RANDOM POLYNOMIALS, RANDOM MATRICES, AND L-FUNCTIONS, II

DAVID W FARMER, FRANCESCO MEZZADRI, AND NINA C SNAITH

Abstract. We show that the Circular Orthogonal Ensemble of random matrices arises naturally from a family of random polynomials. This sheds light on the appearance of random matrix statistics in the zeros of the Riemann zeta-function.

1. Introduction

The statistics of eigenvalues of unitary matrices, chosen uniformly with respect to Haar measure on $U(N)$, are observed to closely match the statistics of zeros of the Riemann zeta-function and other L-functions [18, 19, 9, 22]. In addition, eigenvalues of other compact classical matrix groups give a good model of the zeros of various families of L-functions [13, 12, 21, 11, 7]. Furthermore, the characteristic polynomials of the matrices provide a good model of the L-functions themselves [16, 3, 15, 4, 10].

The L-functions studied in number theory are Dirichlet series having a functional equation and an Euler product. In this paper we are concerned with a wider class of Dirichlet series which have a functional equation but do not have an Euler product. It has been suggested that such functions can be modeled by random self-reciprocal polynomials [8]. Since these functions do not have an Euler product, they are not expected to satisfy the Riemann hypothesis. However, it is possible that occasionally such functions will satisfy the Riemann hypotheses or, as is more likely, will have a large number of consecutive zeros, say the first 100 trillion of them, on the critical line.

This paper is motivated by the following questions: do the Riemann zeta-function and the other L-functions of number theory behave differently than random Dirichlet series with functional equation which just happen to satisfy the Riemann hypothesis? That is, does the Euler product have any effect on the zeros beyond forcing them to be on the critical line? Our results suggest that the answer is 'yes', and the Euler product also has an effect on the local statistics of the zeros.

1.1. Random polynomials.

In random matrix theory, the characteristic polynomial of a random matrix can be viewed as a random polynomial where the randomness is explicitly encoded in the zeros. For example, consider the Weyl integration formula for the classical compact groups [24, 12] or the $\beta = 1, 2, 4$ ensembles of random matrix theory [17]. On the other hand, usually in the study of random polynomials (for a review of the subject see, e.g., Farahmand [6]) the randomness is explicitly encoded in the coefficients of the polynomials.

Date: September 23, 2005.

Research supported by the American Institute of Mathematics and the Focused Research Group grant (0244660) from the NSF. This work was started during the program on Random Matrix Applications in Number Theory at the Isaac Newton Institute for Mathematical Sciences. The second and third authors were also supported by a Royal Society Dorothy Hodgkin Fellowship, and the third author was partially supported by EPSRC.

We will show that there is a simple but surprising connection between these two perspectives, and we believe this connection is relevant to the appearance of random matrix statistics in the zeros of the Riemann zeta-function and other L-functions.

The polynomials we consider are of the form

$$f(z) = z^N + a_1 z^{N-1} + \cdots + a_N \tag{1.1}$$

with $|a_N| = 1$ which have the symmetry

$$f(z) = a_N z^N \overline{f}\left(\frac{1}{z}\right), \tag{1.2}$$

where $\overline{f}(z) := \overline{f(\overline{z})}$. Such polynomials are called *self-reciprocal.* Equation (1.2) ensures that the zeros of f occur either on the unit circle or in pairs located symmetrically with respect to the unit circle. The symmetry (1.2) is analogous to the functional equation of L-functions which arise in number theory, see (1.10) below.

The change of variables $z = e^{ix}$ in the function $a_N^{-\frac{1}{2}} z^{-N/2} f(z)$ converts a self-reciprocal algebraic polynomial into a *real* trigonometric polynomial

$$\sum_{0 \le n \le N/2} c_n \cos\left(\left(N/2 - n\right) x\right) + d_n \sin\left(\left(N/2 - n\right) x\right), \tag{1.3}$$

where c_n and d_n are real. And if f has real coefficients then the associated trigonometric polynomial is an even function of x, having only cosine terms in its expansion. The roots of these trigonometric polynomials are either on the real line or in complex conjugate pairs. This is somewhat closer to the symmetry of an L-function, but in this paper we phrase everything in terms of algebraic polynomials (1.1) to emphasize the comparison with characteristic polynomials of random matrices.

When the coefficients c_n and d_n of the polynomial (1.3) are independent standard normal random variables, Dunnage [5] discovered that the expected number of real zeros is given by

$$\frac{N}{\sqrt{3}} + O\left(N^{11/13} (\log N)^{3/13}\right). \tag{1.4}$$

Since real trigonometric polynomials are equivalent to self-reciprocal polynomials, formula (1.4) also gives the number of the zeros of (1.2) that lie on the unit circle. In this context it was rederived by Bogomolny *et al.* [2] who also computed the two-point correlation function of such zeros.

1.2. L-functions.

An L-function is a Dirichlet series,

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \tag{1.5}$$

with $a_n = O_\epsilon(n^\epsilon)$ for every $\epsilon > 0$, which has an analytic continuation to the complex plane (except for a possible pole at $s = 1$) along with two additional properties. First, it has a functional equation

$$\xi_L(s) := \gamma_L(s) L(s) = \varepsilon \overline{\xi_L}(1 - s), \tag{1.6}$$

with $|\varepsilon| = 1$ and γ_L of the form

$$\gamma_L(s) = P(s) Q^s \prod_{j=1}^{w} \Gamma(w_j s + \mu_j), \tag{1.7}$$

where $Q > 0$, $w_j > 0$, $\mathrm{Re}\mu_j \geq 0$, and P is a polynomial whose only zeros in $\sigma > 0$ are at the poles of $L(s)$. Second, it has an Euler product representation of the form

$$L(s) = \prod_p L_p(1/p^s), \tag{1.8}$$

where the product is over the primes p, and

$$L_p(1/p^s) = \sum_{k=0}^{\infty} \frac{a_{p^k}}{p^{ks}} = \exp\left(\sum_{k=1}^{\infty} \frac{b_{p^k}}{p^{ks}}\right), \tag{1.9}$$

where $b_n = O(n^\theta)$ with $\theta < \frac{1}{2}$. It is conjectured that such L-functions satisfy the *Riemann hypothesis*, which is the assertion that the nontrivial zeros lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$.

The functional equation is a symmetry with respect to the line $\mathrm{Re}(s) = \frac{1}{2}$, so the nontrivial zeros of L are on the line $\mathrm{Re}(s) = \frac{1}{2}$, or they are located symmetrically on either side of it. To show the analogy with self-reciprocal polynomials (1.2) it is more convenient to write the functional equation in asymmetric form:

$$L(s) = \varepsilon X_L(s)\overline{L}(1-s), \tag{1.10}$$

where $X_L(s) = \overline{\gamma_L}(1-s)/\gamma_L(s)$. Note that $|X_L| = 1$ on the $\frac{1}{2}$-line, that is, on the line of symmetry of the L-function. Thus, there is a perfect analogy between the self-reciprocal property of f and the functional equation of L.

1.3. **Results and discussion.** For Dirichlet series with a functional equation but no Euler product, it is not expected that all zeros lie on the critical line. However, it is possible that even if such a Dirichlet series does not satisfy the Riemann hypothesis, it could have, say, its first 100 trillion zeros on the critical line. Our motivating question is this: is the Riemann zeta-function, with it's Euler product, distinguishable from a random Dirichlet series with functional equation that just happens to have its first 100 trillion zeros on the critical line? By considering the analogous case of random self-reciprocal polynomials, we suggest that the answer is 'yes'.

Let us consider the space $\mathcal{R}$ of all the self-reciprocal polynomials. Given a nonvanishing probability distribution on $\mathcal{R}$, the subset $\mathcal{C}$ of those polynomials which happen to have all of their zeros on the unit circle has positive measure. The proof is quite simple. The zeros of a self-reciprocal polynomial lie either on the unit circle or in pairs symmetric with respect to the unit circle, and they are continuous functions of the coefficients. Two zeros on the circle must "collide" in order to move off the circle, so there is a small open neighborhood of coefficients in which the zeros remain on the circle. This neighborhood has positive measure. Therefore, the restriction to $\mathcal{C}$ of the probability distribution on $\mathcal{R}$ is unique and well defined.

Our interest is mainly concentrated on the distribution of the roots of those polynomials whose zeros lie all on the unit circle. Therefore, our approach will be to define a joint probability density function on the coefficients of the polynomials in $\mathcal{R}$ and to study its restriction to $\mathcal{C}$. The coefficient a_n of polynomials with all their zeros on the circle is bounded by $\binom{N}{n}$, therefore not only has $\mathcal{C}$ positive measure, but it is also compact. Thus, the most natural choice is to put a distribution on the coefficients a_n which is uniform on a bounded disk containing $\mathcal{C}$ and zero outside. The following theorem gives the joint probability density function for the roots of such polynomials. Peterson and Sinclair [20] have found some interesting geometric properties of the coefficients of these polynomials.

