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Mean curvature and the heat equation

M. van den Berg[1], **J.F. Le Gall**[2]

[1] Department of Mathematics, Heriot-Watt University, Riccarton, Edinburgh EH14 4AS, United Kingdom

[2] Laboratoire de Probabilités, Université Pierre et Marie Curie, 4, Place Jussieu, Tour 56-66, F-75252 Paris Cedex 05, France

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1 Introduction

Consider the following problem in heat conduction: suppose that a compact set K in $\mathbb{R}^m$ is held at temperature one for all positive times t, while $\mathbb{R}^m \setminus K$ has temperature zero at time $t = 0$; what is the asymptotic behaviour as $t \downarrow 0$ of $E_K(t)$, the amount of heat which has flowed from K into $\mathbb{R}^m \setminus K$ up to time t? So far, this problem has been studied in detail only in the special case in which $m = 2$, K is connected and its boundary is polygonal [2]; in contrast, the problem of determining the asymptotic behaviour of $E_K(t)$ as $t \to \infty$ has been studied exhaustively [9, 10, 13].

In this paper, we determine the first and second term in the asymptotic behaviour as $t \downarrow 0$ of $E_K(t)$ for a compact connected set K in $\mathbb{R}^m$ for $m = 2, 3, \ldots$ under the condition that the boundary ∂K of K is C^3. We also obtain an estimate for the remainder.

There is a second problem in heat conduction which is closely related to the one we have described. Let D be an open, bounded and connected set in $\mathbb{R}^m$ with boundary ∂D and suppose that D has temperature one at time $t = 0$, while $\mathbb{R}^m \setminus D$ is held at temperature zero for all positive times t; what is the asymptotic behaviour as $t \downarrow 0$ of $Q_D(t)$, the amount of heat in D at time t?

In this paper, we determine the first three terms in the asymptotic behaviour and an estimate for the remainder of $Q_D(t)$ as $t \downarrow 0$ for an open, bounded and connected set D in $\mathbb{R}^m$ for $m = 2, 3, \ldots$ under the condition that ∂D is C^3; this improves a result of [1].

Let Δ_D be the Dirichlet laplacian for an open set D, and let $u: D \times [0, \infty) \to \mathbb{R}$ be the unique solution of

$$\Delta_D u = \frac{\partial u}{\partial t}, \quad t > 0, \tag{1.1}$$

$$u = 1, \quad t = 0. \tag{1.2}$$

We define for $t \geqq 0$

$$Q_D(t) = \int_D u(x; t)\, dx. \tag{1.3}$$

Similarly, v: $\mathbb{R}^m \backslash K \times [0, \infty) \to \mathbb{R}$ is the unique solution of

$$\Delta_{\mathbb{R}^m \backslash K} v = \frac{\partial v}{\partial t}, \quad t > 0, \tag{1.4}$$

$$v = 1, \quad t = 0. \tag{1.5}$$

We define for $t \geqq 0$

$$E_K(t) = \int_{\mathbb{R}^m \backslash K} (1 - v(x; t))\, dx. \tag{1.6}$$

The main results of this paper are the following:

Theorem 1.1 *Let D be an open, bounded and connected set in $\mathbb{R}^m$ ($m = 2, 3, \ldots$) with a C^3 boundary ∂D. Let ∂D be oriented with a smooth, inward, unit normal vector field $\mathbb{N}$ and denote the mean curvature at a point $s \in \partial D$ by $H(s)$. Let D have m-dimensional Lebesgue measure $|D|_m$ and let ∂D have $(m-1)$-dimensional Lebesgue $|\partial D|_{m-1}$. Then there exists a constant C, depending on D such that for all $t \geqq 0$*

$$\left| Q_D(t) - |D|_m + 2(t/\pi)^{1/2} |\partial D|_{m-1} - 2^{-1}(m-1)t \int_{\partial D} H(s)\, ds \right| \leqq C t^{3/2}. \tag{1.7}$$

Theorem 1.1 improves a previous result (Theorem 6.2 of [1]), where the first two terms in the asymptotic expansion of $Q_D(t)$ as $t \downarrow 0$ were obtained, with an $O(t)$ estimate for the remainder. Theorem 1.1 also implies that for a planar, open, bounded and connected set D with a C^3 boundary ∂D

$$Q_D(t) = |D|_2 - 2(t/\pi)^{1/2} |\partial D|_1 + \pi t \chi(D) + O(t^{3/2}), \tag{1.8}$$

where $\chi(D)$ is the Euler–Poincaré characteristic for D (i.e. $\chi(D) = 1 - \#$(holes in D)).

Theorem 1.2 *Let K be a compact, connected set in $\mathbb{R}^m$ ($m = 2, 3, \ldots$) with a C^3 boundary ∂K. Let ∂K be oriented with a smooth, inward, unit normal vector field $\mathbb{N}$, and denote the mean curvature at a point $s \in \partial K$ by $H(s)$. Then there exists a constant C depending on K such that for all $t \geqq 0$*

$$\left| E_K(t) - 2(t/\pi)^{1/2} |\partial K|_{m-1} - 2^{-1}(m-1)t \int_{\partial K} H(s)\, ds \right| \leqq C t^{3/2}, \tag{1.9}$$

where $|\partial K|_{m-1}$ is the $(m-1)$-dimensional Lebesgue measure of ∂K.

Theorem 1.2 implies that for a planar, connected and compact set K with a C^3 boundary ∂K

$$E_K(t) = 2(t/\pi)^{1/2} |\partial K|_1 + \pi t \chi(K) + O(t^{3/2}), \tag{1.10}$$

where χ is the Euler–Poincaré characteristic for K. Formulas (1.8) and (1.10) have been conjectured in [2] on the basis of a polygonal approximation.

A heuristic derivation of (1.9) (under the additional assumption that K is convex) has been given in Sect. 2(d) of [12].

These results have a probabilistic interpretation. Let $(B(t), t \geqq 0; \mathbb{P}_x, x \in \mathbb{R}^m)$ be a brownian motion associated with $-\Delta + \frac{\partial}{\partial t}$, where Δ is the Laplace operator for $\mathbb{R}^m$. Since the generator of the brownian motion is the Laplace operator, the covariance matrix of $B(t)$ is $2tI$, where I is the identity matrix. As a consequence, several formulas below are slightly different from the classical ones valid for a standard brownian motion with covariance tI. For an open set D in $\mathbb{R}^m$, $D \ni x$, we define the first exit time by

$$T_D = \inf\{t > 0: B(t) \in \mathbb{R}^m \setminus D\} . \tag{1.11}$$

Then it is well-known [13] that

$$u(x; t) = \mathbb{P}_x[T_D > t] , \tag{1.12}$$

$$v(x; t) = \mathbb{P}_x[T_{\mathbb{R}^m \setminus K} > t] . \tag{1.13}$$

Moreover, define the Wiener sausage associated with the set K up to time t by

$$S_K(t) = \{x \in \mathbb{R}^m: x = B(s) + k, 0 \leqq s \leqq t, k \in K\} . \tag{1.14}$$

Then $S_K(t)$ is Lebesgue measurable for all $t > 0$, almost surely, and

$$\mathbb{E}_0(|S_K(t)|_m) = |K|_m + E_K(t) , \tag{1.15}$$

where the left hand side of (1.15) is the expectation (under $\mathbb{P}_0$) of the volume of the Wiener sausage up to time t, and $|K|_m$ is the volume of the compact set K.

The probabilistic interpretation of $Q_D(t)$ is given in the following:

Proposition 1.3 *Let D be an open, bounded and connected set in $\mathbb{R}^m$ with boundary ∂D. Let $\bar{D} = D \cup \partial D$. Then for all $t > 0$*

$$Q_D(t) = \mathbb{E}_0(|S_{\bar{D}}(t)|_m - |S_{\partial D}(t)|_m) , \tag{1.16}$$

where $S_{\bar{D}}(t)$ and $S_{\partial D}(t)$ are the Wiener sausages up to time t associated to $\bar{D}$ and ∂D, respectively.

Proof. Since D is bounded, ∂D is bounded in $\mathbb{R}^m$. But $\bar{D}$ and ∂D are closed sets in $\mathbb{R}^m$, hence they are compact. Consider the heat equation (1.4), (1.5), with $K = \partial D$. Then

$$1 - v(x; t) = \mathbb{P}_x[T_{\mathbb{R}^m \setminus \partial D} \leqq t] = \begin{cases} \mathbb{P}_x[T_D \leqq t], & x \in D , \\ \mathbb{P}_x[T_{\mathbb{R}^m \setminus \bar{D}} \leqq t], & x \in \mathbb{R}^m \setminus \bar{D} . \end{cases} \tag{1.17}$$

Hence by (1.6), (1.17) and (1.3)

$$\begin{aligned} E_{\partial D}(t) &= \int_{\mathbb{R}^m \setminus \partial D} dx \, \mathbb{P}_x[T_{\mathbb{R}^m \setminus \partial D} \leqq t] \\ &= \int_D dx \, \mathbb{P}_x[T_D \leqq t] + \int_{\mathbb{R}^m \setminus \bar{D}} dx \, \mathbb{P}_x[T_{\mathbb{R}^m \setminus \bar{D}} \leqq t] \\ &= |D|_m - Q_D(t) + E_{\bar{D}}(t) . \end{aligned} \tag{1.18}$$

Then (1.16) follows from (1.15) and (1.18) since

$$|\bar{D}|_m = |D|_m + |\partial D|_m . \tag{1.19}$$

While the behaviour for $t \downarrow 0$ of $Q_D(t)$ is very similar to the behaviour for $t \downarrow 0$ of $E_K(t)$, they are very different for $t \to \infty$. For a compact set K with positive newtonian capacity $E_K(t) \to \infty$ as $t \to \infty$ [9, 13], while for an open set D with finite volume $Q_D(t) \to 0$ as $t \to \infty$. More precisely, we have the following:

Proposition 1.4 *Let D be an open, bounded and connected set in $\mathbb{R}^m$. Let λ_1 denote the first eigenvalue of $-\Delta_D$ with a corresponding normalized eigenfunction ψ_1 in*

$L^2(D)$. *Then*

$$Q_D(t) = e^{-t\lambda_1} \|\psi_1\|_1^2 \{1 + O(t^{-m/2})\}, \quad t \to \infty , \tag{1.20}$$

and in particular

$$\lim_{t \to \infty} t^{-1} \log \mathbb{E}_0(|S_{\bar{D}}(t)|_m - |S_{\partial D}(t)|_m) = -\lambda_1 . \tag{1.21}$$

Proof. Since D is bounded, $|D|_m < \infty$. Hence the spectrum of $-\Delta_D$ is discrete: $\lambda_1 < \lambda_2 \leqq \ldots$, with a corresponding orthonormal set of eigenfunctions $\psi_1, \psi_2, \ldots$ in $L^2(D)$. The heat kernel $p_D(x, y; t)$ of $e^{t\Delta_D}$ is a positive C^∞ function on $D \times D \times (0, \infty)$ and has an eigenfunction expansion

$$p_D(x, y; t) = \sum_{j=1}^{\infty} e^{-t\lambda_j} \psi_j(x)\psi_j(y) . \tag{1.22}$$

The solution u of (1.3) and (1.4) is (by Fubini's theorem) given by

$$u(x; t) = \sum_{j=1}^{\infty} e^{-t\lambda_j} \psi_j(x) \int_D dy\, \psi_j(y) , \tag{1.23}$$

and hence by Fubini's theorem

$$Q_D(t) = \sum_{j=1}^{\infty} e^{-t\lambda_j} \left\{ \int_D dx\, \psi_j(x) \right\}^2 . \tag{1.24}$$

Then

$$\begin{aligned} Q_D(t) &\geqq e^{-t\lambda_1} \left\{ \int_D dx\, \psi_1(x) \right\}^2 \\ &= e^{-t\lambda_1} \|\psi_1\|_1^2 , \end{aligned} \tag{1.25}$$

since ψ_1 does not change sign on D. On the other hand, by Cauchy–Schwarz's inequality and (1.24)

$$Q_D(t) \leqq e^{-t\lambda_1} \|\psi_1\|_1^2 + \sum_{j=2}^{\infty} e^{-t\lambda_j} |D|_m . \tag{1.26}$$

Since D is connected, λ_1 has multiplicity 1, so that $\lambda_2 - \lambda_1 > 0$. Hence

$$\begin{aligned} \sum_{j=2}^{\infty} e^{-t\lambda_j} &= e^{-t\lambda_1} \sum_{j=2}^{\infty} e^{-t(\lambda_j - \lambda_1)} \\ &\leqq e^{-t\lambda_1} \sum_{j=1}^{\infty} e^{-t(\lambda_2 - \lambda_1)\lambda_j/\lambda_2} \\ &= e^{-t\lambda_1} \operatorname{trace}(e^{t(\lambda_2 - \lambda_1)\Delta_D/\lambda_2}) \\ &= e^{-t\lambda_1} \int_D dx\, p_D(x, x; t(\lambda_2 - \lambda_1)/\lambda_2) . \end{aligned} \tag{1.27}$$

But

$$p_D(x, y; t) \leqq (4\pi t)^{-m/2} e^{-|x-y|^2/(4t)}, \quad x \in D,\ y \in D,\ t > 0 , \tag{1.28}$$

so that by (1.26), (1.27) and (1.28)

$$Q_D(t) \leqq e^{-t\lambda_1} \{ \|\psi_1\|_1^2 + \{\lambda_2/(4\pi t(\lambda_2 - \lambda_1))\}^{m/2} \cdot |D|_m^2 \} . \tag{1.29}$$

Then (1.20) follows by (1.25) and (1.29). Finally (1.21) follows by (1.20) and Proposition 1.3.

