that $d(x, y) \geqq |x - y|$. The example of the region S_λ in Theorem 5.10 shows that $d(x, y)$ may be much larger than the euclidean distance. We note that in that example (4.1) holds with $\alpha = \frac{1}{\lambda} - 1 > 0$ so that Theorem 4.2 is applicable to this region. Note also that $d(x, 0) \sim |x|^{1/\lambda}$ as $|x| \to \infty$ since any curve from 0 to x in S_λ must follow the spiral.

5. Horn-shaped Regions

In previous papers [1, 2] we have obtained a theorem for $Z_F(t)$, where F is a horn-shaped region in $\mathbb{R}^m$. In this section we prove a corresponding theorem for $Q_F(t)$. First we recall the notation and definitions of [1].

Notation. A point in $\mathbb{R}^m$ ($m = 2, 3 \ldots$) is denoted by (x, y) where $y \in \mathbb{R}^{m-1}$ (orthogonal to the x-axis). Let P_x be the plane through $(x, 0)$ orthogonal to the x axis and let $F(x)$ be the orthogonal projection of $P_x \cap F$ onto P_0 where F is an open set in $\mathbb{R}^m$.

Definition 5.1. An open set F in $\mathbb{R}^m$ is (one-sided) horn-shaped if

(1) F is connected,

(2) $F(x) \subset F(x')$ for all $x \geqq x' > 0$, $F(x)$ is empty for $x \leqq 0$,

(3) $\int_0^\delta |F(x)|\, dx < \infty$ for $\delta \in [0, \infty)$.

($|F(x)|$ is the $(m-1)$-dimensional volume of $F(x)$).

Definition 5.2. Let $p_F((x_1, y_1), (x_2, y_2); t)$ be the heat kernel for $-\Delta_F + \frac{\partial}{\partial t}$ and let $p_{F(x)}(y_1, y_2; t)$ be the heat kernel for $-\Delta_{F(x)} + \frac{\partial}{\partial t}$, where $-\Delta_{F(x)}$ is the $(m-1)$-dimensional Dirichlet laplacian for $F(x)$, $y_1, y_2 \in F(x)$. Let $P_{(x,y)}[T_F > t]$ be the probability with respect to Wiener measure that a brownian motion $(x(\cdot), y(\cdot))$ in $\mathbb{R}^m$ with $(x(0), y(0)) = (x, y) \in F$ does not leave F up to time t. We define $P_y[T_{F(x)} > t]$ for a brownian motion $y(\cdot)$ with $y(0) = y \in F(x)$ in $\mathbb{R}^{m-1}$ similarly.

By Definition 5.2, (1.4) and (1.9) we have

$$P_{(x,y)}[T_F>t]=\int_0^\infty dx_1 \int_{F(x_1)} dy_1\, p_F((x,y),(x_1,y_1);t), \tag{5.1}$$

$$P_y[T_{F(x)}>t]=\int_{F(x)} dy_1\, p_{F(x)}(y,y_1;t), \tag{5.2}$$

$$Q_{F(x)}(t)=\int_{F(x)} dy\, P_y[T_{F(x)}>t]. \tag{5.3}$$

Theorem 5.3. *Let F be (one-sided) horn-shaped in $\mathbb{R}^m (m=2, 3, \ldots)$. Then for all t for which $\int_0^\infty dx\, Q_{F(x)}(t)$ is finite one has*

$$\int_0^\infty dx\, Q_{F(x)}(t)-\frac{2}{(\pi t)^{1/2}}\int_0^\infty dx\, Q_{F(x)}(t)\int_x^\infty dq\, e^{-q^2/(4t)}\leqq Q_F(t)\leqq \int_0^\infty dx\, Q_{F(x)}(t). \tag{5.4}$$

The proof of this theorem starts at Lemma 5.7. For horn-shaped regions in $\mathbb{R}^2$ it implies the following.

Theorem 5.4. *Let f_1, f_2 be positive, continuous and decreasing on $(0,\infty)$ such that*

$$\int_0^\delta (f_1(x)+f_2(x))\,dx<\infty, \qquad \delta\in(0,\infty), \tag{5.5}$$

$$\int_1^\infty dx(f_1(x)+f_2(x))\,e^{-t\pi^2/(f_1(x)+f_2(x))^2}<\infty, \tag{5.6}$$

and let

$$F=\{(x,y)\,|\,x>0,\ -f_1(x)<y<f_2(x)\}. \tag{5.7}$$

Then

$$\left|Q_F(t)-\frac{8}{\pi^2}\int_0^\infty dx\, f(x)\sum_{k=1,3,\ldots} k^{-2}e^{-t\pi^2k^2/f^2(x)}\right|\leqq 4\int_0^{t^{1/2}} dx\, f(x), \tag{5.8}$$

where

$$f(x)=f_1(x)+f_2(x). \tag{5.9}$$

Proof. Since f_1, f_2 are positive, continuous and decreasing and satisfy (5.5) H is horn-shaped. Furthermore $F(x)=(-f_1(x), f_2(x))$ and $p_{F(x)}(y_1, y_2; t)$ is the one-dimensional heat kernel with eigenfunction expansion

$$p_{F(x)}(y_1,y_2;t)=\frac{2}{f(x)}\sum_{k=1}^\infty e^{-t\pi^2k^2/f^2(x)}\left(\sin\frac{\pi k(f_1(x)+y_1)}{f(x)}\right)\left(\sin\frac{\pi k(f_1(x)+y_2)}{f(x)}\right). \tag{5.10}$$