In what follows we denote by $\Delta(x_1,\ldots,x_N)$ the Vandermonde determinant, i.e.

$$\Delta(x_1,\ldots,x_N)=\prod_{j<k}(x_k-x_j), \tag{1.11}$$

and we denote by e_n the nth elementary symmetric function

$$e_n(x_1,\ldots,x_N)=\sum_{1\le i_1<\cdots<i_n\le N} x_{i_1}\cdots x_{i_n}. \tag{1.12}$$

Theorem 1.1. *Suppose N is odd. Consider random monic polynomials $z^N+\sum_{n=1}^N a_n z^{N-n}$ satisfying the self-reciprocal property $a_{N-n}=a_N\overline{a}_n$, with $a_1,\ldots,a_{(N-1)/2}$ chosen independently and uniformly in $|a_n|\le\binom{N}{n}$, and with ϕ chosen uniformly in $[0,2\pi)$, where $a_N=e^{i\phi}$, and restrict to those polynomials having all zeros on the unit circle. The joint probability density function of the set of zeros $e^{i\delta_1},\ldots,e^{i\delta_N}$ is given, up to a normalization constant, by*

$$|\Delta(e^{i\delta_1},\ldots,e^{i\delta_N})|=\prod_{j<k}|e^{i\delta_k}-e^{i\delta_j}|. \tag{1.13}$$

For N even, consider self-reciprocal random monic polynomials with $a_1,\ldots,a_{N/2}$ chosen independently and uniformly in $|a_n|\le\binom{N}{n}$, and restrict to those polynomials having all zeros on the unit circle. The joint probability density function for the set of zeros $e^{i\delta_1},\ldots,e^{i\delta_N}$ is given, up to a normalization constant, by

$$|a_{N/2}\Delta(e^{i\delta_1},\ldots,e^{i\delta_N})|=|a_{N/2}|\prod_{j<k}|e^{i\delta_k}-e^{i\delta_j}|, \tag{1.14}$$

where the coefficient $a_{N/2}$ is $(-1)^{N/2}$ times the $N/2$ elementary symmetric function in the variables $e^{i\delta_1},\ldots,e^{i\delta_N}$.

In particular, the joint probability density function for odd N is the same as that for eigenvalues of a randomly chosen matrix in the Circular Orthogonal Ensemble $COE(N)$.

The theorem suggests that if a Dirichlet series with functional equation is chosen at random, and all of the zeros in a particular interval of the critical strip happen to lie on the critical line, then those zeros should have similar statistics to those of eigenvalues of matrices from the COE. The COE is the symmetric space $U(N)/O(N)$ with the measure induced from Haar measure on $U(N)$. Thus the joint probability density function for the eigenvalues is

$$\frac{1}{(4\sqrt{\pi})^N\Gamma(1+N/2)}|\Delta(e^{i\delta_1},\ldots,e^{i\delta_N})|=\frac{1}{(4\sqrt{\pi})^N\Gamma(1+N/2)}\prod_{j<k}|e^{i\delta_k}-e^{i\delta_j}|. \tag{1.15}$$

All numerical calculations of zeros of L-functions having an Euler product show the statistics of the CUE (the group $U(N)$ with Haar measure), so this suggests that the L-functions from number theory are not typical Dirichlet series with a functional equation.

Thus, the Euler product which is considered a necessary condition for a Dirichlet series with functional equation to satisfy the Riemann hypothesis does more then just force the zeros onto the critical line. The Euler product fundamentally changes the nature of the spacing of the zeros, in particular changing the linear repulsion of zeros of random polynomials and the COE into the quadratic repulsion of the CUE, whose joint probability density function is given by

$$\frac{1}{(2\pi)^N N!}|\Delta(e^{i\delta_1},\dots,e^{i\delta_N})|^2 = \frac{1}{(2\pi)^N N!}\prod_{j<k}|e^{i\delta_k}-e^{i\delta_j}|^2. \tag{1.16}$$

In random matrix theory it has long been conjectured that in the limit $N \to \infty$ the local correlations of the eigenvalues of random matrices depend exclusively on the invariance properties of the probability distribution that defines the ensemble and not on the explicit form of the measure itself. This random matrix hypothesis is one of the most important features of the subject. Mathematically it translates into the statement that the local correlations are mainly determined by the absolute value of (powers) of the Vandermonde, whose origin is essentially geometrical. Therefore, the local statistics of the roots of the polynomials in $\mathcal{C}$ will be *independent* of our choice of the joint probability density function for the coefficients of the polynomials in $\mathcal{S}$. Indeed, because of this reason the extra factor $|a_{N/2}|$ appearing in equation (1.14) of theorem 1.1 when N is even should not affect the local correlations of the roots in the limit $N \to \infty$.

Bogomolny *et al.* [2] studied self-reciprocal polynomials whose coefficients are independent complex Gaussian random variables and computed the two-point correlation function $R_2(\delta)$ of the subset of zeros that lie on the unit circle. As $\delta \to 0$ they observed linear repulsion between the phases of the zeros. Such level repulsion is a direct consequence of the Vandermonde that appears in equations (1.13) and (1.14) of theorem 1.1 (or more appropriately for their case in equations (3.1) and (3.2) of lemma 3.1) and supports our previous observation.

The appearance of CUE statistics for arithmetic L-functions has been compared to the appearance of CUE statistics in a chaotic system without time-reversal symmetry [14, 1]. Indeed, the appearance of the CUE statistics for zeros of L-functions has been heuristically explained by the analogy between the periodic orbit sum for the density of states of a classically chaotic system with no time-reversal symmetry and the density of zeros of the Riemann zeta function expressed as a sum over primes. Our observation on the effect of the Euler product can be viewed as further evidence for that point of view.

We also consider the case of real self-reciprocal polynomials. That is, polynomials of the form (1.1) satisfying (1.2), where the a_n are real. These polynomials have their zeros in complex conjugate pairs, and in particular the zeros near $z = 1$ would be expected to have somewhat different behavior than the bulk of the zeros. Anomalous behavior of the low-lying zeros occurs for families of arithmetic L-functions, and this behavior has been modeled by the low-lying eigenvalues of matrices from the classical Symplectic and Orthogonal groups.

Theorem 1.2. *Let $N = 2M$. Consider random monic polynomials $z^N + \sum_{n=1}^N a_n z^{N-n}$ satisfying the self-reciprocal property $a_n = a_{N-n}$, with a_n chosen independently and uniformly in the interval $|a_n| \le \binom{N}{n}$, and restrict to those polynomials having all zeros on the unit circle. The joint probability density function of the set of zeros $e^{it_1}, e^{-it_1}, \dots, e^{it_M}, e^{-it_M}$ is given, up to a normalization constant, by*

$$\prod_m |e^{it_m} - e^{-it_m}| \prod_{j<k} |e^{it_k} - e^{it_j}|\,|e^{it_k} - e^{-it_j}|. \tag{1.17}$$

In the case where the polynomial has odd degree $N = 2M + 1$ with roots -1, $e^{it_1}, e^{-it_1}, \dots, e^{it_M}, e^{-it_M}$ then the joint probability density function of $e^{it_1}, e^{-it_1}, \dots, e^{it_M}, e^{-it_M}$ is the same as given above.

The above measure can be written as, up to a normalization constant,

$$\left| \prod_m \sin(t_m) \prod_{j<k} \sin\left(\frac{t_k - t_j}{2}\right) \sin\left(\frac{t_k + t_j}{2}\right) \right|. \tag{1.18}$$

We note the this is the *square root* of Haar measure of $USp(N)$. Thus, random real self-reciprocal polynomials, restricted to have all their zeros on the unit circle, do show anomalous spacings in their low lying zeros. But it is not the same anomalous spacing that has previously been found in families of arithmetic L-functions. This suggests that for the low-lying zeros of a family of L-functions with real coefficients, the Euler product has an effect on the vertical spacing of the zeros, and those L-functions behave differently than random real Dirichlet series with functional equation which just happen to have their first few zeros on the critical line.

In the next section we prove the Theorems and in the following sections we give the Jacobian calculations required in the proofs. We thank Christopher Sinclair for helpful information.