We conclude this introduction with a sketch of the proof of Theorem 1.1, and refer to the remainder of this paper (Sect. 2, . . . , 8) for the details. We omit the proof of Theorem 1.2 since it follows similar lines. For any $\varepsilon > 0$ we define the open set D_ε by

$$D_\varepsilon = \left\{ x \in D\colon \min_{y \in \mathbb{R}^m \setminus D} |y - x| < \varepsilon \right\}. \tag{1.30}$$

Since ∂D is C^3 and compact, there exists a $\delta > 0$ such that for each $x \in D_\delta$, there exists a unique point $s \in \partial D$ for which

$$|s - x| = \min_{y \in \mathbb{R}^m \setminus D} |y - x| . \tag{1.31}$$

By the principle of not feeling the boundary [8]

$$\int_{D \setminus D_\delta} dx\, \mathbb{P}_x[T_D > t] = |D|_m - |D_\delta|_m + O(e^{-\delta^2/(8t)}) . \tag{1.32}$$

Let $x \in D_\delta$,

$$B^{(1)}(\tau) = B(\tau) \cdot \frac{(s - x)}{|s - x|}, \quad \tau \geqq 0 , \tag{1.33}$$

and let τ_t be the unique time (almost surely) such that

$$B^{(1)}(\tau_t) = \sup_{0 \leqq \tau \leqq t} B^{(1)}(\tau) . \tag{1.34}$$

Then

$$\mathbb{P}_x[T_D > t] \leqq \mathbb{P}_x[B(\tau_t) \in D] . \tag{1.35}$$

In Sect. 3 we will obtain a good approximation for the right hand side of (1.35). Using this approximation we prove in Sect. 5 that

$$\begin{aligned} \int_{D_\delta} dx\, \mathbb{P}_x[B(\tau_t) \in D] = {} & |D_\delta|_m - 2(t/\pi)^{1/2} |\partial D|_{m-1} \\ & + 2^{-1}(m-1)t \int_{\partial D} H(s)\, ds + O(t^{3/2}) . \end{aligned} \tag{1.36}$$

It turns out that the right hand side of (1.35) is also a very good approximation for $\mathbb{P}_x[T_D > t]$. To verify this we need to bound

$$\mathbb{P}_x[B(\tau_t) \in D] - \mathbb{P}_x[T_D > t] = \mathbb{P}_x[\tau_t < T_D \leqq t] + \mathbb{P}_x[T_D < \tau_t \leqq t,\, B(\tau_t) \in D] . \tag{1.37}$$

In Sect. 6 we will show that, using the strong Markov property at T_D,

$$\int_{D_\delta} dx\, \mathbb{P}_x[T_D < \tau_t \leqq t,\, B(\tau_t) \in D] = O(t^{3/2}) . \tag{1.38}$$

The proof that

$$\int_{D_\delta} dx\, \mathbb{P}_x[\tau_t < T_D \leqq t] = O(t^{3/2}) , \tag{1.39}$$

(Sect. 7) relies on properties of brownian meanders; these will be given in Sect. 4. Finally in Sect. 8, we combine all the estimates, and complete the proof of Theorem 1.1.

2 Geometric preliminaries

We recall the following from p. 395 of [5].

Definition 2.1 A boundary ∂D of an open set D in $\mathbb{R}^m$ $(m = 2, 3, \ldots)$ is of class C^k $(k = 0, 1, 2, \ldots)$ if (a) D is the interior of its closure, and (b) given any point $s \in \partial D$ there exists an open set $U(s)$ containing s, local cartesian coordinates $(y_1, \ldots, y_m) = (y', y_m)$, where $y' = (y_1, \ldots, y_{m-1})$, with $y = 0$ at $x = s$, an open ball $G(s)$ in $\mathbb{R}^{m-1}$, and a function $h(\cdot\,; s) \in C^k(G)$ such that $\partial D \cap U(s)$ has representation $y_m = h(y'; s)$, $y' \in G(s)$.

Remark 2.2 If D is an open, bounded and connected set in $\mathbb{R}^m$ $(m = 2, 3, \ldots)$, condition (a) is implied by condition (b); see Remark 1 on p. 396 of [5].

Let D be an open, bounded and connected set in $\mathbb{R}^m$ $(m = 2, 3, \ldots)$ with boundary ∂D of class C^3, oriented with an inward unit normal vector field $\mathbb{N}$: $\partial D \to \mathbb{R}^m$. We denote the tangent space at $s \in \partial D$ by T_s. It is possible to choose the local cartesian coordinates $(y_1, \ldots, y_m)$ at s such that $\mathbb{N}(s) = (0, \ldots, 0, 1)$. Then the Weingarten map is the self-adjoint linear map L_s: $T_s \to T_s$ defined by

$$L_s(v) = -(\nabla_v \mathbb{N})(s), \quad v \in T_s\,, \tag{2.1}$$

where ∇_v is the derivative with respect to v. We denote the $m-1$ eigenvalues of L_s (the principal curvatures at s) by $k_1(s), \ldots, k_{m-1}(s)$, $k_1(s) \leqq \cdots \leqq k_{m-1}(s)$, and a corresponding orthonormal set of eigenvectors (the principal curvature directions) by $v_1(s), \ldots, v_{m-1}(s)$ (See Chap. 9 in [14]). The mean curvature at s is defined by

$$H(s) = \frac{1}{m-1}\operatorname{trace}(L_s) = \frac{1}{m-1}\sum_{i=1}^{m-1} k_i(s)\,. \tag{2.2}$$

Finally we define for non empty sets A and B in $\mathbb{R}^m$

$$d(A, B) = \inf_{x \in A,\, y \in B} |x - y|\,, \tag{2.3}$$

so that for a point x in a non empty proper subset D of $\mathbb{R}^m$

$$d(x) = d(\{x\}, \mathbb{R}^m \backslash D)\,. \tag{2.4}$$

Since D is bounded ∂D is compact, and it is possible to choose a family $\{U(s), G(s), h(\cdot\,; s); s \in \partial D\}$ and a constant $\delta_0 > 0$ (independently of s) such that $G(s) \supset \{y' \in \mathbb{R}^{m-1}: |y'| < \delta_0\}$, $U(s) \supset \{y \in \mathbb{R}^m: |y - s| < \delta_0\}$.

Lemma 2.3 *Let D be an open, bounded and connected set in $\mathbb{R}^m$ $(m = 2, 3, \ldots)$ with a boundary ∂D of class C^3, oriented by an inward unit normal vector field $\mathbb{N}$. Then there exists a constant $\delta \in (0, \delta_0)$ such that* (a) *for all $x \in D_\delta$ there exists a unique point $s = s(x) \in \partial D$ with $|s(x) - x| = d(x)$, and* (b) *for all $y' \in \mathbb{R}^{m-1}$ with $|y'| < \delta$*

$$\left| h(y'; s) - \sum_{i=1}^{m-1} k_i(s) y_i^2/2 \right| \leqq \delta^{-2} |y'|^3\,, \tag{2.5}$$

where

$$y_i = y' \cdot v_i(s), \quad i = 1, \ldots, m-1 . \tag{2.6}$$

Proof. See Theorem 3.5 in [5] and the standard theory of focal points in Chap. 16 of [14].

Lemma 2.4 *Let D and δ be as in Lemma 2.3. The map Q: $D_\delta \to [0, \delta) \times \partial D$ given by $Q(x) = (s, r)$ where $s = s(x)$, $r = d(x)$ is C^1 and has jacobian*

$$J(s, r) = (\det dQ)(s, r) = \prod_{i=1}^{m-1} (1 - k_i(s)r) > 0 . \tag{2.7}$$

Proof. See 3.1–3.4 and 10.1–10.4 in [7].

Lemma 2.5 *Let D and δ be as in Lemma 2.3. Then for all $s \in \partial D$ and $r \in [0, \delta)$*

$$|J(s, r) - 1| \leqq (m-1)rK(1 + \delta K)^{m-2} , \tag{2.8}$$

$$|J(s, r) - 1 + (m-1)rH(s)| \leqq 2^{-1}(m-2)(m-1)r^2K^2(1 + \delta K)^{m-3} , \tag{2.9}$$

where

$$K = \max_{i \in \{1, \ldots, m-1\}, s \in \partial D} |k_i(s)| . \tag{2.10}$$

Proof. By (2.7)

$$k_i(s)r < 1, \tag{2.11}$$

for $s \in \partial D$ and $r \in [0, \delta)$. Hence

$$\begin{aligned} J(s, r) &\geqq \prod_{\{i:\, k_i(s) > 0\}} (1 - k_i(s)r) \geqq 1 - \sum_{\{i:\, k_i(s) > 0\}} k_i(s)r \\ &\geqq 1 - rK \cdot \#\{i: k_i(s) > 0\} \geqq 1 - (m-1)rK . \end{aligned} \tag{2.12}$$

Moreover

$$J(s, r) \leqq (1 + rK)^{m-1} \leqq 1 + (m-1)rK(1 + \delta K)^{m-2} , \tag{2.13}$$

and (2.8) follows from (2.12) and (2.13). By expanding the product in (2.7) we have

$$J(s, r) = 1 - (m-1)rH(s) + \sum_{\ell=2}^{m-1} (-r)^\ell \sum_{(i_1, \ldots, i_\ell) \in I(\ell; m)} k_{i_1}(s) \ldots k_{i_\ell}(s) , \tag{2.14}$$

where the index set $I(\ell; m)$ is given by

$$\begin{aligned} I(\ell; m) = \{&(i_1, \ldots, i_\ell) \in \{1, \ldots, m-1\}^\ell : \\ &i_p \neq i_q,\ p = 1, \ldots, \ell,\ q = 1, \ldots, \ell,\ p \neq q\} . \end{aligned} \tag{2.15}$$

Then

$$\begin{aligned} &\left| \sum_{\ell=2}^{m-1} (-r)^\ell \sum_{(i_1, \ldots, i_\ell) \in I(\ell; m)} k_{i_1}(s) \ldots k_{i_\ell}(s) \right| \\ &\leqq \sum_{\ell=2}^{m-1} r^\ell K^\ell \binom{m-1}{\ell} = (1 + rK)^{m-1} - 1 - (m-1)rK \\ &\leqq 2^{-1}(m-2)(m-1)r^2K^2(1 + \delta K)^{m-3} , \end{aligned} \tag{2.16}$$

which proves (2.9).