Hence by (5.3) and (5.10)

$$Q_{F(x)}(t)=\frac{8f(x)}{\pi^2}\sum_{k=1,3,\ldots} k^{-2}e^{-t\pi^2k^2/f^2(x)}. \qquad (5.11)$$

Finally $Q_{F(x)}\leqq f(x)$ by (1.5) so that

$$\frac{2}{(\pi t)^{1/2}}\int_0^\infty dx\,Q_{F(x)}(t)\int_x^\infty dq\,e^{-q^2/(4t)}\leqq\frac{2}{(\pi t)^{1/2}}\int_0^\infty dx\,f(x)\int_x^\infty dq\,e^{-q^2/(4t)}$$

$$\leqq\frac{2}{(\pi t)^{1/2}}\int_0^{t^{1/2}} dx\,f(x)\int_0^\infty dq\,e^{-q^2/(4t)}+\frac{2}{(\pi t)^{1/2}}\int_{t^{1/2}}^\infty dx\,f(t^{1/2})\int_x^\infty dq\,e^{-q^2/(4t)}$$

$$=2\int_0^{t^{1/2}} dx\,f(x)+4\pi^{-1/2}t^{1/2}f(t^{1/2})\int_1^\infty dq\,q^{-2}e^{-q^2/4}\leqq 4\int_0^{t^{1/2}} dx\,f(x). \qquad (5.12)$$

Theorem 5.5. *Let F be as in Theorem* 5.4 *with* $f_1(x)=0$ *and*

$$f(x)=f_2(x)=\pi\{\log(3+x)+\beta\log\log(3+x)\}^{-1/2}, \qquad \beta>0,\ x>0. \qquad (5.13)$$

Then

$$Q_F(t)<\infty \quad \text{if and only if } t>1,\ \beta>0 \quad \text{or} \quad t=1,\ \beta>\tfrac{1}{2}, \qquad (5.14)$$

and

$$Z_F(t)<\infty \quad \text{if and only if } t>1,\ \beta>0 \quad \text{or} \quad t=1,\ \beta>1. \qquad (5.15)$$

Proof. The term $f(x)e^{-t\pi^2/f^2(x)}$ behaves like $(3+x)^{-t}(\log(3+x))^{-\beta t-1/2}(1+o(1))$ for $x\to+\infty$. Hence $f(x)e^{-t\pi^2/f^2(x)}$ is integrable on $[0,\infty)$ if and only if $t>1$, $\beta>0$ or $t=1$ and $\beta>\frac{1}{2}$. The remainder series $\sum_{k=3,5,\ldots} f(x)e^{-t\pi^2k^2/f^2(x)}/k^2$ is integrable for all $t>\frac{1}{9}$. To prove (5.15) we use Theorem 3 and (43) from [2]

$$\left|Z_F(t)-\frac{1}{(4\pi t)^{1/2}}\int_0^\infty dx\sum_{k=1}^\infty e^{-t\pi^2k^2/f^2(x)}\right|\leqq\frac{f(0)}{4(\pi t)^{1/2}}. \qquad (5.16)$$

The leading term $e^{-t\pi^2/f^2(x)}$ behaves like $(3+x)^{-t}(\log(3+x))^{-\beta t}(1+o(1))$ for $x\to\infty$. Hence $e^{-t\pi^2/f^2(x)}$ is integrable on $[0,\infty)$ if and only if $t>1$, $\beta>0$ or $t=1$, $\beta>1$. The remaining series $\sum_{k=2}^\infty e^{-t\pi^2k^2/f^2(x)}$ is integrable for all $t>\frac{1}{4}$.

Suppose F is horn-shaped as in Theorem 5.4 and f is integrable on $[0, \infty)$. Then

$$\lim_{t\downarrow 0} Q_F(t) = \int_0^\infty dx\, f(x). \tag{5.17}$$

In this case Theorem 5.4 provides the leading term of

$$Q_F(t) - \int_0^\infty dx\, f(x) \quad \text{as } t\downarrow 0.$$

Example 5.6. Let $F = \{(x, y) | x > 0, 0 < y < e^{-x}\}$. Then

$$\left| Q_F(t) - 1 - \frac{2t^{1/2}}{\pi^{1/2}} \log t \right| \leqq 14 t^{1/2}, \qquad 0 < t \leqq 1. \tag{5.18}$$

Proof. Note that $y \to (e^{-\alpha y^2} - 1)/y^2$ is strictly increasing on $[0, \infty)$ for $\alpha > 0$. By Theorems 5.3 and 5.4

$$\begin{aligned} Q_F(t) &\leqq \sum_{k=1,3,\ldots} \frac{8}{\pi^2 k^2} \int_0^\infty dx\, e^{-x - t\pi^2 k^2 e^{2x}} \\ &= 1 + \sum_{k=1,3,\ldots} \frac{8}{\pi^2 k^2} \int_0^\infty dx\, e^{-x} (e^{-t\pi^2 k^2 e^{2x}} - 1) \\ &\leqq 1 + \sum_{k=1}^\infty \frac{4}{\pi^2 k^2} \int_0^\infty dx\, e^{-x} (e^{-t\pi^2 k^2 e^{2x}} - 1) \\ &\leqq 1 + \int_1^\infty dk \frac{4}{\pi^2 k^2} \int_0^\infty dx\, e^{-x} (e^{-t\pi^2 k^2 e^{2x}} - 1). \end{aligned} \tag{5.19}$$