2. Proofs of the theorems

We are given a measure on a set of polynomials described in terms of the coefficients of the polynomial, and we wish to describe the measure in terms of the roots of the polynomial. Therefore we must compute the Jacobian of the change of variables from the coefficients to the roots. It is well known that the Jacobian is the Vandermonde in the roots in the case of polynomials with real coefficients, and the Vandermonde squared in the case of complex coefficients. Thus we expect the Jacobian to be close to a Vandermonde in the case of self-reciprocal polynomials, but we were unable to find our specific case in the literature so we give some details.

If X is a random variable then we let $\langle X \rangle$ denote the expected value of X. In our case f will be a random polynomial and $X = M[f]$ is some function of f, and we will need to compute $\langle M[f] \rangle$.

In Theorem 1.1, consider N even so $a_1, \ldots, a_{N/2}$ determine f. If $\rho_{\text{coeffs}}(a_1, \ldots, a_{N/2})$ is a probability measure on the coefficients of f, which is supported on the set S, then

$$\langle M[f] \rangle = C \int_S M[f] \rho_{\text{coeffs}}(a_1, \ldots, a_{N/2}) da_1 \cdots da_{N/2}. \tag{2.1}$$

Here, and for the remainder of the paper, C stands for a normalization constant that may be different at each occurrence.

We can express $\langle M[f] \rangle$ in terms of the zeros $z_1, \ldots, z_N$ of f. Set

$$\rho_{\text{zeros}}(z_1, \ldots, z_N) = \rho_{\text{coeffs}}(a_1(z_1, \ldots, z_N), \ldots, a_N(z_1, \ldots, z_N)) \tag{2.2}$$

and let $J_{\mathbb{C}}(z_1, \ldots, z_N)$ be the Jacobian of the transformation from the coefficients to the zeros. Then

$$\langle M[f] \rangle = C \int_{S'} M[f] \rho_{\text{zeros}}(z_1, \ldots, z_N) |J_{\mathbb{C}}(z_1, \ldots, z_N)| dz_1 \cdots dz_N, \tag{2.3}$$

where S' is the image of S under the coordinate change. To prove the Theorems we merely specialize this discussion to our particular cases.

Proof of Theorem 1.1. We have that ρ_{coeffs} is constant and S is the set of coefficients of self-reciprocal polynomials having all their zeros on the unit circle. So ρ_{zeros} is also constant and $S' = S^1 \times \cdots \times S^1$ where S^1 is the unit circle. So we only require the Jacobian of the transformation, which is given in the following Lemma.

Lemma 2.1. *If the self-reciprocal polynomial $f(z) = z^N + \sum_{n=1}^{N} a_n z^{N-n}$ has all its zeros on the unit circle, $e^{i\delta_1}, \dots, e^{i\delta_N}$, then when N is odd, the absolute value of the Jacobian of the transformation from coefficient variables $\mathrm{Re}a_1, \mathrm{Im}a_1, \dots, \mathrm{Re}a_{(N-1)/2}, \mathrm{Im}a_{(N-1)/2}, \phi$ (where $a_N = e^{i\phi}$) to the zero variables $\delta_1, \dots, \delta_N$ is given by*

$$\left|J_{\mathbb{C}}(e^{i\delta_1}, \dots, e^{i\delta_N})\right| = \left|2^{-\frac{N-1}{2}}\ \Delta(e^{i\delta_1}, \dots, e^{i\delta_N})\right|. \tag{2.4}$$

When N is even, the absolute value of the Jacobian of the transformation from coefficients $\mathrm{Re}a_1, \mathrm{Im}a_1, \dots, \mathrm{Re}a_{N/2}, \mathrm{Im}a_{N/2}$ to zeros $\delta_1, \dots, \delta_N$ is given by

$$\left|J_{\mathbb{C}}(e^{i\delta_1}, \dots, e^{i\delta_N})\right| = \left|2^{-\frac{N}{2}} a_{N/2}\ \Delta(e^{i\delta_1}, \dots, e^{i\delta_N})\right|, \tag{2.5}$$

where the coefficient $a_{N/2}$ is $(-1)^{N/2}$ times the $N/2$ elementary symmetric function in the variables $e^{i\delta_1}, \dots, e^{i\delta_N}$.

A proof of the Lemma can be found in Section 3.

Assembling the pieces we have, for example for odd N,

$$\langle M[f] \rangle = C \int_0^{2\pi} \cdots \int_0^{2\pi} M[f] |\Delta(e^{i\delta_1}, \dots, e^{i\delta_N})| d\delta_1 \cdots d\delta_N, \tag{2.6}$$

which is equivalent to Theorem 1.1. □

The proof of Theorem 1.2 is identical except that we require the following lemma, which is proven in Section 4.

Lemma 2.2. *If the real self-reciprocal polynomial f has even degree $N = 2M$ and has all its zeros on the unit circle, then the absolute value of the Jacobian of the transformation from coefficients $a_1, \dots, a_M$ to zeros $e^{it_1}, e^{-it_1}, \dots, e^{it_M}, e^{-it_M}$ is given by*

$$|J_{\mathbb{R}}(e^{it_1}, e^{-it_1}, \dots, e^{it_M}, e^{-it_M})| = \left| \prod_m (e^{it_m} - e^{-it_m}) \prod_{j<k} (e^{it_k} - e^{it_j})(e^{it_k} - e^{-it_j}) \right|. \tag{2.7}$$

In the case the degree $N = 2M + 1$ is odd, and f has zeros at -1, $e^{it_1}, e^{-it_1}, \dots, e^{it_M}, e^{-it_M}$, the Jacobian is again given by the above formula.

A proof of the Lemma can be found in Section 4.

3. Calculation of the Jacobian: complex case

We prove the following generalization of Lemma 2.1.

Lemma 3.1. *Let the roots of a self-reciprocal polynomial f be $\alpha_1 = e^{i\delta_1}, \dots, \alpha_L = e^{i\delta_L}$, for those roots on the unit circle, and $\beta_1 = \rho_1 e^{i\theta_1}, \frac{1}{\overline{\beta}_1} = \frac{e^{i\theta_1}}{\rho_1}, \dots, \beta_M = \rho_M e^{i\theta_M}, \frac{1}{\overline{\beta}_M} = \frac{e^{i\theta_M}}{\rho_M}$ for the roots occurring in pairs off the unit circle. When $N = L + 2M$ is odd, the absolute value of* *the Jacobian of the transformation from coefficients* $\mathrm{Re}a_1, \mathrm{Im}a_1, \ldots, \mathrm{Re}a_{(N-1)/2}, \mathrm{Im}a_{(N-1)/2}, \phi$ *(where* $a_N = e^{i\phi}$*) to zeros* $\rho_1, \theta_1, \rho_2, \theta_2, \ldots, \rho_M, \theta_M, \delta_1, \ldots, \delta_L$ *is given by*

$$\begin{aligned} &\left|J_{\mathbb{C}}(\rho_1, \theta_1, \rho_2, \theta_2, \ldots, \rho_M, \theta_M, \delta_1, \ldots, \delta_L)\right| \\ &\qquad = \left| 2^{M-\frac{N-1}{2}} \left(\prod_{m=1}^{M} \frac{1}{\rho_m}\right) \Delta(\beta_1, \tfrac{1}{\overline{\beta}_1}, \beta_2, \tfrac{1}{\overline{\beta}_2}, \ldots, \beta_m, \tfrac{1}{\overline{\beta}_m}, \alpha_1, \ldots, \alpha_L) \right|. \end{aligned} \tag{3.1}$$

When $N = L + 2M$ *is even, the absolute value of the Jacobian of the transformation from coefficients* $\mathrm{Re}a_1, \mathrm{Im}a_1, \ldots, \mathrm{Re}a_{N/2}, \mathrm{Im}a_{N/2}$ *to zeros* $\rho_1, \theta_1, \rho_2, \theta_2, \ldots, \rho_M, \theta_M, \delta_1, \ldots, \delta_L$ *is given by*

$$\begin{aligned} &\left|J_{\mathbb{C}}(\rho_1, \theta_1, \rho_2, \theta_2, \ldots, \rho_M, \theta_M, \delta_1, \ldots, \delta_L)\right| \\ &\qquad = \left| 2^{M-\frac{N}{2}} a_{N/2} \left(\prod_{m=1}^{M} \frac{1}{\rho_m}\right) \Delta(\beta_1, \tfrac{1}{\overline{\beta}_1}, \beta_2, \tfrac{1}{\overline{\beta}_2}, \ldots, \beta_m, \tfrac{1}{\overline{\beta}_m}, \alpha_1, \ldots, \alpha_L) \right|, \end{aligned} \tag{3.2}$$

where the coefficient $a_{N/2}$ *is* $(-1)^{N/2}$ *times the* $N/2$ *elementary symmetric function in the variables* $\alpha_1, \ldots, \alpha_L$, $\beta_1, \frac{1}{\overline{\beta}_1}, \ldots, \beta_M, \frac{1}{\overline{\beta}_M}$.