Lemma 2.6 *Let D and δ be as in Lemma 2.3. Then there exists a constant $\delta_1 \in (0, \delta/2)$ such that $x = (s, r) \in D_\delta$, $\sigma \in \partial D$, $|\sigma - x| < \delta_1$, $|w - \sigma| < \delta_1$ with*

$$\sigma = s + \sum_{i=1}^{m-1} \lambda_i v_i(s) + \lambda_m \mathbb{N}(s) , \tag{2.17}$$

$$w = s + \sum_{i=1}^{m-1} w_i v_i(s) + w_m \mathbb{N}(s) , \tag{2.18}$$

and

$$w_m \leqq \lambda_m + \nabla h(\lambda'; s) \cdot (w' - \lambda') - K|w' - \lambda'|^2 , \tag{2.19}$$

implies (i) $\delta > |\lambda'|$, (ii) $\lambda_m = h(\lambda'; s)$, (iii) $w \notin D$, *and* (iv)

$$|\nabla h(\lambda'; s)| \leqq 2K|\lambda'| . \tag{2.20}$$

Proof. We first establish (i), (ii), (iii) and (iv) for s fixed and a constant $\delta_1(s) \in (0, \delta/2)$. For any $\delta_1(s) \in (0, \delta/2)$

$$\delta > 2\delta_1(s) > 2|\sigma - x| \geqq |\sigma - x| + |s - x| \geqq |\sigma - s| \geqq |\lambda'| . \tag{2.21}$$

This proves (i). By Definition 2.1 there exists an open set $U(s)$ containing s such that $\partial D \cap U(s)$ is represented by $y_m = h(y'; s)$, where $h(\cdot; s)$ is of class C^3. By (2.21) $|\sigma - s| < 2\delta_1(s) < \delta < \delta_0$ and $\lambda_m = h(\lambda'; s)$ by the choice of δ_0. This proves (ii). By Taylor's expansion about λ'

$$\begin{aligned} y_m = h(y'; s) &= h(\lambda'; s) + \nabla h(\lambda'; s) \cdot (y' - \lambda') \\ &+ \frac{1}{2} \sum_{i=1}^{m-1} \sum_{j=1}^{m-1} \frac{\partial^2 h(\lambda'; s)}{\partial y_i \partial y_j} (y_i - \lambda_i)(y_j - \lambda_j) + R(s, \lambda', y') , \end{aligned} \tag{2.22}$$

where

$$R(s, \lambda', y') = O(|y' - \lambda'|^3) . \tag{2.23}$$

Note that

$$\left| \sum_{i=1}^{m-1} \sum_{j=1}^{m-1} \frac{\partial^2 h(0; s)}{\partial y_i \partial y_j} y_i y_j \right| \leqq \max_i |k_i(s)| |y'|^2 \leqq K|y'|^2 . \tag{2.24}$$

Therefore if $|\lambda'| \leqq |\sigma - s|$ is sufficiently small we have

$$\left| \sum_{i=1}^{m-1} \sum_{j=1}^{m-1} \frac{\partial^2 h(\lambda'; s)}{\partial y_i \partial y_j} (y_i - \lambda_i)(y_j - \lambda_j) \right| \leqq \frac{3}{2} K|y' - \lambda'|^2 , \tag{2.25}$$

and provided that $|y' - \lambda'|$ is also small

$$|R(s, \lambda', y')| \leqq \frac{1}{4} K|y' - \lambda'|^2 . \tag{2.26}$$

It follows that

$$h(y'; s) \geqq h(\lambda'; s) + \nabla h(\lambda'; s) \cdot (y' - \lambda') - K|y' - \lambda'|^2 , \tag{2.27}$$

and (2.19) implies $w_m \leqq h(w'; s)$. For $\delta_1(s)$ sufficiently small $w \in U(s)$, and hence $w \notin D$. This proves (iii). Since $h(\cdot; s) \in C^3$, $\dfrac{\partial h(\cdot; s)}{\partial v_i} \in C^2$ for $i = 1, \ldots, m - 1$ where

$v_1, \ldots, v_{m-1}$ are the principal curvature directions. Hence

$$\frac{\partial h(\lambda'; s)}{\partial v_i} = \lambda_i k_i(s) + O(|\lambda'|^2), \tag{2.28}$$

and

$$|\nabla h(\lambda'; s)|^2 = \sum_{i=1}^{m-1} \lambda_i^2 k_i^2(s) + O(|\lambda'|^3) \leqq K^2 |\lambda'|^2 + O(|\lambda'|^3) \leqq 4K^2 |\lambda'|^2, \tag{2.29}$$

if $\delta_1(s)$ is sufficiently small. This proves (iv). A compactness argument completes the proof of the existence of a constant $\delta_1 \in (0, \delta/2)$ independent of s.

3 Bounds for $\mathbb{P}_x[B(\tau_t) \in D]$

In this section we obtain upper and lower bounds for $\mathbb{P}_x[B(\tau_t) \in D]$. First we recall some elementary lemmas for a brownian motion associated with $-\Delta + \frac{\partial}{\partial t}$.

Lemma 3.1 *Let $(B(t),\ t \geqq 0;\ \mathbb{P}_x,\ x \in \mathbb{R}^m)$ be a brownian motion associated with $-\Delta + \frac{\partial}{\partial t}$. Then for any $\delta > 0$*

$$\mathbb{P}_x[|B(t) - x| \geqq \delta] \leqq 2^{m/2} e^{-\delta^2/(8t)}. \tag{3.1}$$

Proof. For any Borel set D in $\mathbb{R}^m$

$$\mathbb{P}_x[B(t) \in D] = (4\pi t)^{-m/2} \int_D e^{-|x-y|^2/(4t)} dy. \tag{3.2}$$

Hence

$$\begin{aligned}\mathbb{P}_x[|B(t) - x| \geqq \delta] &= (4\pi t)^{-m/2} \int_{\{|x-y| \geqq \delta\}} e^{-|x-y|^2/(4t)} dy \\ &\leqq (4\pi t)^{-m/2} e^{-\delta^2/(8t)} \int_{\{|x-y| \geqq \delta\}} e^{-|x-y|^2/(8t)} dy \\ &\leqq 2^{m/2} e^{-\delta^2/(8t)}. \end{aligned} \tag{3.3}$$

Lemma 3.2 *Let $(B^{(1)}(\tau),\ \tau \geqq 0;\ \mathbb{P}_0)$ be a brownian motion associated with $-\frac{\partial^2}{\partial x_m^2} + \frac{\partial}{\partial \tau}$ with $B^{(1)}(0) = 0$. Let $t > 0$ be fixed and let τ_t be the unique time (almost surely) such that*

$$B^{(1)}(\tau_t) = \sup_{0 \leqq \tau \leqq t} B^{(1)}(\tau). \tag{3.4}$$

Then the density of $(B^{(1)}(\tau_t), \tau_t)$ is given by

$$\Phi(\xi, \tau; t) = \frac{\xi e^{-\xi^2/(4\tau)}}{2\pi \tau^{3/2}(t-\tau)^{1/2}} 1_{[0,t)}(\tau)\, 1_{[0,\infty)}(\xi). \tag{3.5}$$

Proof. See p. 510 in [6]. Formula (3.5) also follows from Lemma 4 in [11].

Lemma 3.3 *Let $(B(t),\ t \geqq 0;\ \mathbb{P}_x,\ x \in \mathbb{R}^m)$ be a brownian motion associated with $-\Delta + \frac{\partial}{\partial t}$. Then for any open set D containing x*

$$\mathbb{P}_x[T_D \leqq t] \leqq 2^{(2+m)/2} e^{-d^2(x)/(8t)} . \tag{3.6}$$

and

$$\mathbb{P}_x\left[\sup_{0 \leqq \tau \leqq t} |B(\tau) - x| > \delta\right] \leqq 2^{(2+m)/2} e^{-\delta^2/(8t)} . \tag{3.7}$$

Proof. See the proof of Lemma 4 in [2].

Lemma 3.4 *Let D and δ be as in Lemma* 2.3, *let $x = (s, r) \in D_\delta$, and let τ_t be as in* (1.34). *Then*

$$\left|\mathbb{P}_x[B(\tau_t) \in D] - \int_0^\infty \mathrm{d}\xi \int_0^t d\tau\, \Phi(\xi, \tau; t)(4\pi\tau)^{-(m-1)/2} \int_{\{h(y';s) < r-\xi,\ |y'| < \delta\}} dy'\, e^{-|y'|^2/(4\tau)}\right|$$
$$\leqq 2^{(m+2)/2} e^{-\delta^2/(8t)} . \tag{3.8}$$

Proof. Let $x \in D_\delta$ be arbitrary, let

$$B(\tau) = x + \sum_{i=1}^{m-1} B_i(\tau)\, v_i(s) - B_m(\tau)\, \mathbb{N}(s) , \tag{3.9}$$

and put

$$B' = (B_1, \ldots, B_{m-1}) , \tag{3.10}$$

where $B_1, \ldots, B_m$ are independent one dimensional brownian motions, with $B_1(0) = \ldots B_m(0) = 0$. Note that $B^{(1)} = B_m$ by (1.33). We will consider the brownian motion B under the probability measure $\mathbb{P}_x$. It should be understood that $B_1, \ldots, B_m$, or τ_t are defined with x as a reference point. Then by Lemma 3.1

$$\begin{aligned}\mathbb{P}_x[B(\tau_t) \in D] &= \mathbb{P}_x\left[B(\tau_t) \in D,\ \sup_{0 \leqq \tau \leqq t} |B(\tau)| < \delta\right] \\ &\quad + \mathbb{P}_x\left[B(\tau_t) \in D,\ \sup_{0 \leqq \tau \leqq t} |B(\tau)| \geqq \delta\right] \\ &\leqq \mathbb{P}_x\left[B(\tau_t) \in D,\ \sup_{0 \leqq \tau \leqq t} |B(\tau)| < \delta\right] + 2^{(m+2)/2} e^{-\delta^2/(8t)} .\end{aligned} \tag{3.11}$$

Furthermore

$$\begin{aligned}&\mathbb{P}_x\left[B(\tau_t) \in D,\ \sup_{0 \leqq \tau \leqq t} |B(\tau)| < \delta\right] \\ &\quad = \mathbb{P}_x\left[h(B'(\tau_t); s) < r - B_m(\tau_t),\ \sup_{0 \leqq \tau \leqq t} |B(\tau)| < \delta\right] \\ &\quad \leqq \mathbb{P}_x[h(B'(\tau_t); s) < r - B_m(\tau_t), |B'(\tau_t)| < \delta] .\end{aligned} \tag{3.12}$$

By Lemma 3.1 we also have

$$\begin{aligned}\mathbb{P}_x[B(\tau_t)\in D] &\geqq \mathbb{P}_x\Big[B(\tau_t)\in D,\ \sup_{0\leqq\tau\leqq t}|B(\tau)|<\delta\Big]\\ &= \mathbb{P}_x\Big[h(B'(\tau_t);s)<r-B_m(\tau_t),\ \sup_{0\leqq\tau\leqq t}|B(\tau)|<\delta\Big]\\ &= \mathbb{P}_x[h(B'(\tau_t);s)<r-B_m(\tau_t),|B'(\tau_t)|<\delta]\\ &\quad-\mathbb{P}_x\Big[h(B'(\tau_t);s)<r-B_m(\tau_t),\ \sup_{0\leqq\tau\leqq t}|B(\tau)|\geqq\delta,|B'(\tau_t)|<\delta\Big]\\ &\geqq \mathbb{P}_x[h(B'(\tau_t);s)<r-B_m(\tau_t),|B'(\tau_t)|<\delta]\\ &\quad-2^{(m+2)/2}e^{-\delta^2/(8t)}\,. \end{aligned}\tag{3.13}$$

By the independence of B' and B_m, and by (3.1), (3.5) we have

$$\begin{aligned}&\mathbb{P}_x[h(B'(\tau_t);s)<r-B_m(\tau_t),|B'(\tau_t)|<\delta]\\ &=\int_0^\infty d\xi\int_0^t d\tau\,\Phi(\xi,\tau;t)(4\pi\tau)^{-(m-1)/2}\int_{\{h(y':s)<r-\xi,\,|y'|<\delta\}}dy'\,e^{-|y'|^2/(4\tau)}\,,\end{aligned}\tag{3.14}$$

and the lemma follows from (3.11)–(3.14).

4 Some estimates for the brownian meander

In this section we obtain some estimates for the brownian meander that will be used in Sect. 7 to complete the proof of (1.39). We denote by $(Z(\tau), 0\leqq\tau\leqq 1)$ a brownian meander on the time interval $[0,1]$, with $Z(0)=0$. We refer to [3] and the references in that paper for a precise definition and the main properties of the brownian meander. We recall the following.