Integration by parts with respect to k gives

$$\begin{aligned} Q_F(t) &\leqq 1 - 8t \int_1^\infty dk \int_0^\infty dx\, e^{x - t\pi^2 k^2 e^{2x}} \\ &= 1 - 8t \int_1^\infty dx\, x^{-1} \int_x^\infty dk\, e^{-t\pi^2 k^2} \\ &= 1 - 8t \int_1^\infty dx (\log x) e^{-t\pi^2 x^2} \\ &\leqq 1 - 8t \int_0^\infty dx (\log x) e^{-t\pi^2 x^2} \\ &\leqq 1 + (4t/\pi)^{1/2} (C + \log(4\pi^2 t)), \end{aligned} \tag{5.20}$$

by 4.333 of [14]. C is Euler's constant. By Theorem 5.4

$$
\begin{aligned}
Q_F(t) &\geqq 1+\sum_{k=1,3,\dots}\frac{8}{\pi^2k^2}\int_0^\infty dx\, e^{-x}(e^{-t\pi^2k^2e^{2x}}-1)-4t^{1/2}\\
&\geqq 1+\sum_{k=2}^{\infty}\frac{4}{\pi^2k^2}\int_0^\infty dx\, e^{-x}(e^{-t\pi^2k^2e^{2x}}-1)-4t^{1/2}\\
&\quad+\frac{8}{\pi^2}\int_0^\infty dx\, e^{-x}(e^{-t\pi^2e^{2x}}-1)\\
&\geqq 1+\int_1^\infty dk.\frac{4}{\pi^2k^2}\int_0^\infty dx\, e^{-x}(e^{-t\pi^2k^2e^{2x}}-1)-4t^{1/2}\\
&\quad+\frac{8}{\pi^2}\int_0^\infty dx\, e^{-x}(e^{-t\pi^2e^{2x}}-1)\\
&=1+(4t/\pi)^{1/2}(C+\log(4\pi^2t))-4t^{1/2}\\
&\quad+\frac{12}{\pi^2}\int_0^\infty dx\, e^{-x}(e^{-t\pi^2e^{2x}}-1)+8t\int_0^1 dx(\log x)e^{-t\pi^2x^2}.
\end{aligned}
\tag{5.21}
$$

Moreover by 3.464 of [14]

$$
\begin{aligned}
\frac{12}{\pi^2}\int_0^\infty dx\, e^{-x}(e^{-t\pi^2e^{2x}}-1) &= \frac{12}{\pi^2}\int_1^\infty dx\, x^{-2}(e^{-t\pi^2x^2}-1)\\
&\geqq \frac{12}{\pi^2}\int_0^\infty dx\, x^{-2}(e^{-t\pi^2x^2}-1)=12\pi^{-1/2}t^{1/2},
\end{aligned}
$$

and for $0<t\leqq 1$

$$8t\int_0^1 dx(\log x)e^{-t\pi^2x^2}\geqq -8t^{1/2}.$$

Combining these inequalities with (5.21) we obtain

$$Q_F(t)\geqq 1+(4t/\pi)^{1/2}(C+\log(4\pi^2t)-6\pi^{1/2}-6),\qquad 0<t\leqq 1, \tag{5.22}$$

which completes the proof.

Lemma 5.7. *Let F be horn-shaped in $\mathbb{R}^m$ and $(x, y)\in F$. Then*

$$P_{(x,y)}[T_F>t]\leqq \int_0^x d\xi\,\rho(\xi;t)\,P_y[T_{F(x-\xi)}>t], \tag{5.23}$$

where $\rho(\xi;t)$ is given by

$$\rho(\xi;t)=\begin{cases}0, & \xi\leqq 0\\ (\pi t)^{-1/2}e^{-\xi^2/(4t)}, & \xi>0\end{cases}. \tag{5.24}$$

Proof. Let $\omega=(x(\cdot), y(\cdot))$ be a brownian path in $\mathbb{R}^m$ with $(x(0), y(0))=(x, y)$. Then since F is horn-shaped

$$\{\omega: (x(\tau), y(\tau))\in F, 0\leqq\tau\leqq t\} \subset \bigcup_{\{\xi: 0\leqq\xi\leqq x\}} \{\omega: \min_{0\leqq\tau\leqq t} x(\tau)>x-\xi, y(\tau)\in F(x-\xi), 0\leqq\tau\leqq t\}.$$

Since $x(\cdot)$ and $y(\cdot)$ are independent and the random variable $\min_{0\leqq\tau\leqq t} x(\tau)$ has probability density $\rho(\xi; t)$ the lemma follows.