Proof. The polynomial $f(z)$ has order $N = L + 2M$:

$$\begin{aligned} f(z) &= (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_L)(z-\beta_1)(z-\tfrac{1}{\overline{\beta}_1})\cdots(z-\beta_M)(z-\tfrac{1}{\overline{\beta}_M}) \\ &= z^N + a_1 z^{N-1} + a_2 z^{N-2} + \cdots + a_{N-2} z^2 + a_{N-1} z + a_N. \end{aligned} \tag{3.3}$$

Note that if we define

$$\alpha_j = e^{i\delta_j}, \qquad j = 1, \ldots, L \tag{3.4a}$$

$$\beta_j = t_j \rho_j = e^{i\theta_j} \rho_j, \qquad j = 1, \ldots, M \tag{3.4b}$$

$$\frac{1}{\overline{\beta}_j} = \frac{t_j}{\rho_j} = \frac{e^{i\theta_j}}{\rho_j}, \qquad j = 1, \ldots, M \tag{3.4c}$$

(with δ_j, ρ_j and θ_j real) then a_N is on the unit circle and

$$a_N = (-1)^N e^{2i\theta_1} e^{2i\theta_2} \cdots e^{2i\theta_M} e^{i\delta_1} \cdots e^{i\delta_L} = e^{i\phi}. \tag{3.5}$$

Also

$$a_j = (-1)^j e_j(\alpha_1, \ldots, \alpha_L, \beta_1, \tfrac{1}{\overline{\beta}_1}, \ldots, \beta_M, \tfrac{1}{\overline{\beta}_M}), \tag{3.6}$$

where $e_j(x_1, \ldots, x_n)$ is the jth elementary symmetric function. In addition we have

$$a_{N-j} = a_N \overline{a}_j, \tag{3.7}$$

since by construction f is self-reciprocal.

For now we take N odd; the slight variation when N is even is described at the end of this section. We want the Jacobian of the transformation from the independent real variables $\mathrm{Re}a_1, \mathrm{Im}a_1, \ldots, \mathrm{Re}a_{\frac{N-1}{2}}, \mathrm{Im}a_{\frac{N-1}{2}}, \phi$ to the real independent variables $\rho_1, \theta_1, \rho_2, \theta_2, \ldots, \rho_M, \theta_M, \delta_1, \ldots, \delta_L$.

So, the Jacobian is:

$$J_{\mathbb{C}} = \begin{vmatrix} \frac{\partial \mathrm{Re} a_1}{\partial \rho_1} & \frac{\partial \mathrm{Re} a_1}{\partial \theta_1} & \cdots & \frac{\partial \mathrm{Re} a_1}{\partial \rho_M} & \frac{\partial \mathrm{Re} a_1}{\partial \theta_M} & \frac{\partial \mathrm{Re} a_1}{\partial \delta_1} & \cdots & \frac{\partial \mathrm{Re} a_1}{\partial \delta_L} \\ \frac{\partial \mathrm{Im} a_1}{\partial \rho_1} & \frac{\partial \mathrm{Im} a_1}{\partial \theta_1} & \cdots & \frac{\partial \mathrm{Im} a_1}{\partial \rho_M} & \frac{\partial \mathrm{Im} a_1}{\partial \theta_M} & \frac{\partial \mathrm{Im} a_1}{\partial \delta_1} & \cdots & \frac{\partial \mathrm{Im} a_1}{\partial \delta_L} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathrm{Re} a_{\frac{N-1}{2}}}{\partial \rho_1} & \frac{\partial \mathrm{Re} a_{\frac{N-1}{2}}}{\partial \theta_1} & \cdots & \frac{\partial \mathrm{Re} a_{\frac{N-1}{2}}}{\partial \rho_M} & \frac{\partial \mathrm{Re} a_{\frac{N-1}{2}}}{\partial \theta_M} & \frac{\partial \mathrm{Re} a_{\frac{N-1}{2}}}{\partial \delta_1} & \cdots & \frac{\partial \mathrm{Re} a_{\frac{N-1}{2}}}{\partial \delta_L} \\ \frac{\partial \mathrm{Im} a_{\frac{N-1}{2}}}{\partial \rho_1} & \frac{\partial \mathrm{Im} a_{\frac{N-1}{2}}}{\partial \theta_1} & \cdots & \frac{\partial \mathrm{Im} a_{\frac{N-1}{2}}}{\partial \rho_M} & \frac{\partial \mathrm{Im} a_{\frac{N-1}{2}}}{\partial \theta_M} & \frac{\partial \mathrm{Im} a_{\frac{N-1}{2}}}{\partial \delta_1} & \cdots & \frac{\partial \mathrm{Im} a_{\frac{N-1}{2}}}{\partial \delta_L} \\ \frac{\partial \phi}{\partial \rho_1} & \frac{\partial \phi}{\partial \theta_1} & \cdots & \frac{\partial \phi}{\partial \rho_M} & \frac{\partial \phi}{\partial \theta_M} & \frac{\partial \phi}{\partial \delta_1} & \cdots & \frac{\partial \phi}{\partial \delta_L} \end{vmatrix}$$

$$(3.8) \quad = (-\tfrac{1}{2})^{\frac{N-1}{2}} \begin{vmatrix} \frac{\partial a_1}{\partial \rho_1} & \frac{\partial a_1}{\partial \theta_1} & \cdots & \frac{\partial a_1}{\partial \rho_M} & \frac{\partial a_1}{\partial \theta_M} & \frac{\partial a_1}{\partial \delta_1} & \cdots & \frac{\partial a_1}{\partial \delta_L} \\ \frac{\partial \overline{a}_1}{\partial \rho_1} & \frac{\partial \overline{a}_1}{\partial \theta_1} & \cdots & \frac{\partial \overline{a}_1}{\partial \rho_M} & \frac{\partial \overline{a}_1}{\partial \theta_M} & \frac{\partial \overline{a}_1}{\partial \delta_1} & \cdots & \frac{\partial \overline{a}_1}{\partial \delta_L} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial a_{\frac{N-1}{2}}}{\partial \rho_1} & \frac{\partial a_{\frac{N-1}{2}}}{\partial \theta_1} & \cdots & \frac{\partial a_{\frac{N-1}{2}}}{\partial \rho_M} & \frac{\partial a_{\frac{N-1}{2}}}{\partial \theta_M} & \frac{\partial a_{\frac{N-1}{2}}}{\partial \delta_1} & \cdots & \frac{\partial a_{\frac{N-1}{2}}}{\partial \delta_L} \\ \frac{\partial \overline{a}_{\frac{N-1}{2}}}{\partial \rho_1} & \frac{\partial \overline{a}_{\frac{N-1}{2}}}{\partial \theta_1} & \cdots & \frac{\partial \overline{a}_{\frac{N-1}{2}}}{\partial \rho_M} & \frac{\partial \overline{a}_{\frac{N-1}{2}}}{\partial \theta_M} & \frac{\partial \overline{a}_{\frac{N-1}{2}}}{\partial \delta_1} & \cdots & \frac{\partial \overline{a}_{\frac{N-1}{2}}}{\partial \delta_L} \\ \frac{\partial \phi}{\partial \rho_1} & \frac{\partial \phi}{\partial \theta_1} & \cdots & \frac{\partial \phi}{\partial \rho_M} & \frac{\partial \phi}{\partial \theta_M} & \frac{\partial \phi}{\partial \delta_1} & \cdots & \frac{\partial \phi}{\partial \delta_L} \end{vmatrix}.$$

This step was achieved in two stages: first, by adding each even row to the one above, and then by multiplying each even row by -1 and adding to it $1/2$ the row above.