Lemma 4.1 *The transition density of the brownian meander Z is given by*

$$\begin{aligned}p(\tau,t;\xi,\eta)&=\left(\frac{1-\tau}{4\pi(t-\tau)(1-t)}\right)^{1/2}\left(e^{-(\eta-\xi)^2/(4t-4\tau)}-e^{-(\eta+\xi)^2/(4t-4\tau)}\right)\\ &\quad\cdot\frac{\int_0^\eta e^{-v^2/(4-4t)}\,dv}{\int_0^\xi e^{-v^2/(4-4\tau)}\,dv}\cdot 1_{[0,\infty)}(\xi)\,1_{[0,\infty)}(\eta)\,,\end{aligned}\tag{4.1}$$

where $0<\tau<t\leqq 1$, *and*

$$p(0,t;0,\eta)=\frac{\eta\,e^{-\eta^2/(4t)}}{(4\pi t^3(1-t))^{1/2}}\int_0^\eta dv\,e^{-v^2/(4-4t)}\,.\tag{4.2}$$

Proof. Note that Z is a time inhomogeneous Markov process, such that for every $u\in(0,1]$, $Z(u)>0$ almost surely, and such that the conditional distribution of $(Z(u+\tau), 0\leqq\tau\leqq 1-u)$, given $Z(u)$ coincides with the distribution of a linear brownian motion $(B_1(\tau), 0<\tau\leqq 1-u)$, with $B_1(0)=Z(u)$, and conditioned by the event $\inf_{0\leqq\tau\leqq 1-u}B_1(\tau)\geqq 0$. Formula (4.1) follows by an application of the reflection principle. Formula (4.2) follows by taking the limit $\xi\downarrow 0$ in (4.1).

Lemma 4.2 *There exists a constant $M_1 \in (0, \infty)$ such that for all $\varepsilon > 0$ and all $\delta \in (0, 1]$*

$$\mathbb{P}\left[\inf_{\delta \leqq u \leqq 1} Z(u) \leqq \varepsilon\right] \leqq M_1 \varepsilon \delta^{-1/2} . \tag{4.3}$$

Proof. We first assume $\delta \leqq 1/2$. Then

$$\mathbb{P}\left[\inf_{\delta \leqq u \leqq 1} Z(u) \leqq \varepsilon\right] = \mathbb{P}[Z(\delta) \leqq \varepsilon] + \mathbb{P}\left[Z(\delta) > \varepsilon, \inf_{\delta \leqq u \leqq 1} Z(u) \leqq \varepsilon\right]$$

$$= \int_0^\varepsilon d\xi\, p(0,\delta;0,\xi) + \int_\varepsilon^\infty d\xi\, p(0,\delta;0,\xi) \frac{\mathbb{P}_\xi\left[0 < \inf_{0 \leqq u \leqq 1-\delta} B_1(u) < \varepsilon\right]}{\mathbb{P}_\xi\left[0 < \inf_{0 \leqq u \leqq 1-\delta} B_1(u)\right]} , \tag{4.4}$$

where B_1 is a one dimensional brownian motion starting at ξ under the probability $\mathbb{P}_\xi$. We have

$$\int_0^\varepsilon d\xi\, p(0,\delta;0,\xi) \leqq \frac{\varepsilon}{(4\pi\delta^3(1-\delta))^{1/2}} \int_0^\varepsilon d\xi\, \xi\, e^{-\xi^2/(4\delta)}$$

$$= \frac{\varepsilon}{(\pi\delta(1-\delta))^{1/2}} (1 - e^{-\varepsilon^2/(4\delta)}) \leqq \varepsilon(2/(\pi\delta))^{1/2} , \tag{4.5}$$

by our assumption on δ. On the other hand using the formula for the distribution of the supremum of a one dimensional brownian motion gives

$$\int_\varepsilon^\infty d\xi\, p(0,\delta;0,\xi) \frac{\mathbb{P}_\xi\left[0 < \inf_{0 \leqq u \leqq 1-\delta} B_1(u) < \varepsilon\right]}{\mathbb{P}_\xi\left[0 < \inf_{0 \leqq u \leqq 1-\delta} B_1(u)\right]}$$

$$= \int_\varepsilon^\infty d\xi\, p(0,\delta;0,\xi) \frac{\int_0^\varepsilon dv\, e^{-(\xi-v)^2/(4-4\delta)}}{\int_0^\xi dv\, e^{-(\xi-v)^2/(4-4\delta)}}$$

$$= (4\pi\delta^3(1-\delta))^{-1/2} \int_\varepsilon^\infty d\xi\, \xi\, e^{-\xi^2/(4\delta)} \int_0^\varepsilon dv\, e^{-(\xi-v)^2/(4-4\delta)}$$

$$\leqq \varepsilon(2/(\pi\delta))^{1/2} . \tag{4.6}$$

Hence (4.3) holds for $\delta \leqq 1/2$ with $M_1 = (8/\pi)^{1/2}$. If we take $M_1 = (16/\pi)^{1/2}$, we obtain the bound for any $\delta \in (0, 1]$ after replacing δ by $\delta/2$. (Note that it is also possible to prove Lemma 4.2 by using the relationship between the brownian meander and the brownian bridge [3, Théorème 8].)

Let $d = m - 1$ and let $R = (R(\tau), \tau \geqq 0)$ be a d-dimensional Bessel process with $R(0) = 0$, independent of the brownian meander Z. Since R is distributed as the euclidean norm of a d-dimensional brownian motion starting at 0 we have by (3.7)

$$\mathbb{P}\left[\sup_{0 \leqq \tau \leqq u} R(\tau) \geqq \xi\right] \leqq 2^{(2+d)/2} e^{-\xi^2/(8u)} . \tag{4.7}$$

Let $h\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$h(\xi) = \gamma + a\xi + b\xi^2 , \tag{4.8}$$

where $\gamma < 0$, $a > 0$ and $b > 0$ are constants, and denote the positive root of $h(\xi) = 0$ by ξ_0. Then

$$\xi_0 = (2b)^{-1}(-a + (a^2 - 4\gamma b)^{1/2}) , \tag{4.9}$$

and we have the following.

Lemma 4.3 *There exist two constants $M_2 \in (0, \infty)$, $M_3 \in (0, \infty)$, that do not depend on a, b, γ such that*

$$\mathbb{P}[\exists \tau \in [0, 1] : Z(\tau) \leqq h(R(\tau))] \leqq M_2(a + b)(1 + \log^+(1/\xi_0))e^{-M_3\xi_0^2} , \tag{4.10}$$

where the $+ (-)$ denotes the positive (negative) part.

Proof. By Lemma 4.2 and (4.7) we have

$$\begin{aligned} &\mathbb{P}[\exists \tau \in [1/2, 1] : Z(\tau) \leqq h(R(\tau))] \\ &\leqq \sum_{n=0}^{\infty} \mathbb{P}\Big[\sup_{0 \leqq \tau \leqq 1} R(\tau) \geqq 2^n\xi_0, \inf_{1/2 \leqq \tau \leqq 1} Z(\tau) \leqq h(2^{n+1}\xi_0)\Big] \\ &\leqq \sum_{n=0}^{\infty} 2^{(2+d)/2} e^{-2^{2n}\xi_0^2/8} \cdot 2^{1/2} M_1 h(2^{n+1}\xi_0) . \end{aligned} \tag{4.11}$$

Since

$$h(2^{n+1}\xi_0) \leqq a2^{n+1}\xi_0 + b2^{2(n+1)}\xi_0^2 , \tag{4.12}$$

we obtain

$$\begin{aligned} &\mathbb{P}[\exists \tau \in [1/2, 1] : Z(\tau) \leqq h(R(\tau))] \\ &\leqq 2^{(7+d)/2} M_1 \sum_{n=0}^{\infty} (a2^n\xi_0 + b2^{2n}\xi_0^2) e^{-2^{2n}\xi_0^2/8} \\ &\leqq M_2'(a + b)e^{-\xi_0^2/16} , \end{aligned} \tag{4.13}$$

with

$$M_2' = 3.2^{(15+d)/2} M_1 . \tag{4.14}$$

Similarly, for any integer $p \geqq 1$,

$$\begin{aligned} &\mathbb{P}[\exists \tau \in [2^{-p-1}, 2^{-p}] : Z(\tau) \leqq h(R(\tau))] \\ &\leqq \sum_{n=0}^{\infty} \mathbb{P}\Big[\sup_{0 \leqq \tau \leqq 2^{-p}} R(\tau) \geqq 2^n\xi_0, \inf_{2^{-p-1} \leqq \tau \leqq 1} Z(\tau) \leqq h(2^{n+1}\xi_0)\Big] \\ &\leqq \sum_{n=0}^{\infty} 2^{(2+d)/2} e^{-2^{2n+p}\xi_0^2/8} \cdot 2^{(p+1)/2} M_1 h(2^{n+1}\xi_0) \\ &\leqq 2^{(7+d)/2} M_1 \sum_{n=0}^{\infty} (a2^n(2^{p/2}\xi_0) + b2^{2n}(2^{p/2}\xi_0)^2) e^{-2^{2n}(2^{p/2}\xi_0)^2/8} \\ &\leqq M_2'(a + b)e^{-2^p\xi_0^2/16} \end{aligned} \tag{4.15}$$

as before (ξ_0 is replaced by $2^{p/2}\xi_0$).

Finally, since $h(R(0)) = h(0) < 0$,

$$\mathbb{P}[\exists\, \tau \in [0, 1]: Z(\tau) \leqq h(R(\tau))] \leqq M_2'(a+b) \sum_{p=0}^{\infty} e^{-2^p \xi_0^2/16}$$

$$\leqq M_2'(a+b) e^{-\xi_0^2/32} \sum_{p=0}^{\infty} e^{-2^p \xi_0^2/32}$$

$$\leqq M_2'(a+b)\left(2 + \frac{\log^+(32/\xi_0^2)}{\log 2}\right) e^{-\xi_0^2/32}, \tag{4.16}$$

which gives the bound of Lemma 4.3, with $M_2 = 7M_2'$ and $M_3 = 1/32$.

We finally recall without proof a theorem of Denisov [4] which relates the brownian meander to the one dimensional brownian motion.

Theorem 4.4 *Let $\beta = (\beta(\tau), \tau \geqq 0)$ be a one dimensional brownian motion with $\beta(0) = 0$ and let $t > 0$. Let τ_t be the almost surely unique time such that $\tau_t \in [0, t]$ and*

$$\beta(\tau_t) = \sup_{0 \leqq \tau \leqq t} \beta(\tau). \tag{4.17}$$

Define for $u \in [0, 1]$

$$Z(u) = (t - \tau_t)^{-1/2} (\beta(\tau_t) - \beta(\tau_t + u(t - \tau_t))), \tag{4.18}$$

$$Z'(u) = (\tau_t)^{-1/2} (\beta(\tau_t) - \beta(\tau_t - u\tau_t)). \tag{4.19}$$

Then the processes Z, Z' are two independent brownian meanders on the time interval $[0, 1]$, and the pair (Z, Z') is independent of τ_t.

Remark 4.5 Since $\beta(\tau_t) = (\tau_t)^{1/2} Z'(1)$, Theorem 4.4 implies in particular that Z is independent of the pair $(\tau_t, \beta(\tau_t))$.

5 Bounds for $\int_{D_\delta} dx\, \mathbb{P}_x[B(\tau_t) \in D]$

In this section we prove the following.