Lemma 5.8. *Let F be horn-shaped in $\mathbb{R}^m$, $(x, y)\in F$. Then*

$$P_{(x,y)}[T_F>t]\geqq \int_{\{\xi: \xi\geqq 0, y\in F(x+\xi)\}} d\xi P_y[T_{F(x+\xi)}>t] \frac{\partial}{\partial\xi} P_x[T_{(0,x+\xi)}>t], \tag{5.25}$$

where

$$P_x[T_{(0,x+\xi)}>t]=P_x[0<x(\tau)<x+\xi, 0\leqq\tau\leqq t]. \tag{5.26}$$

Proof. Let ω be as in the proof of Lemma 5.7. Then since F is horn-shaped

$$\{\omega: (x(\tau), y(\tau))\in F, 0\leqq\tau\leqq t\} \supset \bigcup_{\xi\geqq 0} \{\omega: \min_{0\leqq\tau\leqq t} x(\tau)\geqq 0, \max_{0\leqq\tau\leqq t} x(\tau)\leqq x+\xi, y(\tau)\in F(x+\xi), 0\leqq\tau\leqq t\}. \tag{5.27}$$

Since $x(\cdot)$, $y(\cdot)$ are independent and the random variable $\max_{0\leqq\tau\leqq t} x(\tau)$ conditioned to $\min_{0\leqq\tau\leqq t} x(\tau)>0$ has probability density $\frac{\partial}{\partial\xi} P_x[T_{(0,x+\xi)}>t]$ on $[x,\infty)$ the lemma follows.

Lemma 5.9. *For $x>0$, $\xi>0$*

$$P_x[T_{(0,x+\xi)}>t]\geqq 1-\int_x^\infty \rho(\xi'; t)\,d\xi'-\int_\xi^\infty \rho(\xi'; t)\,d\xi'. \tag{5.28}$$

Proof.

$$\begin{aligned} P_x[T_{(0,x+\xi)}>t] &= P_x[0<x(\tau)<x+\xi, 0\leqq\tau\leqq t] \\ &\geqq P_x[\max_{0\leqq\tau\leqq t} x(\tau)<x+\xi]-P_x[\min_{0\leqq\tau\leqq t} x(\tau)<0] \\ &= \int_0^\xi \rho(\xi'; t)\,d\xi'-\int_x^\infty \rho(\xi'; t)\,d\xi'. \end{aligned} \tag{5.29}$$

Proof of Theorem 5.3. By Lemma 5.7 and Fubini's theorem

$$\begin{aligned}\int_{F(x)} dy\, P_{(x,y)}[T_F>t] &\leqq \int_0^x d\xi\, \rho(\xi;t) \int_{F(x)} dy\, P_y[T_{F(x-\xi)}>t]\\ &\leqq \int_0^x d\xi\, \rho(\xi;t) \int_{F(x-\xi)} dy\, P_y[T_{F(x-\xi)}>t]\\ &=\{\rho(\cdot;t)*Q_{F(\cdot)}(t)\}(x), \end{aligned} \tag{5.30}$$

where $*$ denotes convolution (with respect to x). Hence

$$Q_F(t)\leqq \int_0^\infty dx\, \rho(\cdot;t)*Q_{F(\cdot)}(t)(x)=\int_0^\infty dx\, Q_{F(x)}(t). \tag{5.31}$$

We define $P_y[T_{F(x+\xi)}>t]$ to be zero if $y\notin F(x+\xi)$. Then by Lemma 5.8, Fubini's theorem and integration by parts we have

$$\begin{aligned}\int_{F(x)} dy\, P_{(x,y)}[T_F>t] &\geqq \int_{F(x)} dy \int_{[0,\infty)} d\xi\, P_y[T_{F(x+\xi)}>t]\, \frac{\partial}{\partial\xi} P_x[T_{(0,x+\xi)}>t]\\ &\geqq \int_{F(x+\xi)} dy \int_{[0,\infty)} d\xi\, P_y[T_{F(x+\xi)}>t]\, \frac{\partial}{\partial\xi} P_x[T_{(0,x+\xi)}>t]\\ &=-\int_0^\infty d\xi\, P_x[T_{(0,x+\xi)}>t]\, \frac{\partial}{\partial\xi} Q_{F(x+\xi)}(t). \end{aligned} \tag{5.32}$$

Since F is horn-shaped $-\frac{\partial}{\partial\xi} Q_{F(x+\xi)}(t)$ is positive. Hence by Lemma 5.9 and further integrations by parts

$$\begin{aligned}\int_{F(x)} dy\, P_{(x,y)}[T_F>t] &\geqq \int_0^\infty d\xi \left\{1-\int_\xi^\infty \rho(\xi';t)\,d\xi'-\int_x^\infty \rho(\xi';t)\,d\xi'\right\}\cdot -\frac{\partial}{\partial\xi} Q_{F(x+\xi)}(t)\\ &=Q_{F(x)}(t)+\int_{[0,\infty)} d\xi\, \{Q_{F(x+\xi)}(t)-Q_{F(x)}(t)\}\, \rho(\xi;t)\\ &\quad -Q_{F(x)}(t)\int_x^\infty \rho(\xi';t)\,d\xi'. \end{aligned} \tag{5.33}$$

Hence

$$\begin{aligned}\int_0^\infty dx \int_{F(x)} dy\, P_{(x,y)}[T_F>t] &\geqq \int_0^\infty dx\, Q_{F(x)}(t)-\int_0^\infty d\xi\, \rho(\xi;t)\int_0^\xi dx\, Q_{F(x)}(t)\\ &\quad -\int_0^\infty dx\, Q_{F(x)}(t)\int_x^\infty \rho(\xi';t)\,d\xi'. \end{aligned} \tag{5.34}$$

A further integration by parts completes the proof.