Now note that

$$(3.9a) \quad \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial a_N}\frac{\partial a_N}{\partial x} = \frac{\partial a_N}{\partial x}\left(\frac{1}{i a_N}\right),$$

$$(3.9b) \quad \frac{\partial a_{N-j}}{\partial x} = a_N \frac{\partial \overline{a}_j}{\partial x} + \overline{a}_j \frac{\partial a_N}{\partial x}$$

$$(3.9c) \quad \frac{\partial x}{\partial \delta_j} = \frac{\partial x}{\partial \alpha_j}\frac{\partial \alpha_j}{\partial \delta_j} = i\alpha_j \frac{\partial x}{\partial \alpha_j}$$

$$(3.9d) \quad \frac{\partial x}{\partial \theta_j} = \frac{\partial x}{\partial t_j}\frac{\partial t_j}{\partial \theta_j} = i t_j \frac{\partial x}{\partial t_j}.$$

So we have

$$J_{\mathbb{C}} = \left(-\frac{1}{2}\right)^{\frac{N-1}{2}} \left(\frac{1}{i a_N}\right) \left(\frac{1}{a_N}\right)^{\frac{N-1}{2}} \prod_{\ell=1}^{L} (i\alpha_\ell) \prod_{m=1}^{M} (i t_m)$$

$$(3.10) \quad \times \begin{vmatrix} \frac{\partial a_1}{\partial \rho_1} & \frac{\partial a_1}{\partial t_1} & \cdots & \frac{\partial a_1}{\partial \rho_M} & \frac{\partial a_1}{\partial t_M} & \frac{\partial a_1}{\partial \alpha_1} & \cdots & \frac{\partial a_1}{\partial \alpha_L} \\ \frac{\partial a_{N-1}}{\partial \rho_1} & \frac{\partial a_{N-1}}{\partial t_1} & \cdots & \frac{\partial a_{N-1}}{\partial \rho_M} & \frac{\partial a_{N-1}}{\partial t_M} & \frac{\partial a_{N-1}}{\partial \alpha_1} & \cdots & \frac{\partial a_{N-1}}{\partial \alpha_L} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial a_{\frac{N-1}{2}}}{\partial \rho_1} & \frac{\partial a_{\frac{N-1}{2}}}{\partial t_1} & \cdots & \frac{\partial a_{\frac{N-1}{2}}}{\partial \rho_M} & \frac{\partial a_{\frac{N-1}{2}}}{\partial t_M} & \frac{\partial a_{\frac{N-1}{2}}}{\partial \alpha_1} & \cdots & \frac{\partial a_{\frac{N-1}{2}}}{\partial \alpha_L} \\ \frac{\partial a_{\frac{N+1}{2}}}{\partial \rho_1} & \frac{\partial a_{\frac{N+1}{2}}}{\partial t_1} & \cdots & \frac{\partial a_{\frac{N+1}{2}}}{\partial \rho_M} & \frac{\partial a_{\frac{N+1}{2}}}{\partial t_M} & \frac{\partial a_{\frac{N+1}{2}}}{\partial \alpha_1} & \cdots & \frac{\partial a_{\frac{N+1}{2}}}{\partial \alpha_L} \\ \frac{\partial a_N}{\partial \rho_1} & \frac{\partial a_N}{\partial t_1} & \cdots & \frac{\partial a_N}{\partial \rho_M} & \frac{\partial a_N}{\partial t_M} & \frac{\partial a_N}{\partial \alpha_1} & \cdots & \frac{\partial a_N}{\partial \alpha_L} \end{vmatrix}.$$

So we have, where ε is a quantity with modulus one that may vary at each occurrence,

$$J_{\mathbb{C}} = \varepsilon \left(\frac{1}{2}\right)^{\frac{N-1}{2}} \begin{vmatrix} \frac{\partial a_1}{\partial \rho_1} & \frac{\partial a_1}{\partial t_1} & \cdots & \frac{\partial a_1}{\partial \rho_M} & \frac{\partial a_1}{\partial t_M} & \frac{\partial a_1}{\partial \alpha_1} & \cdots & \frac{\partial a_1}{\partial \alpha_L} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial a_N}{\partial \rho_1} & \frac{\partial a_N}{\partial t_1} & \cdots & \frac{\partial a_N}{\partial \rho_M} & \frac{\partial a_N}{\partial t_M} & \frac{\partial a_N}{\partial \alpha_1} & \cdots & \frac{\partial a_N}{\partial \alpha_L} \end{vmatrix}. \tag{3.11}$$

Following the method in [23], for a given m,

$$a_n = (-1)^n \left(\frac{\beta_m}{\overline{\beta}_m} e'_{n-2,m} + (\beta_m + \tfrac{1}{\overline{\beta}_m}) e'_{n-1,m} + e'_{n,m}\right), \tag{3.12}$$

with the convention that

$$e_0 = e_{0,m} = e'_{0,m} = 1 \tag{3.13a}$$

$$e_{-j} = e_{-j,m} = e'_{-j,m} = 0 \qquad \text{for } j > 0 \text{ and} \tag{3.13b}$$

$$e_n = e'_{n,m} = e_{n,m} = 0 \qquad \text{if } n > \#\text{ variables.} \tag{3.13c}$$

Here $e_{n,\ell}$ is the nth symmetric function in all the α's, β's and $\frac{1}{\overline{\beta}}$'s except for α_ℓ, and $e'_{n,m}$ is the nth symmetric function in all the α's, β's and $\frac{1}{\overline{\beta}}$'s except for β_m and $\frac{1}{\overline{\beta}_m}$.

Therefore,

$$\frac{\partial a_n}{\partial \rho_m} = (-1)^n \left(t_m - \frac{t_m}{\rho_m^2}\right) e'_{n-1,m} \tag{3.14a}$$

$$\frac{\partial a_n}{\partial t_m} = (-1)^n \left(2t_m e'_{n-2,m} + (\rho_m + \tfrac{1}{\rho_m}) e'_{n-1,m}\right) \tag{3.14b}$$

$$\frac{\partial a_n}{\partial \alpha_\ell} = (-1)^n e_{n-1,\ell}, \tag{3.14c}$$

so the jth row of our determinant (3.11) is

$$\Bigg(\left(t_1 - \frac{t_1}{\rho_1^2}\right) e'_{j-1,1},\ 2t_1 e'_{j-2,1} + \left(\rho_1 + \frac{1}{\rho_1}\right) e'_{j-1,1}, \cdots$$
$$\cdots, \left(t_M - \frac{t_M}{\rho_M^2}\right) e'_{j-1,M},\ 2t_M e'_{j-2,M} + \left(\rho_M + \frac{1}{\rho_M}\right) e'_{j-1,M},\ e_{j-1,1},\ e_{j-1,2}, \ldots, e_{j-1,L}\Bigg). \tag{3.15}$$

Alternately, if we write this in terms of the β variables, the jth row is

$$\left(\prod_{m=1}^{M} (t_m - \frac{t_m}{\rho_m^2})\right) \prod_{m=1}^{M} \frac{1}{t_m} \Bigg(e'_{j-1,1},\ 2\frac{\beta_1}{\overline{\beta}_1} e'_{j-2,1} + \left(\beta_1 + \frac{1}{\overline{\beta}_1}\right) e'_{j-1,1}, \cdots$$
$$\cdots, e'_{j-1,M},\ 2\frac{\beta_M}{\overline{\beta}_M} e'_{j-2,M} + \left(\beta_M + \frac{1}{\overline{\beta}_M}\right) e'_{j-1,M},\ e_{j-1,1},\ e_{j-1,2}, \ldots, e_{j-1,L}\Bigg). \tag{3.16}$$

Now we define the following polynomials:

$$f_\ell(x) = \prod_{\substack{k=1\\k\neq\ell}}^{L}(x-\alpha_k)\ \prod_{m=1}^{M}(x-\beta_m)(x-\tfrac{1}{\overline{\beta}_m}) = \sum_{n=0}^{N-1}(-1)^n e_{n,\ell}\ x^{N-1-n} \tag{3.17a}$$

$$\begin{aligned} h_m(x) &= x\ \prod_{\ell=1}^{L}(x-\alpha_\ell)\ \prod_{\substack{k=1\\k\neq m}}^{M}(x-\beta_k)(x-\tfrac{1}{\overline{\beta}_k}) = \sum_{n=0}^{N-2}(-1)^n e'_{n,m}\ x^{N-1-n} \\ &= \sum_{n=0}^{N-1}(-1)^n e'_{n,m}\ x^{N-1-n}, \end{aligned} \tag{3.17b}$$

where the last line follows from the convention that $e'_{N-1,m}=0$, and

$$\begin{aligned} g_m(x) &= \left(-2\frac{\beta_m}{\overline{\beta}_m} + x(\beta_m+\tfrac{1}{\overline{\beta}_m})\right)\left(\prod_{\ell=1}^{L}(x-\alpha_\ell)\ \prod_{\substack{k=1\\k\neq m}}(x-\beta_k)(x-\tfrac{1}{\overline{\beta}_k})\right) \\ &= \left(-2\frac{\beta_m}{\overline{\beta}_m} + x(\beta_m+\tfrac{1}{\overline{\beta}_m})\right)\left(\sum_{n=0}^{N-2}(-1)^n e'_{n,m}\ x^{N-2-n}\right) \\ &= -2\frac{\beta_m}{\overline{\beta}_m}\sum_{n=0}^{N-2}(-1)^n e'_{n,m}\ x^{N-2-n} + \left(\beta_m+\tfrac{1}{\overline{\beta}_m}\right)\sum_{n=0}^{N-2}(-1)^n e'_{n,m}\ x^{N-1-n} \\ &= \left(\beta_m+\tfrac{1}{\overline{\beta}_m}\right)x^{N-1} + \sum_{n=1}^{N-2}(-1)^n\left(2\frac{\beta_m}{\overline{\beta}_m}e'_{n-1,m} + (\beta_m+\tfrac{1}{\overline{\beta}_m})e'_{n,m}\right)x^{N-1-n} \\ &\quad + 2(-1)^{N-1}e'_{N-2,m}\frac{\beta_m}{\overline{\beta}_m} \\ &= \sum_{n=0}^{N-1}(-1)^n\left(2\frac{\beta_m}{\overline{\beta}_m}e'_{n-1,m} + (\beta_m+\tfrac{1}{\overline{\beta}_m})e'_{n,m}\right)x^{N-1-n}. \end{aligned} \tag{3.18}$$