Lemma 5.1 *Let D and δ be as in Lemma 2.3. Then there exists a constant $C_1 \in (0, \infty)$ depending on K, $|\partial D|_{m-1}$, $|D|_m$, and δ such that for all $t > 0$*

$$\left| \int_{D_\delta} dx\, \mathbb{P}_x[B(\tau_t) \in D] - |D_\delta|_m + 2(t/\pi)^{1/2} |\partial D|_{m-1} - 2^{-1}(m-1)t \int_{\partial D} H(s)\, ds \right|$$
$$\leqq C_1 t^{3/2}. \tag{5.1}$$

Proof. By Lemma 3.4 and Lemma 2.4

$$\left| \int_{D_\delta} dx\, \mathbb{P}_x[B(\tau_t) \in D] \right.$$
$$\left. - \int_0^\delta dr \int_{\partial D} ds\, J(s, r) \int_0^\infty d\xi \int_0^t d\tau\, \Phi(\xi, \tau; t) (4\pi\tau)^{(1-m)/2} \int_{\{h(y';s) < r - \xi,\, |y'| < \delta\}} dy' e^{-|y'|^2/(4\tau)} \right|$$
$$\leqq 2^{(m+2)/2} e^{-\delta^2/(8t)} |D_\delta|_m \leqq 2^{(m+8)/2} (3/e)^{3/2} |D_\delta|_m \delta^{-3} t^{3/2}. \tag{5.2}$$

Moreover

$$\int_0^\delta dr \int_{\partial D} ds\, J(s,r) \int_0^\infty d\xi \int_0^t d\tau\, \Phi(\xi,\tau;t)(4\pi\tau)^{(1-m)/2} \times \int_{\{h(y';s)<r-\xi,\,|y'|<\delta\}} dy' e^{-|y'|^2/(4\tau)} = \sum_{i=1}^8 A_i(t)\,, \tag{5.3}$$

where

$$A_1(t) = \int_0^\delta dr \int_{\partial D} ds\, J(s,r) \int_0^r d\xi \int_0^t d\tau\, \Phi(\xi,\tau;t)\,, \tag{5.4}$$

$$A_2(t) = -\int_0^\infty dr \int_{\partial D} ds \int_0^r d\xi \int_0^t d\tau\, \Phi(\xi,\tau;t)(4\pi\tau)^{(1-m)/2} \times \int_{\{h(y';s)>r-\xi,\,|y'|<\delta\}} dy' e^{-|y'|^2/(4\tau)}\,, \tag{5.5}$$

$$A_3(t) = \int_\delta^\infty dr \int_{\partial D} ds \int_0^r d\xi \int_0^t d\tau\, \Phi(\xi,\tau;t)(4\pi\tau)^{(1-m)/2} \times \int_{\{h(y';s)>r-\xi,\,|y'|<\delta\}} dy' e^{-|y'|^2/(4\tau)}\,, \tag{5.6}$$

$$A_4(t) = -\int_0^\delta dr \int_{\partial D} ds\, J(s,r) \int_0^r d\xi \int_0^t d\tau\, \Phi(\xi,\tau;t)(4\pi\tau)^{(1-m)/2} \times \int_{\{|y'|>\delta\}} dy' e^{-|y'|^2/(4\tau)}\,, \tag{5.7}$$

$$A_5(t) = \int_0^\delta dr \int_{\partial D} ds\,(1-J(s,r)) \int_0^r d\xi \int_0^t d\tau\, \Phi(\xi,\tau;t)(4\pi\tau)^{(1-m)/2} \times \int_{\{h(y';s)>r-\xi,\,|y'|<\delta\}} dy' e^{-|y'|^2/(4\tau)}\,, \tag{5.8}$$

$$A_6(t) = \int_0^\infty dr \int_{\partial D} ds \int_r^\infty d\xi \int_0^t d\tau\, \Phi(\xi,\tau;t)(4\pi\tau)^{(1-m)/2} \times \int_{\{h(y';s)<r-\xi,\,|y'|<\delta\}} dy' e^{-|y'|^2/(4\tau)}\,, \tag{5.9}$$

$$A_7(t) = -\int_\delta^\infty dr \int_{\partial D} ds \int_r^\infty d\xi \int_0^t d\tau\, \Phi(\xi,\tau;t)(4\pi\tau)^{(1-m)/2} \times \int_{\{h(y';s)<r-\xi,\,|y'|<\delta\}} dy' e^{-|y'|^2/(4\tau)}\,, \tag{5.10}$$

$$A_8(t) = \int_0^\delta dr \int_{\partial D} ds\,(J(s,r)-1) \int_r^\infty d\xi \int_0^t d\tau\, \Phi(\xi,\tau;t)(4\pi\tau)^{(1-m)/2} \times \int_{\{h(y';s)<r-\xi,\,|y'|<\delta\}} dy' e^{-|y'|^2/(4\tau)}\,. \tag{5.11}$$

Since

$$\int_0^\delta dr \int_{\partial D} ds\, J(s,r) = |D_\delta|_m . \tag{5.12}$$

and

$$\int_0^r d\xi \int_0^t d\tau\, \Phi(\xi,\tau;t) = 1 - (\pi t)^{-1/2} \int_r^\infty e^{-\xi^2/(4t)} d\xi , \tag{5.13}$$

we have for $A_1(t)$

$$A_1(t) = |D_\delta|_m - 2(t/\pi)^{1/2} |\partial D|_{m-1} + (m-1)t \int_{\partial D} H(s) ds + C_1(t) + C_2(t) , \tag{5.14}$$

where

$$C_1(t) = (\pi t)^{-1/2} \int_\delta^\infty dr \int_{\partial D} ds (1 - (m-1) r H(s)) \int_r^\infty e^{-\xi^2/(4t)} d\xi , \tag{5.15}$$

$$C_2(t) = (\pi t)^{-1/2} \int_0^\delta dr \int_{\partial D} ds (1 - (m-1) r H(s) - J(s,r)) \int_r^\infty e^{-\xi^2/(4t)} d\xi . \tag{5.16}$$

But for $r \geqq \delta$

$$|1 - (m-1) r H(s)| \leqq r^2 (\delta^{-2} + (m-1) K \delta^{-1}) , \tag{5.17}$$

so that

$$\begin{aligned} |C_1(t)| &\leqq (\pi t)^{-1/2} \int_0^\infty dr \int_{\partial D} ds\, r^2 (\delta^{-2} + (m-1) K \delta^{-1}) \int_r^\infty e^{-\xi^2/(4t)} d\xi \\ &= 8(9\pi)^{-1/2} |\partial D|_{m-1} (\delta^{-2} + (m-1) K \delta^{-1}) t^{3/2} . \end{aligned} \tag{5.18}$$

Furthermore by Lemma 2.5

$$\begin{aligned} |C_2(t)| &\leqq (\pi t)^{-1/2} \int_0^\infty dr \int_{\partial D} ds\, 2^{-1} (m-2)(m-1)(1+\delta K)^{m-3} K^2 r^2 \int_r^\infty d\xi\, e^{-\xi^2/(4t)} \\ &= 4(9\pi)^{-1/2} |\partial D|_{m-1} (m-2)(m-1)(1+\delta K)^{m-3} K^2 t^{3/2} . \end{aligned} \tag{5.19}$$

The integrand in the right hand side of (5.5) as a function of r is a convolution. Hence by using the formula for the integral (with respect to r on $[0,\infty)$) of a convolution in r we obtain

$$A_2(t) = - \int_{\partial D} ds \int_0^t d\tau\, \pi^{-1} \tau^{-1/2} (t-\tau)^{-1/2} (4\pi\tau)^{(1-m)/2} \int_{\{|y'|<\delta\}} dy' h^+(y';s) e^{-|y'|^2/(4\tau)} . \tag{5.20}$$

We obtain an upper bound for $|A_3(t)|$ if we replace $\int_\delta^\infty dr$ in (5.6) by $\int_0^\infty (r/\delta) dr$ and note that

$$\int_0^\infty dr\, r \int_0^r f(\xi) g(r-\xi) d\xi = \int_0^\infty r f(r) dr \int_0^\infty g(\xi) d\xi + \int_0^\infty f(r) dr \int_0^\infty \xi g(\xi) d\xi , \tag{5.21}$$

for $f \in L^1[0,\infty)$, $g \in L^1[0,\infty)$ and

$$\int_0^\infty r|f(r)|\,dr < \infty, \quad \int_0^\infty r|g(r)|\,dr < \infty. \tag{5.22}$$

Hence

$$|A_3(t)| \leqq \delta^{-1} \int_{\partial D} ds \int_0^t d\tau \int_0^\infty d\xi\, \xi \Phi(\xi,\tau;t)(4\pi\tau)^{(1-m)/2} \int_0^\infty dr$$

$$\times \int_{\{h(y';s)>r,\,|y'|<\delta\}} dy' e^{-|y'|^2/(4\tau)}$$

$$+ \delta^{-1} \int_{\partial D} ds \int_0^t d\tau \int_0^\infty d\xi\, \Phi(\xi,\tau;t)(4\pi\tau)^{(1-m)/2} \int_0^\infty r\,dr$$

$$\times \int_{\{h(y';s)>r,\,|y'|<\delta\}} dy' e^{-|y'|^2/(4\tau)}$$

$$= \delta^{-1} \int_{\partial D} ds \int_0^t d\tau\, \pi^{-1/2}(t-\tau)^{-1/2}(4\pi\tau)^{(1-m)/2} \int_{\{|y'|<\delta\}} dy' h^+(y';s)e^{-|y'|^2/(4\tau)}$$

$$+ (2\delta)^{-1} \int_{\partial D} ds \int_0^t d\tau\, \pi^{-1}\tau^{-1/2}(t-\tau)^{-1/2}(4\pi\tau)^{(1-m)/2}$$

$$\times \int_{\{|y'|<\delta\}} dy' (h^+(y';s))^2 e^{-|y'|^2/(4\tau)}. \tag{5.23}$$

By (2.5) we have for $|y'| < \delta$

$$h^+(y';s) \leqq \left(\frac{K}{2} + \frac{1}{\delta}\right)|y'|^2, \tag{5.24}$$

$$(h^+(y';s))^2 \leqq \left(\frac{K}{2} + \frac{1}{\delta}\right)^2 \delta |y'|^3. \tag{5.25}$$

Replacing $\{|y'| < \delta\}$ in (5.23) by $\mathbb{R}^{m-1}$, gives together with (5.24) and (5.25)

$$|A_3(t)| \leqq \delta^{-1}\left(\frac{K}{2} + \frac{1}{\delta}\right)|\partial D|_{m-1} \cdot 8(m-1)(9\pi)^{-1/2} t^{3/2}$$

$$+ \left(\frac{K}{2} + \frac{1}{\delta}\right)^2 |\partial D|_{m-1} \cdot 16\Gamma((m+2)/2)(3\pi\Gamma((m-1)/2))^{-1} t^{3/2}. \tag{5.26}$$

Replacing $\int_{\{|y'|>\delta\}} dy'$ by $\int_{\mathbb{R}^{m-1}} \frac{|y'|^3}{\delta^3} dy'$ and $-\int_0^r d\xi$ *by* $\int_0^\infty d\xi$ *in* (5.7) gives

$$|A_4(t)| \leqq \delta^{-3}|D_\delta|_m 32\Gamma((m+2)/2) \cdot (3\pi\Gamma((m-1)/2))^{-1} t^{3/2}. \tag{5.27}$$

In order to estimate $|A_5(t)|$ we first use (2.8), and we subsequently replace $\int_0^\delta dr$ in (5.8) by $\int_0^\infty dr$. Hence

$$|A_5(t)| \leqq \int_0^\infty dr \int_{\partial D} ds\,(m-1)rK(1+\delta K)^{m-2} \int_0^r d\xi \int_0^t d\tau\, \Phi(\xi,\tau;t)$$

$$\cdot (4\pi\tau)^{(1-m)/2} \int_{\{h(y';s)>r-\xi,\,|y'|<\delta\}} dy' e^{-|y'|^2/(4\tau)}. \tag{5.28}$$

The integrand in the right hand side of (5.28) (as a function of r) is of the type (5.21). We obtain as in (5.23)–(5.26)

$$|A_5(t)| \leqq (m-1)^2 K(1+\delta K)^{m-2}\left(\frac{K}{2}+\frac{1}{\delta}\right)|\partial D|_{m-1}\, 8(9\pi)^{-1/2} t^{3/2}$$
$$+ (m-1)\delta K(1+\delta K)^{m-2}\left(\frac{K}{2}+\frac{1}{\delta}\right)^2 |\partial D|_{m-1} \frac{16\,\Gamma((m+2)/2)}{3\pi\,\Gamma((m-1)/2)} t^{3/2} . \tag{5.29}$$