It is possible to obtain the precise asymptotic behaviour of $Z(t)$, $Q(t)$ as $t\downarrow 0$ for other unbounded regions. In particular we have the following.

Theorem 5.10. *Let $S_\lambda\subset\mathbb{R}^2$ be the complement of the range of the curve $\gamma\colon [0,\infty)\to\mathbb{R}^2$ defined by*

$$\gamma(s)=(s^\lambda\cos s,\, s^\lambda \sin s), \qquad 0<\lambda<1. \tag{5.35}$$

Then S_λ is open and dense in $\mathbb{R}^2$ and has infinite volume. Furthermore

$$\lim_{t\downarrow 0} Q_{S_\lambda}(t)\, t^{\lambda/(1-\lambda)} = \frac{2}{\pi}\, \lambda^{2\lambda/(1-\lambda)}(4^{1/(1-\lambda)}-1)\,\Gamma(1/(1-\lambda))\,\zeta(2/(1-\lambda)), \tag{5.36}$$

$$\lim_{t\downarrow 0} Z_{S_\lambda}(t)\, t^{1/(1-\lambda)} = \frac{1}{\pi^{1/2}(1-\lambda)}\, \lambda^{(1+\lambda)/(1-\lambda)}\, 2^{(3\lambda-1)/(1-\lambda)}\, \Gamma((1+\lambda)/(2-2\lambda)) \cdot \zeta((1+\lambda)/(1-\lambda)), \tag{5.37}$$

where ζ, Γ are the Riemann function, gamma function respectively.

Sketch of Proof. By "unrolling" the spiral one obtains a (one-sided) horn-shaped region of the form (5.7) with $f(x)\sim(s+2\pi)^\lambda - s^\lambda \sim 2\pi\lambda s^{\lambda-1}$ as $s\uparrow\infty$ and $dx \sim (s^{2\lambda}+\lambda^2 s^{2\lambda-2})^{1/2}\, ds \sim s^\lambda ds$ as $s\uparrow\infty$. Formulas (5.8) and (5.16) yield (5.36) and (5.37) respectively.

Theorem 5.11. *Let F be horn-shaped in $\mathbb{R}^2$. Then for $0\leqq t<\infty$ the following are equivalent. (i) $Q_F(s)<\infty$ for all $s>t$, (ii) $Z_F(s)<\infty$ for all $s>t$.*

Proof. Because of Lemma 2.6 we only have to prove that $Q_F(s)<\infty$ for $s>t$ implies $Z_F(s)<\infty$ for $s>t$. From (5.8) and [2] we obtain (for one-sided horn-shaped regions in $\mathbb{R}^2$)

$$\begin{aligned}
Z_F(s) &\leqq \frac{1}{(4\pi s)^{1/2}} \int_0^\infty dx \sum_{k=1}^\infty e^{-s\pi^2 k^2/f^2(x)} \\
&\leqq \frac{1}{4\pi s}\int_0^{t^{1/2}} dx\, f(x) + \frac{1}{(4\pi s)^{1/2}} \int_{t^{1/2}}^\infty dx \sum_{k=1}^\infty e^{-s\pi^2k^2/f^2(x)} \\
&\leqq \frac{1}{4\pi s}\int_0^{t^{1/2}} dx\, f(x) + \frac{2}{(4\pi s)^{1/2}} \int_{t^{1/2}}^\infty dx \sum_{k=1,3,\dots} e^{-s\pi^2k^2/f^2(x)} \\
&\leqq \frac{1}{4\pi s}\int_0^{t^{1/2}} dx\, f(x) + \frac{2}{(4\pi s)^{1/2}} \int_{t^{1/2}}^\infty dx \sum_{k=1,3,\dots} e^{-(s+t)\pi^2k^2/(2f^2(x))-(s-t)\pi^2k^2/(2f^2(x))} \\
&\leqq \frac{1}{4\pi s}\int_0^{t^{1/2}} dx\, f(x) + \frac{4}{(4\pi s)^{1/2}}\frac{1}{(s-t)\pi^2} \int_{t^{1/2}}^\infty dx\, f^2(x) \sum_{k=1,3,\dots} k^{-2} e^{-(s+t)\pi^2k^2/(2f^2(x))} \\
&\leqq \frac{1}{4\pi s}\int_0^{t^{1/2}} dx\, f(x) + \frac{f(t^{1/2})}{(16\pi s)^{1/2}(s-t)} \int_0^\infty dx\, f(x) \sum_{k=1,3,\dots} \frac{8}{\pi^2 k^2} e^{-(s+t)\pi^2k^2/(2f^2(x))} \\
&\leqq \frac{1}{4\pi s}\int_0^{t^{1/2}} dx\, f(x) + \frac{f(t^{1/2})}{(16\pi s)^{1/2}(s-t)} \left\{Q_F((s+t)/2) + 4\int_0^{t^{1/2}} dx\, f(x)\right\}.
\end{aligned} \tag{5.38}$$

This is finite since f is integrable at 0 and $Q_F((s+t)/2)$ is finite by $(s+t)/2>t$.

6. Bounded Regions with R-smooth Boundaries

Before we state the main theorem of this section we make the following definition.