With the definition $\Delta(x_1,\dots,x_n)=\prod_{1\le j<k\le n}(x_k-x_j)$, if we multiply the determinant (3.16) by

$$\Delta(\beta_1,\tfrac{1}{\beta_1},\dots,\beta_M,\tfrac{1}{\beta_M},\alpha_1,\dots,\alpha_L) = \begin{vmatrix} \beta_1^{N-1} & -\beta_1^{N-2} & \beta_1^{N-3} & -\beta_1^{N-4} & \cdots & -\beta_1 & 1 \\ (\frac{1}{\beta_1})^{N-1} & -(\frac{1}{\beta_1})^{N-2} & (\frac{1}{\beta_1})^{N-3} & -(\frac{1}{\beta_1})^{N-4} & \cdots & -\frac{1}{\beta_1} & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_M^{N-1} & -\beta_M^{N-2} & \beta_M^{N-3} & -\beta_M^{N-4} & \cdots & -\beta_M & 1 \\ (\frac{1}{\beta_M})^{N-1} & -(\frac{1}{\beta_M})^{N-2} & (\frac{1}{\beta_M})^{N-3} & -(\frac{1}{\beta_M})^{N-4} & \cdots & -\frac{1}{\beta_M} & 1 \\ \alpha_1^{N-1} & -\alpha_1^{N-2} & \alpha_1^{N-3} & -\alpha_1^{N-4} & \cdots & -\alpha_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_L^{N-1} & -\alpha_L^{N-2} & \alpha_L^{N-3} & -\alpha_L^{N-4} & \cdots & -\alpha_L & 1 \end{vmatrix} \tag{3.19}$$

on the left then we get, with some $|\varepsilon|=1$,

$$\varepsilon\,(\tfrac{1}{2})^{\frac{N-1}{2}}\prod_{m=1}^{M}\left(t_m-\frac{t_m}{\rho_m^2}\right) \times \begin{vmatrix} h_1(\beta_1) & g_1(\beta_1) & \cdots & h_M(\beta_1) & g_M(\beta_1) & f_1(\beta_1) & \cdots & f_L(\beta_1) \\ h_1(\frac{1}{\beta_1}) & g_1(\frac{1}{\beta_1}) & \cdots & h_M(\frac{1}{\beta_1}) & g_M(\frac{1}{\beta_1}) & f_1(\frac{1}{\beta_1}) & \cdots & f_L(\frac{1}{\beta_1}) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ h_1(\beta_M) & g_1(\beta_M) & \cdots & h_M(\beta_M) & g_M(\beta_M) & f_1(\beta_M) & \cdots & f_L(\beta_M) \\ h_1(\frac{1}{\beta_M}) & g_1(\frac{1}{\beta_M}) & \cdots & h_M(\frac{1}{\beta_M}) & g_M(\frac{1}{\beta_M}) & f_1(\frac{1}{\beta_M}) & \cdots & f_L(\frac{1}{\beta_M}) \\ h_1(\alpha_1) & g_1(\alpha_1) & \cdots & h_M(\alpha_1) & g_M(\alpha_1) & f_1(\alpha_1) & \cdots & f_L(\alpha_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ h_1(\alpha_L) & g_1(\alpha_L) & \cdots & h_M(\alpha_L) & g_M(\alpha_L) & f_1(\alpha_L) & \cdots & f_L(\alpha_L) \end{vmatrix}. \tag{3.20}$$

However, many of these matrix elements are zero:

$$h_j(\beta_i)=h_j(\tfrac{1}{\beta_i})=0 \qquad \text{if } i\ne j, \tag{3.21a}$$
$$h_j(\alpha_i)=0 \qquad \forall\, i,j, \tag{3.21b}$$
$$g_j(\beta_i)=g_j(\tfrac{1}{\beta_i})=0 \qquad \text{if } i\ne j, \tag{3.21c}$$
$$g_j(\alpha_i)=0 \qquad \forall\, i,j, \tag{3.21d}$$
$$f_j(\alpha_i)=0 \qquad \text{if } i\ne j, \tag{3.21e}$$
$$f_j(\beta_i)=f_j(\tfrac{1}{\beta_i})=0 \qquad \forall\, i,j. \tag{3.21f}$$

So, we have a matrix with M 2×2 blocks on the diagonal and then L diagonal entries and everything else zero.

The determinant (3.20) is

$$\varepsilon\,(\tfrac{1}{2})^{\frac{N-1}{2}}\left(\frac{1}{\prod_{m=1}^{M}\rho_m}\right)\left(\prod_{m=1}^{M}(\beta_m-\tfrac{1}{\overline{\beta}_m})\right)\prod_{m=1}^{M}\begin{vmatrix} h_m(\beta_m) & g_m(\beta_m)\\ h_m(\frac{1}{\overline{\beta}_m}) & g_m(\frac{1}{\overline{\beta}_m})\end{vmatrix}\prod_{\ell=1}^{L}f_\ell(\alpha_\ell), \tag{3.22}$$

and we note that

$$\begin{aligned}
&h_m(\beta_m)g_m(\tfrac{1}{\overline{\beta}_m})-h_m(\tfrac{1}{\overline{\beta}_m})g_m(\beta_m)\\
&\quad=\beta_m\prod_{\ell=1}^{L}(\beta_m-\alpha_\ell)\prod_{\substack{k=1\\k\neq m}}^{M}(\beta_m-\beta_k)(\beta_m-\tfrac{1}{\overline{\beta}_k})\\
&\qquad\times(-2\tfrac{\beta_m}{\overline{\beta}_m}+\tfrac{1}{\overline{\beta}_m}(\beta_m+\tfrac{1}{\overline{\beta}_m}))\prod_{\ell=1}^{L}(\tfrac{1}{\overline{\beta}_m}-\alpha_\ell)\prod_{k=1}^{M}(\tfrac{1}{\overline{\beta}_m}-\beta_k)(\tfrac{1}{\overline{\beta}_m}-\tfrac{1}{\overline{\beta}_k})\\
&\qquad-\frac{1}{\overline{\beta}_m}\prod_{\ell=1}^{L}(\tfrac{1}{\overline{\beta}_m}-\alpha_\ell)\prod_{\substack{k=1\\k\neq m}}^{M}(\tfrac{1}{\overline{\beta}_m}-\beta_k)(\tfrac{1}{\overline{\beta}_m}-\tfrac{1}{\overline{\beta}_k})\\
&\qquad\times(-2\tfrac{\beta_m}{\overline{\beta}_m}+\beta_m(\beta_m+\tfrac{1}{\overline{\beta}_m}))\prod_{\ell=1}^{L}(\beta_m-\alpha_\ell)\prod_{\substack{k=1\\k\neq m}}^{M}(\beta_m-\beta_k)(\beta_m-\tfrac{1}{\overline{\beta}_k})\\
&\quad=\left(\prod_{\ell=1}^{L}(\beta_m-\alpha_\ell)(\tfrac{1}{\overline{\beta}_m}-\alpha_\ell)\right)\left(\prod_{\substack{k=1\\k\neq m}}^{M}(\beta_m-\beta_k)(\beta_m-\tfrac{1}{\overline{\beta}_k})(\tfrac{1}{\overline{\beta}_m}-\beta_k)(\tfrac{1}{\overline{\beta}_m}-\tfrac{1}{\overline{\beta}_k})\right)\\
&\qquad\times\left(2\frac{\beta_m}{\overline{\beta}_m^2}-2\frac{\beta_m^2}{\overline{\beta}_m}\right)\\
&\quad=\left(\prod_{\ell=1}^{L}(\beta_m-\alpha_\ell)(\tfrac{1}{\overline{\beta}_m}-\alpha_\ell)\right)\left(\prod_{\substack{k=1\\k\neq m}}^{M}(\beta_m-\beta_k)(\beta_m-\tfrac{1}{\overline{\beta}_k})(\tfrac{1}{\overline{\beta}_m}-\beta_k)(\tfrac{1}{\overline{\beta}_m}-\tfrac{1}{\overline{\beta}_k})\right)\\
&\qquad\times\left(-2\frac{\beta_m}{\overline{\beta}_m}\right)\left(\beta_m-\frac{1}{\overline{\beta}_m}\right).
\end{aligned}\tag{3.23}$$