Applying Fubini's theorem in (5.9) with respect to the integrals over r and ξ gives

$$A_6(t) = \int_0^\infty d\xi \int_{\partial D} ds \int_0^t d\tau\, \Phi(\xi,\tau;t)\,(4\pi\tau)^{(1-m)/2} \int_{\{|y'|<\delta\}} dy' \min\{\xi, h^-(y';s)\}\, e^{-|y'|^2/(4\tau)}$$
$$= C_3(t) + C_4(t) , \tag{5.30}$$

where

$$C_3(t) = \int_{\partial D} ds \int_0^t d\tau\, \pi^{-1} \tau^{-1/2} (t-\tau)^{-1/2} (4\pi\tau)^{(1-m)/2} \int_{\{|y'|<\delta\}} dy'\, h^-(y';s) e^{-|y'|^2/(4\tau)} , \tag{5.31}$$

$$C_4(t) = \int_{\partial D} ds \int_0^t d\tau\, (4\pi\tau)^{(1-m)/2} \int_{\{|y'|<\delta\}} dy' \int_{[0,h^-(y';s))} d\xi (\xi - h^-(y';s))\, \Phi(\xi,\tau;t)$$
$$\cdot\, e^{-|y'|^2/(4\tau)} . \tag{5.32}$$

In order to estimate $|C_4(t)|$ we note that for $\xi \in [0, h^-(y';s))$

$$|\xi - h^-(y^-;s)| \leqq h^-(y';s) , \tag{5.33}$$

and

$$\int_{[0,h^-(y';s))} d\xi\, \Phi(\xi,\tau;t) \leqq \frac{(h^-(y';s))^2}{4\pi\tau^{3/2}(t-\tau)^{1/2}} , \tag{5.34}$$

and for $|y'| < \delta$

$$(h^-(y';s))^3 \leqq \left(\frac{K}{2}+\frac{1}{\delta}\right)^3 \delta |y'|^5 . \tag{5.35}$$

Hence

$$|C_4(t)| \leqq \int_{\partial D} ds \int_0^t d\tau\, (4\pi)^{-1} \tau^{-3/2} (t-\tau)^{-1/2} \left(\frac{K}{2}+\frac{1}{\delta}\right)^3 \delta \int_{\mathbb{R}^{m-1}} dy' |y'|^5 e^{-|y'|^2/(4\tau)}$$
$$= \left(\frac{K}{2}+\frac{1}{\delta}\right)^3 \delta\, |\partial D|_{m-1} \frac{32\Gamma((m+4)/2)}{3\pi\Gamma((m-1)/2)} t^{3/2} . \tag{5.36}$$

Moreover, by (5.20) and (5.31)

$$A_2(t) + C_3(t) = -\int_{\partial D} ds \int_0^t d\tau\, \pi^{-1} \tau^{-1/2} (t-\tau)^{-1/2} (4\pi\tau)^{(1-m)/2}$$
$$\cdot \int_{\{|y'|<\delta\}} dy'\, h(y';s)\, e^{-|y'|^2/(4\tau)}$$
$$= -2^{-1}(m-1)t \int_{\partial D} H(s) ds + C_5(t) + C_6(t) , \tag{5.37}$$

where

$$C_5(t) = \int_{\partial D} ds \int_0^t d\tau (2\pi)^{-1} \tau^{-1/2} (t-\tau)^{-1/2} (4\pi\tau)^{(1-m)/2} \times \int_{\{|y'|>\delta\}} \sum_{i=1}^{m-1} k_i(s) y_i^2 e^{-|y'|^2/(4\tau)}, \tag{5.38}$$

$$C_6(t) = \int_{\partial D} ds \int_0^t d\tau \pi^{-1} \tau^{-1/2} (t-\tau)^{-1/2} (4\pi\tau)^{(1-m)/2} \times \int_{\{|y'|<\delta\}} dy' \left\{ \sum_{i=1}^{m-1} k_i(s) y_i^2/2 - h(y';s) \right\} e^{-|y'|^2/(4\tau)}. \tag{5.39}$$

Replacing $\sum_{i=1}^{m-1} k_i(s) y_i^2$ by $\delta^{-1} K|y'|^3$ and $\{|y'|>\delta\}$ by $\mathbb{R}^{m-1}$ in (5.38) gives the following bound:

$$|C_5(t)| \leqq |\partial D|_{m-1} K\delta^{-1} \frac{16\Gamma((m+2)/2)}{3\pi\Gamma((m-1)/2)} t^{3/2}. \tag{5.40}$$

Replacing $\left\{\sum_{i=1}^{m-1} k_i(s) y_i^2/2 - h(y';s)\right\}$ by $\delta^{-2}|y'|^3$ and $\{|y'|<\delta\}$ by $\mathbb{R}^{m-1}$ in (5.39) gives

$$|C_6(t)| \leqq |\partial D|_{m-1} \delta^{-2} \frac{32\Gamma((m+2)/2)}{3\pi\Gamma((m-1)/2)} t^{3/2}. \tag{5.41}$$

Replacing $-\int_\delta^\infty dr$ by $\int_0^\infty (r/\delta) dr$ in (5.10) and replacing $J(s,r)-1$ by $(m-1)rK(1+\delta K)^{m-2}$ and $\int_0^\delta dr$ by $\int_0^\infty dr$ in (5.11) gives

$$\begin{aligned}
|A_7(t)| + |A_8(t)| &\leqq \{\delta^{-1} + (m-1)K(1+\delta K)^{m-2}\} \\
&\quad \cdot \int_{\partial D} ds \int_0^\infty r dr \int_r^\infty d\xi \int_0^t d\tau\, \Phi(\xi,\tau;t)(4\pi\tau)^{(1-m)/2} \\
&\quad \times \int_{\{h(y';s)<r-\xi, |y'|<\delta\}} dy' e^{-|y'|^2/(4\tau)} \\
&= \{\delta^{-1} + (m-1)K(1+\delta K)^{m-2}\} \\
&\quad \cdot \int_{\partial D} ds \int_0^\infty d\xi \int_0^\xi r dr \int_0^t d\tau\, \Phi(\xi,\tau;t)(4\pi\tau)^{(1-m)/2} \\
&\quad \times \int_{\{h(y';s)<r-\xi, |y'|<\delta\}} dy' e^{-|y'|^2/(4\tau)} \\
&\leqq \{\delta^{-1} + (m-1)K(1+\delta K)^{m-2}\} \\
&\quad \cdot \int_{\partial D} ds \int_0^\infty \xi\, d\xi \int_{-\infty}^\xi dr \int_0^t d\tau\, \Phi(\xi,\tau;t)(4\pi\tau)^{(1-m)/2} \\
&\quad \times \int_{\{h(y';s)<r-\xi, |y'|<\delta\}} dy' e^{-|y'|^2/(4\tau)}
\end{aligned}$$

$$= \{\delta^{-1} + (m-1)K(1+\delta K)^{m-2}\}$$

$$\cdot \int_{\partial D} ds \int_0^t d\tau\, \pi^{-1/2}(t-\tau)^{-1/2}(4\pi\tau)^{(1-m)/2}$$

$$\times \int_{\{|y'|<\delta\}} dy'\, h^-(y';s)e^{-|y'|^2/(4\tau)}\,. \tag{5.42}$$

By (2.5) we have for $|y'| < \delta$

$$h^-(y';s) \leqq \left(\frac{K}{2} + \frac{1}{\delta}\right)|y'|^2\,. \tag{5.43}$$

Hence

$$|A_7(t)| + |A_8(t)| \leqq \{\delta^{-1} + (m-1)K(1+\delta K)^{m-2}\}\left\{\frac{K}{2} + \frac{1}{\delta}\right\}|\partial D|_{m-1}$$

$$\cdot 8(m-1)(9\pi)^{-1/2}t^{3/2}\,. \tag{5.44}$$

This concludes the proof of Lemma 5.1 with a constant C_1 which follows from (5.2), (5.18), (5.19), (5.26), (5.27), (5.29), (5.36), (5.40), (5.41) and (5.44)

6 An upper bound for $\int_{D_\delta} dx\, \mathbb{P}_x[T_D \leqq \tau_t \leqq t, B(\tau_t) \in D]$

In this section we will prove the following.

Lemma 6.1 *Let D and δ be as in Lemma 2.3. Then there exists a constant $C_2 \in (0, \infty)$ depending on K, $|\partial D|_{m-1}$, $|D|_m$, and δ such that for all $t > 0$*

$$\int_{D_\delta} dx\, \mathbb{P}_x[T_D \leqq \tau_t \leqq t, B(\tau_t) \in D] \leqq C_2 t^{3/2}\,. \tag{6.1}$$

Proof. Let $x = (s, r) \in D_\delta$ be arbitrary and let $B(\tau)$, $\tau \geqq 0$ be given by (3.9). For every $u \geqq 0$ we define

$$\tau_u = \inf\left\{\tau : B_m(\tau) = \sup_{0 \leqq v \leqq u} B_m(v)\right\}. \tag{6.2}$$

We will consider the brownian motion B under the probability measure $\mathbb{P}_x$ (as in Lemma 3.4), but also under $\mathbb{P}_\sigma$ for $\sigma \neq x$. It should be understood that $B_1, \ldots, B_m$, or τ_u are in all cases defined with x as a reference point. Let δ_1 be as in Lemma 2.6. Then

$$\mathbb{P}_x[T_D \leqq \tau_t \leqq t, B(\tau_t) \in D] \leqq \mathbb{P}_x[|B(T_D) - x| \geqq \delta_1]$$

$$+ \mathbb{P}_x[T_D \leqq \tau_t \leqq t, |B(T_D) - x| < \delta_1, B(\tau_t) \in D]\,. \tag{6.3}$$

By Lemma 3.3

$$\mathbb{P}_x[|B(T_D) - x| \geqq \delta_1] \leqq \mathbb{P}_x\left[\sup_{0 \leqq s \leqq t} |B(s) - x| \geqq \delta_1\right]$$

$$\leqq 2^{(2+m)/2}\, e^{-\delta_1^2/(8t)} \leqq 2^{(8+m)/2}\,(3/e)^{3/2}\,\delta_1^{-3}\,t^{3/2}\,. \tag{6.4}$$

Moreover, by the strong Markov property at T_D

$$\begin{aligned} &\mathbb{P}_x[T_D \leqq \tau_t \leqq t, |B(T_D) - x| < \delta_1, B(\tau_t) \in D] \\ &\leqq \mathbb{E}_x[1_{\{T_D \leqq t, |B(T_D) - x| < \delta_1\}} \phi_x(B(T_D), t - T_D)] , \end{aligned} \tag{6.5}$$

where for $\sigma \in \partial D$ and $u \geqq 0$

$$\phi_x(\sigma, u) = \mathbb{P}_\sigma[B(\tau_u) \in D] \tag{6.6}$$

(we write ϕ_x to emphasize the choice of x as a reference point, the definition of τ_u depends on $\mathbb{N}(s)$ and thus also on x).