Definition 6.1. A boundary ∂D of an open set D in $\mathbb{R}^m$, $m=2, 3, \dots$ is R-smooth if for each point $x_0\in\partial D$ there exist two open balls B_1, B_2 with radius R such that $B_1\subset D$, $B_2\subset\mathbb{R}^m\backslash(D\cup\partial D)$, $\partial B_1\cap\partial B_2 = x_0$.

Theorem 6.2. *Let D be an open, bounded and connected set in $\mathbb{R}^m$, $m=2, 3, \ldots$ with R-smooth boundary ∂D. Then for all $t>0$*

$$\left|Q_D(t)-|D|+\frac{2}{\pi^{1/2}}t^{1/2}|\partial D|\right|\leqq 10^m |D| t R^{-2}, \tag{6.1a}$$

$$\left|(4\pi t)^{m/2} Z_D(t)-|D|+\frac{\pi^{1/2}}{2}t^{1/2}|\partial D|\right|\leqq 2^m m^4 |D| t R^{-2}, \tag{6.1b}$$

where $|\partial D|$ is the area of the boundary ∂D.

The proof of (6.1b) has been given in [2]. Here we will prove (6.1a). In Lemma 6.3 and Corollary 6.4 we will use Levy's maximal inequality for brownian motion to obtain estimates for $P_x[T_B>t]$ where B is an open ball in $\mathbb{R}^m$. In Lemmas 6.5 and 6.6 we obtain a lower bound and upper bound respectively for x near the boundary ∂D. In Lemma 6.7 we recall a result on areas of parallel surfaces. Then we complete the proof of (6.1a).

Lemma 6.3. *Let B be an open ball in $\mathbb{R}^m$ with radius R and centre 0. Then*

$$1\geqq P_0[T_B>t]\geqq 1-\frac{2}{\Gamma(m/2)}\int_{R^2/(4t)}^{\infty} y^{(m-2)/2}e^{-y}dy. \tag{6.2}$$

Proof. By Levy's maximal inequality (Theorem 3.6.5 of [19])

$$\begin{aligned}P_0[T_B<t]&=P_0[\max_{0\leqq\tau\leqq t}|x(\tau)|>R]\leqq 2P_0[|x(t)|>R]\\&=\frac{2}{(4\pi t)^{m/2}}\int_{|x|>R} dx\, e^{-|x|^2/(4t)}.\end{aligned} \tag{6.3}$$

Corollary 6.4. *Let B be as in Lemma 6.3. Then*

$$P_0[T_B<t]\leqq 2^{1+m/2}e^{-R^2/(8t)}. \tag{6.4}$$

Lemma 6.5. *Let D be open in $\mathbb{R}^m$ with R-smooth boundary ∂D. Let $x\in D$ such that $d(x)<R$. Then*

$$\begin{aligned}P_x[T_D>t]\geqq&\int_0^{d(x)}\frac{d\xi}{(\pi t)^{1/2}}e^{-\xi^2/(4t)}\left\{1-\frac{2}{\Gamma((m-1)/2)}\int_{R(d(x)-\xi)/(4t)}^{\infty} dy\, e^{-y}y^{(m-3)/2}\right\}\\&-\frac{2}{\Gamma(m/2)}\int_{Rd(x)/(4t)}^{\infty} dy\, e^{-y}y^{(m-2)/2}.\end{aligned} \tag{6.5}$$

Proof. Since ∂D is R-smooth and $d(x)<R$ there exists a ball B with radius R and centre 0 such that $0=(-(R-d(x)), 0)$ and $x=(0,0)$ in a cartesian frame

(x_1, y), $(x_1 \in \mathbb{R}, y \in \mathbb{R}^{m-1})$. Let $\partial_1 B = \{z \in \partial B \mid x_1(z) \geqq 0\}$, $\partial_2 B = \partial B \setminus \partial_1 B$. Then by monotonicity

$$P_x[T_D > t] \geqq P_x[T_B > t] \geqq P_{(0,0)}[x(\cdot) \text{ does not hit } \partial_1 B \text{ up to time } t] - P_{(0,0)}[x(\cdot) \text{ does hit } \partial_2 B \text{ in } [0,t]]. \tag{6.6}$$

Since the distance from $(0,0)$ to $\partial_2 B$ is bounded from below by $(Rd(x))^{1/2}$ we have by Lemma 6.3

$$P_{(0,0)}[x(\cdot) \text{ does hit } \partial_2 B \text{ in } [0,t]] \leqq P_{(0,0)}[\max_{0 \leqq \tau \leqq t} |x(\tau)|^2 > Rd(x)]$$

$$\leqq \frac{2}{\Gamma(m/2)} \int_{Rd(x)/(4t)}^{\infty} dy\, e^{-y} y^{(m-2)/2}. \tag{6.7}$$