Thus the determinant (3.20) is

$$\begin{aligned}
&\varepsilon\, (\tfrac{1}{2})^{\frac{N-1}{2}}(-2)^M \left(\prod_{m=1}^{M} (\tfrac{t_m^2}{\rho_m})(\beta_m - \tfrac{1}{\overline{\beta}_m})^2\right) \\
&\qquad \times \prod_{m=1}^{M}\left(\left(\prod_{\ell=1}^{L}(\beta_m-\alpha_\ell)(\tfrac{1}{\overline{\beta}_m}-\alpha_\ell)\right)\left(\prod_{\substack{k=1\\k\neq m}}(\beta_m-\beta_k)(\beta_m-\tfrac{1}{\overline{\beta}_k})(\tfrac{1}{\overline{\beta}_m}-\beta_k)(\tfrac{1}{\overline{\beta}_m}-\tfrac{1}{\overline{\beta}_k})\right)\right) \\
&\qquad \times \prod_{\ell=1}^{L}\left(\prod_{\substack{k=1\\k\neq \ell}}^{L}(\alpha_\ell-\alpha_k)\ \prod_{m=1}^{M}(\alpha_\ell-\beta_m)(\alpha_\ell-\tfrac{1}{\overline{\beta}_m})\right) \\
&\qquad = \varepsilon\,(-1)^M(-1)^{\frac{L(L-1)}{2}}2^{M-\frac{N-1}{2}}\prod_{m=1}^{M}\left(\frac{t_m^2}{\rho_m}\right)\Delta^2(\beta_1,\tfrac{1}{\overline{\beta}_1},\ldots,\beta_M,\tfrac{1}{\overline{\beta}_M},\alpha_1,\ldots,\alpha_L).
\end{aligned}$$

We need to divide by the determinant of the Vandermonde (3.19) so, with ε a factor with modulus one, we have that the Jacobian is

$$J_{\mathbb{C}} = \varepsilon\, 2^{M-\frac{N-1}{2}}\left(\prod_{m=1}^{M}\frac{1}{\rho_m}\right)\Delta(\beta_1,\tfrac{1}{\overline{\beta}_1},\beta_2,\tfrac{1}{\overline{\beta}_2},\ldots,\beta_m,\tfrac{1}{\overline{\beta}_m},\alpha_1,\ldots,\alpha_L). \tag{3.24}$$

This completes the proof in the case N is odd.

When N is even, we want the Jacobian of the transformation from the real, independent variables $\mathrm{Re}a_1, \mathrm{Im}a_1, \ldots, \mathrm{Re}a_{\frac{N}{2}}, \mathrm{Im}a_{\frac{N}{2}}$ to the real independent variables $\rho_1,\theta_1,\rho_2,\theta_2,\ \ldots,$ $\rho_M,\theta_M,\delta_1,\ldots,\delta_L$.

So, we start with

$$\begin{aligned}
J_{\mathbb{C}} &= \begin{vmatrix}
\frac{\partial \mathrm{Re}a_1}{\partial\rho_1} & \frac{\partial \mathrm{Re}a_1}{\partial\theta_1} & \cdots & \frac{\partial \mathrm{Re}a_1}{\partial\rho_M} & \frac{\partial \mathrm{Re}a_1}{\partial\theta_M} & \frac{\partial \mathrm{Re}a_1}{\partial\delta_1} & \cdots & \frac{\partial \mathrm{Re}a_1}{\partial\delta_L} \\
\frac{\partial \mathrm{Im}a_1}{\partial\rho_1} & \frac{\partial \mathrm{Im}a_1}{\partial\theta_1} & \cdots & \frac{\partial \mathrm{Im}a_1}{\partial\rho_M} & \frac{\partial \mathrm{Im}a_1}{\partial\theta_M} & \frac{\partial \mathrm{Im}a_1}{\partial\delta_1} & \cdots & \frac{\partial \mathrm{Im}a_1}{\partial\delta_L} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial \mathrm{Re}a_{\frac{N}{2}}}{\partial\rho_1} & \frac{\partial \mathrm{Re}a_{\frac{N}{2}}}{\partial\theta_1} & \cdots & \frac{\partial \mathrm{Re}a_{\frac{N}{2}}}{\partial\rho_M} & \frac{\partial \mathrm{Re}a_{\frac{N}{2}}}{\partial\theta_M} & \frac{\partial \mathrm{Re}a_{\frac{N}{2}}}{\partial\delta_1} & \cdots & \frac{\partial \mathrm{Re}a_{\frac{N}{2}}}{\partial\delta_L} \\
\frac{\partial \mathrm{Im}a_{\frac{N}{2}}}{\partial\rho_1} & \frac{\partial \mathrm{Im}a_{\frac{N}{2}}}{\partial\theta_1} & \cdots & \frac{\partial \mathrm{Im}a_{\frac{N}{2}}}{\partial\rho_M} & \frac{\partial \mathrm{Im}a_{\frac{N}{2}}}{\partial\theta_M} & \frac{\partial \mathrm{Im}a_{\frac{N}{2}}}{\partial\delta_1} & \cdots & \frac{\partial \mathrm{Im}a_{\frac{N}{2}}}{\partial\delta_L}
\end{vmatrix} \\
&= (-\tfrac{1}{2})^{\frac{N}{2}}\begin{vmatrix}
\frac{\partial a_1}{\partial\rho_1} & \frac{\partial a_1}{\partial\theta_1} & \cdots & \frac{\partial a_1}{\partial\rho_M} & \frac{\partial a_1}{\partial\theta_M} & \frac{\partial a_1}{\partial\delta_1} & \cdots & \frac{\partial a_1}{\partial\delta_L} \\
\frac{\partial \overline{a}_1}{\partial\rho_1} & \frac{\partial \overline{a}_1}{\partial\theta_1} & \cdots & \frac{\partial \overline{a}_1}{\partial\rho_M} & \frac{\partial \overline{a}_1}{\partial\theta_M} & \frac{\partial \overline{a}_1}{\partial\delta_1} & \cdots & \frac{\partial \overline{a}_1}{\partial\delta_L} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial a_{\frac{N}{2}}}{\partial\rho_1} & \frac{\partial a_{\frac{N}{2}}}{\partial\theta_1} & \cdots & \frac{\partial a_{\frac{N}{2}}}{\partial\rho_M} & \frac{\partial a_{\frac{N}{2}}}{\partial\theta_M} & \frac{\partial a_{\frac{N}{2}}}{\partial\delta_1} & \cdots & \frac{\partial a_{\frac{N}{2}}}{\partial\delta_L} \\
\frac{\partial \overline{a}_{\frac{N}{2}}}{\partial\rho_1} & \frac{\partial \overline{a}_{\frac{N}{2}}}{\partial\theta_1} & \cdots & \frac{\partial \overline{a}_{\frac{N}{2}}}{\partial\rho_M} & \frac{\partial \overline{a}_{\frac{N}{2}}}{\partial\theta_M} & \frac{\partial \overline{a}_{\frac{N}{2}}}{\partial\delta_1} & \cdots & \frac{\partial \overline{a}_{\frac{N}{2}}}{\partial\delta_L}
\end{vmatrix}.
\end{aligned} \tag{3.25}$$

Now we want to transform the derivatives of $\overline{a}_j$ into derivatives of a_{N-j}, with the exception of $\overline{a}_{N/2}$ that should give us derivatives of a_N. Note that a_N has modulus 1 and

$$\frac{a_{N/2}}{\overline{a}_{N/2}} = a_N. \tag{3.26}$$

Therefore, we have

$$\frac{\partial a_N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{a_{N/2}}{\overline{a}_{N/2}}\right) \tag{3.27}$$

$$= \frac{1}{\overline{a}_{N/2}}\frac{\partial a_{N/2}}{\partial x} - \frac{a_{N/2}}{(\overline{a}_{N/2})^2}\frac{\partial \overline{a}_{N/2}}{\partial x}. \tag{3.28}$$