For $\sigma \in \partial D$ with $|\sigma - x| < \delta_1$, we have by Lemma 2.6

$$\sigma = x + \sum_{i=1}^{m-1} \lambda_i v_i(s) + (h(\lambda'; s) - r)\mathbb{N}(s) , \tag{6.7}$$

for a unique $\lambda' = (\lambda_1, \ldots, \lambda_{m-1}) \in \mathbb{R}^{m-1}$ with $|\lambda'| < \delta_1$. Let $w \in \mathbb{R}^m$ be such that $|w - \sigma| < \delta_1$, and

$$w = x + \sum_{i=1}^{m-1} w_i v_i(s) - \xi \mathbb{N}(s) . \tag{6.8}$$

By Lemma 2.6

$$\xi \geqq r - h(\lambda'; s) - \nabla h(\lambda'; s) \cdot (y' - \lambda') + K|y' - \lambda'|^2 \tag{6.9}$$

implies $w \notin D$. Furthermore by Lemma 2.6 and (6.7)

$$|\nabla h(\lambda'; s)| \leqq 2K|\lambda'| \leqq 2K|\sigma - x| . \tag{6.10}$$

It follows that

$$\begin{aligned} \phi_x(\sigma, u) \leqq\; &\mathbb{P}_\sigma\left[\sup_{0 \leqq s \leqq u} |B(s) - \sigma| \geqq \delta_1\right] \\ &+ \mathbb{P}_\sigma[B_m(\tau_u) < r - h(\lambda'; s) - \nabla h(\lambda'; s) \cdot (B'(\tau_u) - \lambda') \\ &+ K|B'(\tau_u) - \lambda'|^2] . \end{aligned} \tag{6.11}$$

By Lemma 3.3

$$\begin{aligned} \mathbb{P}_\sigma\left[\sup_{0 \leqq s \leqq u} |B(s) - \sigma| \geqq \delta_1\right] &\leqq 2^{(2+m)/2} e^{-\delta_1^2/(8u)} \\ &\leqq 2^{(8+m)/2} (3/e)^{3/2} \delta_1^{-3} t^{3/2} . \end{aligned} \tag{6.12}$$

Moreover, we note that by Lemma 3.2 the joint distribution of $(r - h(\lambda'; s) - B_m(\tau_u), B'(\tau_u) - \lambda', \tau_u)$ under $\mathbb{P}_\sigma$ has a density given by

$$\rho_u(\xi, y', \tau) = \Phi(\xi, \tau; u)(4\pi\tau)^{(1-m)/2} e^{-|y'|^2/(4\tau)} 1_{[0,\infty)}(\xi) 1_{[0,u]}(\tau) . \tag{6.13}$$

Hence the second term in the right hand side of (6.11) is equal to

$$\begin{aligned} &\int_0^u d\tau \int_{\mathbb{R}^{m-1}} dy' \int_{\{0 < \xi < K|y'|^2 - \nabla h(\lambda'; s) \cdot y'\}} d\xi \quad \Phi(\xi, \tau; u)(4\pi\tau)^{(1-m)/2} e^{-|y'|^2/(4\tau)} \\ &= \int_0^u d\tau\, 4 \cdot (4\pi)^{-(m+1)/2} \tau^{-m/2} (u - \tau)^{-1/2} \int_{\mathbb{R}^{m-1}} dy'\, e^{-|y'|^2/(4\tau)} \\ &\quad \cdot \left(1 - \exp - \frac{1}{4\tau}((\nabla h(\lambda'; s) \cdot y' - K|y'|^2)^-)^2\right) . \end{aligned} \tag{6.14}$$

Since

$$1 - \exp - \frac{1}{4\tau}((\nabla h(\lambda'; s)\cdot y' - K|y'|^2)^-)^2 \leqq \frac{1}{2\tau}(|\nabla h(\lambda'; s)|^2|y'|^2 + K^2|y'|^4), \tag{6.15}$$

we obtain the following upper bound for (6.14):

$$(m-1)|\nabla h(\lambda'; s)|^2 + (m^2-1)K^2 u \leqq 4(m-1)K^2|\sigma - x|^2 + (m^2-1)K^2 t. \tag{6.16}$$

Hence by (6.3), (6.4), (6.5), (6.12) and (6.16)

$$\begin{aligned} &\mathbb{P}_x[T_D \leqq \tau_t \leqq t,\ B(\tau_t)\in D] \\ &\quad \leqq 2^{(10+m)/2}(3/e)^{3/2}\delta_1^{-3}t^{3/2} \\ &\qquad + \mathbb{E}_x[1_{\{T_D \leqq t, |B(T_D)-x|<\delta_1\}}(4(m-1)K^2|B(T_D)-x|^2 + (m^2-1)K^2 t)] \\ &\quad \leqq 2^{(10+m)/2}(3/e)^{3/2}\delta_1^{-3}t^{3/2} \\ &\qquad + \mathbb{E}_x\left[1_{\{T_D\leqq t\}}\left(4(m-1)K^2\sup_{0\leqq s\leqq t}|B(s)-x|^2 + (m^2-1)K^2 t\right)\right] \\ &\quad \leqq 2^{(10+m)/2}(3/e)^{3/2}\delta_1^{-3}t^{3/2} + (m^2-1)K^2 t\,\mathbb{P}_x[T_D\leqq t] \\ &\qquad + 4(m-1)K^2(\mathbb{E}_x[1_{\{T_D\leqq t\}}])^{1/2}\left(\mathbb{E}_x\left[\sup_{0\leqq s\leqq t}|B(s)-x|^4\right]\right)^{1/2}. \end{aligned} \tag{6.17}$$

By Doob's inequality we have

$$\mathbb{E}_x\left[\sup_{0\leqq s\leqq t}|B(s)-x|^4\right] \leqq \left(\frac{4}{3}\right)^4 \mathbb{E}_x[|B(t)-x|^4] = 4^5\,3^{-4}\,m(m+2)t^2. \tag{6.18}$$

By Lemma 3.3, (6.17) and (6.18) we obtain

$$\begin{aligned} \mathbb{P}_x[T_D \leqq \tau_t \leqq t, B(\tau_t)\in D] \leqq\ & 2^{(10+m)/2}(3/e)^{3/2}\delta_1^{-3}t^{3/2} \\ &+ 2^{(2+m)/2}(m^2-1)K^2 t\, e^{-d^2(x)/(8t)} \\ &+ 2^{(30+m)/4}3^{-2}(m-1)(m^2+2m)^{1/2}K^2 t \\ &\times e^{-d^2(x)/(16t)}. \end{aligned} \tag{6.19}$$

Finally for $\alpha > 0$

$$\begin{aligned} \int_{D_\delta} dx\, e^{-d^2(x)/(\alpha t)} &= \int_0^\delta dr \int_{\partial D} ds\, J(s,r)\, e^{-r^2/(\alpha t)} \\ &\leqq (1+\delta K)^{m-1}|\partial D|_{m-1}\int_0^\delta dr\, e^{-r^2/(\alpha t)} \\ &\leqq (1+\delta K)^{m-1}|\partial D|_{m-1}(\pi\alpha/4)^{1/2}t^{1/2}. \end{aligned} \tag{6.20}$$

This completes the proof of Lemma 6.1 with C_2 given by

$$C_2 = 2^{(10+m)/2}(3/e)^{3/2}\,\delta_1^{-3}|D|_m + 2^{(3+m)/2}\pi^{1/2}(m^2-1)K^2(1+\delta K)^{m-1}|\partial D|_{m-1} + 2^{(34+m)/4}3^{-2}\pi^{1/2}(m-1)(m^2+2m)^{1/2}K^2(1+\delta K)^{m-1}|\partial D|_{m-1}\,. \quad (6.21)$$

7 An upper bound for $\int_{D_\delta} dx\, \mathbb{P}_x[\tau_t < T_D \leqq t]$

In this section we will use the estimates of Sect. 4 to prove the following.

Lemma 7.1 *Let D and δ be as in Lemma 2.3. Then there exists a constant $C_3 \in (0, \infty)$ depending on K, $|\partial D|_{m-1}$* and δ *such that for all* $t > 0$

$$\int_{D_\delta} dx\, \mathbb{P}_x[\tau_t < T_D \leqq t] \leqq C_3 t^{3/2}\,. \quad (7.1)$$

Proof. Using the parametrization of Lemma 2.3 we have

$$\begin{aligned}\int_{D_\delta} dx\, \mathbb{P}_x[\tau_t < T_D \leqq t] &= \int_0^\delta dr \int_{\partial D} ds\, J(s,r)\,\mathbb{P}_{(s,r)}[\tau_t < T_D \leqq t] \\ &\leqq (1+\delta K)^{m-1}\int_0^\delta dr \int_{\partial D} ds\, \mathbb{P}_{(s,r)}[\tau_t < T_D \leqq t]\,. \quad (7.2)\end{aligned}$$

In what follows we fix $s \in \partial D$ and we will bound

$$\int_0^\delta dr\, \mathbb{P}_{(s,r)}[\tau_t < T_D \leqq t]$$

independently of s. Let $B(\tau)$, $\tau \geqq 0$ be given by (3.9) and let $\tilde{x} \in \mathbb{R}^m$ be given by

$$\tilde{x} = x + \sum_{i=1}^{m-1} y_i v_i(s) - \xi \mathbb{N}(s)\,. \quad (7.3)$$

Then, provided that $|\tilde{x} - x| < \delta$, Lemma 2.3 implies that if $\xi < r - H(s, y') - \delta^{-2}|y'|^3$, then $\tilde{x} \in D$ and if $\xi \geqq r - H(s, y') + \delta^{-2}|y'|^3$, then $\tilde{x} \notin D$, where

$$H(s, y') = \sum_{i=1}^{m-1} k_i(s) y_i^2/2\,. \quad (7.4)$$

It follows that

$$\begin{aligned}\mathbb{P}_x[\tau_t < T_D \leqq t] \leqq\ & \mathbb{P}_x\Big[\sup_{0<v\leqq t} |B(v) - x| \geqq \delta\Big] \\ &+ \mathbb{P}_x[B_m(\tau_t) < r - H(s, B'(\tau_t)) + \delta^{-2}|B'(\tau_t)|^3\,, \\ &\exists u \in [\tau_t, t]\colon B_m(u) \geqq r - H(s, B'(\tau_t)) - \delta^{-2}|B'(\tau_t)|^3]\,. \quad (7.5)\end{aligned}$$

By Lemma 3.3,

$$\mathbb{P}_x\Big[\sup_{0<v\leqq t} |B(v) - x| \geqq \delta\Big] \leqq 2^{(2+m)/2} e^{-\delta^2/(8t)} \leqq 2^{(8+m)/2}(3/e)^{3/2}\delta^{-3} t^{3/2}\,. \quad (7.6)$$

It remains to bound the second term in the right hand side of (7.5). First we introduce some notation. For $u \in [0, 1]$,

$$X(u) = (t - \tau_t)^{-1/2}(B_m(\tau_t) - B_m(\tau_t + u(t - \tau_t))) , \tag{7.7}$$

$$Y'(u) = (Y_1(u), \ldots , Y_{m-1}(u)) \tag{7.8}$$

where, for $i = 1, \ldots , m - 1$

$$Y_i(u) = (t - \tau_t)^{-1/2}(B_i(\tau_t + u(t - \tau_t)) - B_i(\tau_t)) . \tag{7.9}$$

Next we apply Theorem 4.4 with $\beta = B_m$ and infer that X is a brownian meander starting at 0, independent of $(\tau_t, B_m(\tau_t))$. By the independence of $B_1, \ldots , B_m$, and the fact that τ_t is a measurable functional of B_m, we also get that $Y_1, \ldots , Y_{m-1}$ are independent brownian motions, independent of B_m, and hence independent of X. Moreover, $(X, Y_1, \ldots , Y_{m-1})$ is independent of $(\tau_t, B(\tau_t))$.