Let $x(\cdot) = (x_1(\cdot), y(\cdot))$ where $x_1(\cdot)$ is a brownian motion along the x_1 axis with $x_1(0) = 0$ and $y(\cdot)$ is an independent brownian motion in $\mathbb{R}^{m-1}$ with $y(0) = 0$. The probability density $\rho(\xi; t)$ of the random variable $\max_{0 \leqq \tau \leqq t} x_1(\tau)$ is given by (5.24). Hence by mononicity and Lemma 6.3

$$\begin{aligned} &P_{(0,0)}[x(\cdot) \text{ does not hit } \partial_1 B \text{ up to time } t] \\ &\geqq \int_0^{d(x)} d\xi\, \rho(\xi; t) P_0[\max_{0 \leqq \tau \leqq t} |y(\tau)|^2 < R^2 - (R - d(x) + \xi)^2] \\ &\geqq \int_0^{d(x)} d\xi\, \rho(\xi; t) P_0[\max_{0 \leqq \tau \leqq t} |y(\tau)|^2 < R(d(x) - \xi)] \\ &\geqq \int_0^{d(x)} d\xi\, \rho(\xi; t) \left\{1 - \frac{2}{\Gamma((m-1)/2)} \int_{R(d(x)-\xi)/(4t)}^{\infty} dy\, e^{-y} y^{(m-3)/2}\right\}. \end{aligned} \tag{6.8}$$

Lemma 6.6. *Let D be open in $\mathbb{R}^m$ with R-smooth boundary ∂D. Let $x \in D$ such that $d(x) < R$. Then*

$$P_x[T_D > t] \leqq \int_0^{d(x)} \frac{d\xi}{(\pi t)^{1/2}} e^{-\xi^2/(4t)} + \frac{4(m-1) t^{1/2}}{\pi^{1/2} R} e^{-d^2(x)/(4t)} + 2^{9/2} t/(eR^2). \tag{6.9}$$

Proof. Since ∂D is R-smooth there exists a closed ball B with radius R in the complement of D such that the distance from x to ∂B is equal to $d(x)$. Define a cartesian frame (x_1, y), $(x_1 \in \mathbb{R}, y \in \mathbb{R}^{m-1})$ such that $x = (0,0)$ and the centre of B is given by $(d(x) + R, 0)$. Then by monotonicity and the decomposition of the brownian motion (as in the proof of Lemma 6.5)

$$\begin{aligned} P_x[T_D > t] \leqq P_x[T_{\mathbb{R}^m \setminus B} > t] &\leqq \int_0^{d(x)} \rho(\xi; t)\, d\xi + \int_{d(x)+R}^{\infty} \rho(\xi; t)\, d\xi \\ &+ \int_{d(x)}^{d(x)+R} \rho(\xi; t)\, d\xi\, P_0[\max_{0 \leqq \tau \leqq t} |y(\tau)|^2 > R^2 - (R - \xi + d(x))^2]. \end{aligned} \tag{6.10}$$

By Lemma 6.3

$$\int_{d(x)}^{d(x)+R} \rho(\xi;t)\,d\xi\, P_0[\max_{0\le\tau\le t}|y(\tau)|^2 > R^2-(R-\xi+d(x))^2]$$

$$\leqq \int_{d(x)}^{d(x)+R} \rho(\xi;t)\,d\xi\, P_0[\max_{0\le\tau\le t}|y(\tau)|^2 > R(\xi-d(x))]$$

$$\leqq \int_{d(x)}^{d(x)+R} \frac{d\xi}{(\pi t)^{1/2}} e^{-d^2(x)/(4t)} \frac{2}{\Gamma((m-1)/2)} \int_{R(\xi-d(x))/(4t)}^{\infty} dy\, e^{-y} y^{(m-3)/2}$$

$$\leqq \frac{2}{(\pi t)^{1/2}} \frac{e^{-d^2(x)/(4t)}}{\Gamma((m-1)/2)} \int_0^\infty d\xi \int_{R\xi/(4t)}^{\infty} dy\, e^{-y} y^{(m-3)/2}. \tag{6.11}$$

Furthermore by Corollary 6.4 for $m=1$

$$\int_{d(x)+R}^{\infty} \rho(\xi;t)\,d\xi \leqq 2^{3/2} e^{-R^2/(8t)} \leqq 2^{9/2} t/(eR^2). \tag{6.12}$$

Lemma 6.7. *Let D be an open, bounded and connected set in $\mathbb{R}^m$, $m=2,3,\ldots$ with R-smooth boundary ∂D. Let ∂D_q denote the boundary of the set $\{x\in D\,|\,d(x)>q\}$ and let $|\partial D_q|$ denote its area. Then*

$$|\partial D|\left(\frac{R-q}{R}\right)^{m-1} \leqq |\partial D_q| \leqq |\partial D|\left(\frac{R}{R-q}\right)^{m-1}, \quad 0\leqq q<R. \tag{6.13}$$

Proof. See Lemma 5 of [2].

Proof of Theorem 6.2. By Lemmas 6.6 and 6.7 we have

$$\begin{aligned}
Q_D(t) &= |D| - \int_D dx\, P_x[T_D<t] \\
&\leqq |D| - \int_{\{x\in D:\, d(x)<R/2\}} dx\, P_x[T_D<t] \\
&\leqq |D| - \int_0^{R/2} |\partial D_q|\, dq \left\{ \int_q^\infty \frac{d\xi}{(\pi t)^{1/2}} e^{-\xi^2/(4t)} - \frac{4(m-1)t^{1/2}}{\pi^{1/2}R} e^{-q^2/(4t)} - \frac{2^{9/2}t}{eR^2} \right\} \\
&\leqq |D| - \int_0^{R/2} |\partial D| \left(1-\frac{q}{R}\right)^{m-1} dq \int_q^\infty \frac{d\xi}{(\pi t)^{1/2}} e^{-\xi^2/(4t)} \\
&\quad + 2^{m-1}|\partial D| \int_0^\infty dq \frac{4(m-1)t^{1/2}}{\pi^{1/2}R} e^{-q^2/(4t)} + |D|\, 2^{9/2} t/(eR^2) \\
&\leqq |D| - \int_0^\infty dq\, |\partial D|(1-2(m-1)q/R) \int_q^\infty \frac{d\xi}{(\pi t)^{1/2}} e^{-\xi^2/(4t)} \\
&\quad + 2^{m+1}(m-1)|\partial D|\, t/R + 2^{9/2}|D|\, t/(eR^2) \\
&= |D| - \frac{2t^{1/2}}{\pi^{1/2}} |\partial D| + \frac{|\partial D|\, t}{R} 2(m-1)(1+2^m) + 2^{9/2}|D|\, t/(eR^2).
\end{aligned} \tag{6.14}$$

From Lemma 6.7 we obtain by integrating with respect to q

$$|\partial D| \leqq \frac{m|D|}{R}. \tag{6.15}$$

The upper bound in Theorem 6.2 follows from (6.14) and (6.15).

To prove the lower bound in Theorem 6.2 we use Corollary 6.4 and Lemmas 6.5 and 6.7. We obtain

$$\begin{aligned}
Q_D(t) \geqq & \int_{\{x\in D:\, d(x)>R/2\}} dx\,(1-2^{1+m/2}e^{-R^2/(32t)}) + \int_{\{x\in D:\, d(x)\leqq R/2\}} dx\, P_x[T_D>t] \\
\geqq & |D| - |D|\, t\, 2^{6+m/2}/(eR^2) - \int_{\{x\in D:\, d(x)\leqq R/2\}} dx \int_{d(x)}^{\infty} \frac{d\xi}{(\pi t)^{1/2}} e^{-\xi^2/(4t)} \\
& - \frac{2}{\Gamma((m-1)/2)} \int_{\{x\in D:\, d(x)\leqq R/2\}} \int_0^{d(x)} \frac{d\xi}{(\pi t)^{1/2}} e^{-\xi^2/(4t)} \int_{R(d(x)-\xi)/(4t)}^{\infty} dy\, e^{-y} y^{(m-3)/2} \\
& - \frac{2}{\Gamma(m/2)} \int_{\{x\in D:\, d(x)\leqq R/2\}} \int_{Rd(x)/(4t)}^{\infty} dy\, e^{-y} y^{(m-2)/2} \\
\geqq & |D| - |D|\, t\, 2^{6+m/2}/(eR^2) - \int_0^{\infty} dq\, |\partial D| (1+(m-1)\, 2^{m-1} q/R) \int_q^{\infty} \frac{d\xi}{(\pi t)^{1/2}} e^{-\xi^2/(4t)} \\
& - \frac{2}{\Gamma((m-1)/2)} \cdot 2^{m-1} |\partial D| \int_0^{\infty} dq \int_0^{q} \frac{d\xi}{(\pi t)^{1/2}} e^{-\xi^2/(4t)} \int_{R(q-\xi)/(4t)}^{\infty} dy\, e^{-y} y^{(m-3)/2} \\
& - \frac{2}{\Gamma(m/2)} \cdot 2^{m-1} |\partial D| \int_0^{\infty} dq \int_{Rq/(4t)}^{\infty} dy\, e^{-y} y^{(m-2)/2} \\
= & |D| - \frac{2t^{1/2}}{\pi^{1/2}} |\partial D| - |D|\, t\, 2^{6+m/2}/(eR^2) - \frac{|\partial D|\, t}{R} (9m-5)\, 2^{m-1}.
\end{aligned} \tag{6.16}$$

The lower bound in Theorem 6.2 follows from (6.15) and (6.16).

Corollary 6.8. *Let D be an open, bounded and connected set in $\mathbb{R}^m$, $m=2, 3, \ldots$ with R-smooth boundary ∂D. Let h: $[0, \infty) \to R$ be C^1 and $h(0)=0$. Let q: $\bar{D} \times [0, \infty) \to R$ be the (classical) solution of*

$$\Delta q = \frac{\partial q}{\partial t} \quad \text{on } D\times(0,\infty), \tag{6.17}$$

$$q(x;t) = h(t), \qquad x\in\partial D, \qquad t\geqq 0, \tag{6.18}$$

$$q(x;0) = 0, \qquad x\in D, \tag{6.19}$$

where Δ is the laplacian. Then for $t>0$

$$\left| \int_D q(x;t)\, dx - \frac{|\partial D|}{\pi^{1/2}} \int_0^t h(\tau)(t-\tau)^{-1/2} d\tau \right| \leqq 10^m \frac{|D|}{R^2} \int_0^t |h'(\tau)|(t-\tau)\, d\tau. \tag{6.20}$$

Proof. The solution of (6.17)–(6.19) is given by

$$q(x;t) = \int_0^t h'(\tau)\, P_x[T_D < t-\tau]\, d\tau. \tag{6.21}$$

Hence by Fubini's theorem

$$\int_D q(x;t)\,dx = \int_0^t h'(\tau)\Big(|D| - Q_D(t-\tau)\Big)\,d\tau. \tag{6.22}$$

Corollary 6.8 follows from (6.22) and Theorem 6.2.

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