Let $\chi_t(\xi, y', u)$, $\xi \in [0, \infty)$, $y' \in \mathbb{R}^{m-1}$, $u \in [0, t]$ denote the density of $(B_m(\tau_t), B_1(\tau_t), \ldots , B_{m-1}(\tau_t), t - \tau_t)$. Then the second term in the right hand side of (7.5) can be written as

$$\begin{aligned} D(s, r, t) = \int_0^\infty d\xi \int_{\mathbb{R}^{m-1}} dy' \int_0^t du\, \chi_t(\xi, y', u)\, 1_{\{\xi < r - H(s, y') + \delta^{-2}|y'|^3\}} \\ \cdot \mathbb{P}[\exists \tau \in [0, 1]: \xi - u^{1/2} X(\tau) \geqq r - H(s, y' + u^{1/2}\, Y'(\tau)) \\ - \delta^{-2}|y' + u^{1/2}\, Y'(\tau)|^3] . \end{aligned} \tag{7.10}$$

Then, using the bound $(a + b)^3 \leqq 8(a^3 + b^3)$ for $a, b \geqq 0$, we have

$$\begin{aligned} & H(s, y' + u^{1/2}\, Y'(\tau)) + \delta^{-2}|y' + u^{1/2}\, Y'(\tau)|^3 \\ & \leqq H(s, y') + 8\delta^{-2}|y'|^3 + u^{1/2}\, W(y', u, \tau) \end{aligned}$$

where

$$W(y', u, \tau) = K|y'||Y'(\tau)| + \frac{K}{2} u^{1/2}|Y'(\tau)|^2 + 8\delta^{-2}u|Y'(\tau)|^3 . \tag{7.11}$$

It follows that:

$$\begin{aligned} D(s, r, t) \leqq \int_0^\infty d\xi \int_{\mathbb{R}^{m-1}} dy' \int_0^t du\, \chi_t(\xi, y', u)\, 1_{\{\xi - r + H(s, y') < \delta^{-2}|y'|^3\}} \\ \cdot \mathbb{P}[\exists \tau \in [0, 1]: X(\tau) \leqq u^{-1/2}(\xi - r + H(s, y') \\ + 8\delta^{-2}|y'|^3) + W(y', u, \tau)] . \end{aligned} \tag{7.12}$$

Before we use Lemma 4.3 we make two preliminary reductions. First we have

$$\begin{aligned} & \int_0^\delta dr \int_0^\infty d\xi \int_{\mathbb{R}^{m-1}} dy' \int_0^t du\, \chi_t(\xi, y', u)\, 1_{\{-8\delta^{-2}|y'|^3 \leqq \xi - r + H(s, y') \leqq \delta^{-2}|y'|^3\}} \\ & \leqq \int_0^\infty d\xi \int_{\mathbb{R}^{m-1}} dy' \int_0^t du\, \chi_t(\xi, y', u) \cdot 9\delta^{-2}|y'|^3 \\ & = 9\delta^{-2}\mathbb{E}[|Y'(\tau_t)|^3] \\ & = 9\delta^{-2}\mathbb{E}[|Y'(1)|^3]\mathbb{E}[\tau_t^{3/2}] \\ & = \frac{96}{\pi}\Gamma((m + 2)/2)(\Gamma((m - 1)/2))^{-1}\delta^{-2}t^{3/2} , \end{aligned} \tag{7.13}$$

independently of s. Secondly we have by Lemma 3.3

$$\mathbb{P}\left[\sup_{0\leqq\tau\leqq 1}|Y'(\tau)|\geqq \delta u^{-1/2}\right]\leqq 2^{(1+m)/2}e^{-\delta^2/(8u)}$$

$$\leqq 2^{(7+m)/2}(3/e)^{3/2}\delta^{-3}t^{3/2}\,. \qquad (7.14)$$

Note that, on the set $\{\sup_{0\leqq\tau\leqq 1}|Y'(\tau)|<\delta u^{-1/2}\}$, we can bound

$$W(y',u,\tau)\leqq K|y'||Y'(\tau)|+\left(\frac{K}{2}+8\delta^{-1}\right)u^{1/2}|Y'(\tau)|^2=:V(y',u,\tau)\,. \qquad (7.15)$$

It follows from (7.5), (7.6), (7.12), (7.13), (7.14) and (7.15) that

$$\int_0^\delta dr\,\mathbb{P}_{(s,r)}[\tau_t<T_D\leqq t]\leqq\left(2^{(7+m)/2}(1+2^{1/2})(3/e)^{3/2}+\frac{96}{\pi}\frac{\Gamma((m+2)/2)}{\Gamma((m-1)/2)}\right)\delta^{-2}t^{3/2}$$

$$+\int_0^\delta dr\int_0^\infty d\xi\int_{\mathbb{R}^{m-1}}dy'\int_0^t du\,\chi_t(\xi,y',u)\cdot\mathbf{1}_{\{\xi-r+H(s,y')<-8\delta^{-2}|y'|^3\}}$$

$$\cdot\,\mathbb{P}[\exists\tau\in[0,1]\colon X(\tau)\leqq u^{-1/2}(\xi-r+H(s,y')$$

$$+\,8\delta^{-2}|y'|^3)+V(y',u,\tau)]\,. \qquad (7.16)$$

Note that X is a brownian meander starting at 0, and that $|Y'|$ is a d-dimensional Bessel process, also starting at 0, and independent of X. Put

$$\gamma=u^{-1/2}(\xi-r+H(s,y')+8\delta^{-2}|y'|^3)\,, \qquad (7.17)$$

$$a=K|y'|\,, \qquad (7.18)$$

$$b=\left(\frac{K}{2}+8\delta^{-1}\right)u^{1/2}\,, \qquad (7.19)$$

and note that $\gamma<0$ on the set of integration. Hence by Lemma 4.3

$$\mathbb{P}[\exists\tau\in[0,1]\colon X(\tau)\leqq\gamma+a|Y'(\tau)|+b|Y'(\tau)|^2]$$

$$\leqq M_2(a+b)(1+\log^+(1/x_0))\,e^{-M_3x_0^2} \qquad (7.20)$$

with

$$x_0=(2b)^{-1}(-a+(a^2-4\gamma b)^{1/2})\,. \qquad (7.21)$$

Denote the multiple integral in the right hand side of (7.16) by $F_s(t)$. Then by (7.18) and (7.20)

$$F_s(t)\leqq\int_0^\delta dr\int_0^\infty d\xi\int_{\mathbb{R}^{m-1}}dy'\int_0^t du\,\chi_t(\xi,y',u)\,1_{\{\xi-r+H(s,y')<-8\delta^{-2}|y'|^3\}}$$

$$\cdot\,M_2(a+b)(1+\log^+(1/x_0))\,e^{-M_3x_0^2}\,. \qquad (7.22)$$

Note that a and b do not depend on r. The change of variables $r\to\gamma$ results into

$$F_s(t)\leqq M_2t^{1/2}\int_0^\infty d\xi\int_{\mathbb{R}^{m-1}}dy'\int_0^t du\,\chi_t(\xi,y',u)\int_{-\infty}^0 d\gamma\,(a+b)(1+\log^+(1/x_0))\,e^{-M_3x_0^2}\,,$$

$$(7.23)$$

using the trivial bound $u^{1/2} \leqq t^{1/2}$. Note that

$$\gamma = (4b)^{-1}(a^2 - (a + 2bx_0)^2) , \tag{7.24}$$

so that the change of variables $\gamma \to x_0$ gives

$$\begin{aligned} F_s(t) &\leqq M_2 t^{1/2} \int_0^\infty d\xi \int_{\mathbb{R}^{m-1}} dy' \int_0^t du\, \chi_t(\xi, y', u) \\ &\quad \times \int_0^\infty dx_0 (a + b)(a + 2bx_0)(1 + \log^+(1/x_0))\, e^{-M_3 x_0^2} \\ &= M_2 t^{1/2} \int_0^\infty d\xi \int_{\mathbb{R}^{m-1}} dy' \int_0^t du\, \chi_t(\xi, y', u)(N_1 a(a + b) + 2N_2 b(a + b)) , \end{aligned} \tag{7.25}$$

where

$$N_1 = \int_0^\infty dv\,(1 + \log^+(1/v))\, e^{-M_3 v^2} , \tag{7.26}$$

$$N_2 = \int_0^\infty dv\, v(1 + \log^+(1/v))\, e^{-M_3 v^2} . \tag{7.27}$$

By (7.18) and (7.19) we have

$$\begin{aligned} \int_0^\infty d\xi \int_{\mathbb{R}^{m-1}} dy' \int_0^t du\, \chi_t(\xi, y', u) a^2 &= K^2 \mathbb{E}[|Y'(\tau_t)|^2] = K^2 \mathbb{E}[|Y'(1)|^2]\, \mathbb{E}[\tau_t] \\ &= K^2 (m - 1) t \end{aligned} \tag{7.28}$$

$$\begin{aligned} \int_0^\infty d\xi \int_{\mathbb{R}^{m-1}} dy' \int_0^t du\, \chi_t(\xi, y', u) ab &= K\left(\frac{K}{2} + 8\delta^{-1}\right) t^{1/2}\, \mathbb{E}[|Y'(\tau_t)|] \\ &\leqq K\left(\frac{K}{2} + 8\delta^{-1}\right)(m - 1)^{1/2}\, t \end{aligned} \tag{7.29}$$

$$\int_0^\infty d\xi \int_{\mathbb{R}^{m-1}} dy' \int_0^t du\, \chi_t(\xi, y', u) b^2 \leqq \left(\frac{K}{2} + 8\delta^{-1}\right)^2 t , \tag{7.30}$$

and by (7.26) and (7.27) we have

$$N_1 \leqq 2^{-1}(\pi/M_3)^{1/2} - \int_0^1 dv \log v \leqq 7 , \tag{7.31}$$

$$N_2 \leqq (2M_3)^{-1} - \int_0^1 dv\, v \log v \leqq 17 , \tag{7.32}$$

since $M_3 = 1/32$. From (7.25), (7.28)–(7.32) we obtain

$$\begin{aligned} F_s(t) &\leqq 41\, m(K + 8\delta^{-1})^2 M_2 t^{3/2} \\ &= 3 \cdot 7 \cdot 41 \cdot \pi^{-1/2} 2^{(18+m)/2}\, m(K + 8\delta^{-1})^2\, t^{3/2} . \end{aligned} \tag{7.33}$$

This completes the proof of Lemma 7.1 with C_3 given by

$$C_3 = \left\{ 2^{(7+m)/2}(1 + 2^{1/2})(3/e)^{3/2}\delta^{-2} + \frac{96}{\pi}\frac{\Gamma((m+2)/2)}{\Gamma((m-1)/2)}\delta^{-2} + 3\cdot 7\cdot 41\cdot \pi^{-1/2}2^{(18+m)/2}\,m(K + 8\delta^{-1})^2 \right\}(1 + K\delta)^{m-1}|\partial D|_{m-1}\,. \tag{7.34}$$

8 Proof of Theorem 1.1

Let D and δ be as in Lemma 2.3. Then

$$\begin{aligned} Q_D(t) &= \int_{D\setminus D_\delta} dx\, \mathbb{P}_x[T_D > t] + \int_{D_\delta} dx\, \mathbb{P}_x[T_D > t] \\ &\leqq |D\setminus D_\delta|_m + \int_{D_\delta} dx\, \mathbb{P}_x[B(\tau_t) \in D] \\ &\leqq |D|_m - 2(t/\pi)^{1/2}|\partial D|_{m-1} + 2^{-1}(m-1)t\int_{\partial D} H(s)\,ds + C_1 t^{3/2}\,, \end{aligned} \tag{8.1}$$

by Lemma 5.1. On the other hand by Lemma 3.3

$$\begin{aligned} \int_{D\setminus D_\delta} dx\, \mathbb{P}_x[T_D > t] &\geqq \int_{D\setminus D_\delta} dx\,[1 - 2^{(2+m)/2}e^{-d^2(x)/(8t)}] \\ &\geqq |D\setminus D_\delta|_m[1 - 2^{(2+m)/2}e^{-\delta^2/(8t)}] \\ &\geqq |D\setminus D_\delta|_m - |D|_m\delta^{-3}2^{(8+m)/2}(3/e)^{3/2}t^{3/2}\,. \end{aligned} \tag{8.2}$$

By (1.37) and Lemmas 5.1, 7.1 and 8.1

$$\begin{aligned} \int_{D_\delta} dx\, \mathbb{P}_x[T_D > t] &= \int_{D_\delta} dx\,[\mathbb{P}_x[B(\tau_t)\in D] \\ &\quad - \mathbb{P}_x[T_D \leqq \tau_t \leqq t, B(\tau_t)\in D] - \mathbb{P}_x[\tau_t < T_D \leqq t]] \\ &\geqq |D_\delta|_m - 2(t/\pi)^{1/2}|\partial D|_{m-1} + 2^{-1}(m-1)t\int_{\partial D} H(s)\,ds \\ &\quad - (C_1 + C_2 + C_3)t^{3/2}\,. \end{aligned} \tag{8.3}$$

This completes the proof of Theorem 1.1 with a constant

$$C = C_1 + C_2 + C_3 + 2^{(8+m)/2}(3/e)^{3/2}\delta^{-3}|D|_m\,. \tag{8.4}$$

In fact, using various elementary inequalities such as

$$|D_\delta| = \int_0^\delta dr \int_{\partial D} ds\, J(s,r) \leqq \delta|\partial D|_{m-1}(1 + \delta K)^{m+1}\,, \tag{8.5}$$

together with the bounds for C_1, C_2 and C_3 from the Sects. 5, 6 and 7 respectively, we obtain

$$C \leqq 2^{100+m}[(1 + \delta K)^{m+1}(\delta^{-2} + \delta_1^{-2} + K^2)|\partial D|_{m-1} + \delta^{-3}|D|_m]\,. \tag{8.6}$$

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