Thus we need to multiply the last row of (3.25) by $-a_{N/2}/(\overline{a}_{N/2})^2$ and add to it $1/\overline{a}_{N/2}$ times the row above. This procedure has multiplied the determinant by a factor $-a_{N/2}/(\overline{a}_{N/2})^2 = -a_N/\overline{a}_{N/2}$. Note that a_N has modulus one, but $a_{N/2}$ does not. Thus, incorporating factors of modulus one into ε, the Jacobian is

$$J_{\mathbb{C}} = \varepsilon\, \overline{a}_{N/2}\left(\frac{1}{2}\right)^{\frac{N}{2}} \begin{vmatrix} \frac{\partial a_1}{\partial \rho_1} & \frac{\partial a_1}{\partial t_1} & \cdots & \frac{\partial a_1}{\partial \rho_M} & \frac{\partial a_1}{\partial t_M} & \frac{\partial a_1}{\partial \alpha_1} & \cdots & \frac{\partial a_1}{\partial \alpha_L} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial a_N}{\partial \rho_1} & \frac{\partial a_N}{\partial t_1} & \cdots & \frac{\partial a_N}{\partial \rho_M} & \frac{\partial a_N}{\partial t_M} & \frac{\partial a_N}{\partial \alpha_1} & \cdots & \frac{\partial a_N}{\partial \alpha_L} \end{vmatrix}. \tag{3.29}$$

This proves Lemma 3.1. □

4. Calculation of the Jacobian: real case

We prove Lemma 2.2. It would be possible to cover a more general case, considering the number of zeros on the unit circle, the number in complex conjugate pairs located symmetrically with respect to the unit circle, and the number of real zeros. But our concern here is with the case that all zeros are on the unit circle so we only consider the case we require in this paper. We describe the odd degree case in detail and then discuss the modifications required for the even degree case.

We have a self-reciprocal polynomial $f(z)$ that has a root at -1 and at $\beta_1, \overline{\beta}_1, \dots, \beta_M, \overline{\beta}_M$ on the unit circle. Thus, $\overline{\beta}_j = 1/\beta_j$. The polynomial $f(z)$ has order $N = 2M + 1$:

$$\begin{aligned} f(z) &= (z+1)(z-\beta_1)(z-\overline{\beta}_1)\cdots(z-\beta_M)(z-\overline{\beta}_M) \\ &= z^N + a_1 z^{N-1} + a_2 z^{N-2} + \cdots + a_{N-2}z^2 + a_{N-1}z + 1. \end{aligned} \tag{4.1}$$

We have the functional equation

$$f(z) = z^N f\left(\frac{1}{z}\right), \tag{4.2}$$

which implies the symmetry of the coefficients

$$a_n = a_{N-n}. \tag{4.3}$$

So the polynomial is determined by $a_1, \dots, a_M$. We also have

$$a_n = (-1)^n e_n(-1, \beta_1, \overline{\beta}_1, \dots, \beta_M, \overline{\beta}_M), \tag{4.4}$$

where e_n is the nth elementary symmetric function.

We want the Jacobian of the transformation from the real variables $a_1, \dots, a_M$ to the real variables $t_1, \dots, t_M$, where $\beta_j = e^{it_j}$.

We start with

$$J_{\mathbb{R}} := \begin{vmatrix} \frac{\partial a_1}{\partial t_1} & \frac{\partial a_1}{\partial t_2} & \cdots & \frac{\partial a_1}{\partial t_M} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial a_M}{\partial t_1} & \frac{\partial a_M}{\partial t_2} & \cdots & \frac{\partial a_M}{\partial t_M} \end{vmatrix} = i^M \beta_1 \cdots \beta_M \begin{vmatrix} \frac{\partial a_1}{\partial \beta_1} & \frac{\partial a_1}{\partial \beta_2} & \cdots & \frac{\partial a_1}{\partial \beta_M} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial a_M}{\partial \beta_1} & \frac{\partial a_M}{\partial \beta_2} & \cdots & \frac{\partial a_M}{\partial \beta_M} \end{vmatrix}, \tag{4.5}$$

where we used the fact that

$$\frac{\partial a_j}{\partial t_k} = \frac{\partial a_j}{\partial \beta_k}\frac{d\beta_k}{dt_k} = i\beta_k \frac{\partial a_j}{\partial \beta_k}. \tag{4.6}$$

As in the previous case, for each m we have

$$a_n = (-1)^n \left(\left(\beta_m + \frac{1}{\beta_m}\right) e'_{n-1,m} + e'_{n-2,m} + e'_{n,m} \right), \tag{4.7}$$

where $e'_{n,m}$ is the nth symmetric function in -1 and all the β_j and $\overline{\beta}_j = \frac{1}{\beta_j}$, $j = 1, \dots, M$, except for β_m and $\frac{1}{\beta_m}$, with the convention that $e_0 = e'_{0,m} = 1$ and $e_n = e'_{n,m} = 0$ if n is greater than the number of variables. Thus, we obtain

$$\frac{\partial a_n}{\partial \beta_m} \quad = \quad (-1)^n \left(\left(1 - \frac{1}{\beta_m^2}\right) e'_{n-1,m} \right) \tag{4.8}$$

so our determinant is

$$J_{\mathbb{R}} = \varepsilon \prod_{m=1}^{M} \left(\beta_m - \frac{1}{\beta_m}\right) \begin{vmatrix} e'_{0,1} & e'_{0,2} & \cdots & e'_{0,M} \\ e'_{1,1} & e'_{1,2} & \cdots & e'_{1,M} \\ \vdots & \vdots & \ddots & \vdots \\ e'_{M-1,1} & e'_{M-1,2} & \cdots & e'_{M-1,M} \end{vmatrix}. \tag{4.9}$$

Here and following we use ε to denote a number with absolute value 1, which may be different at each occurrence.

Now we define the following polynomials:

$$\begin{aligned} f_\ell(x) &= (x+1) \prod_{\substack{m=1 \\ m\neq \ell}}^{M} (x-\beta_m)(x - \frac{1}{\beta_m}) = \sum_{n=0}^{N-2} (-1)^n e'_{n,\ell}\, x^{N-2-n} \\ &= \sum_{n=0}^{M-1} (-1)^n e'_{n,\ell} \, \left(x^{N-2-n} + x^n\right) \\ &= x^{M-\frac{1}{2}} \sum_{n=0}^{M-1} (-1)^n e'_{n,\ell} \, \left(x^{M-\frac{1}{2}-n} + x^{-M+\frac{1}{2}+n}\right), \end{aligned} \tag{4.10}$$

where the next-to-last step used the fact that $e'_{n,\ell} = -e'_{N-2-n,\ell}$, since the roots come in conjugate pairs except for the extra root at -1.

Note that $f_\ell(\beta_m) = 0$ if $\ell \neq m$. Also we have

$$f_\ell(e^{it}) = 2e^{i(M-\frac{1}{2})t} \sum_{n=0}^{M-1} (-1)^n e'_{n,\ell} \cos((M - \tfrac{1}{2} - n)t). \tag{4.11}$$

Therefore, with

$$V(t_1, \dots, t_M) = \tag{4.12}$$

$$\begin{vmatrix} \cos((M-\frac{1}{2})t_1) & -\cos((M-\frac{3}{2})t_1) & \cdots & (-1)^{M-2}\cos(\frac{3}{2}t_1) & (-1)^{M-1}\cos(\frac{1}{2}t_1) \\ \cos((M-\frac{1}{2})t_2) & -\cos((M-\frac{3}{2})t_2) & \cdots & (-1)^{M-2}\cos(\frac{3}{2}t_2) & (-1)^{M-1}\cos(\frac{1}{2}t_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \cos((M-\frac{1}{2})t_M) & -\cos((M-\frac{3}{2})t_M) & \cdots & (-1)^{M-2}\cos(\frac{3}{2}t_M) & (-1)^{M-1}\cos(\frac{1}{2}t_M) \end{vmatrix}$$

we have

$$V(t_1,\ldots,t_M)J_{\mathbb{R}} = C_M \begin{vmatrix} f_1(\beta_1) & 0 & \cdots & 0 \\ 0 & f_2(\beta_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_M(\beta_M) \end{vmatrix} \tag{4.13}$$

where

$$C_M = \varepsilon 2^{-M} \prod_{m=1}^{M} \left(\beta_m - \frac{1}{\beta_m}\right). \tag{4.14}$$

Since

$$2\sin(\tfrac{1}{2}t)\cos((M-\tfrac{1}{2}-n)t) = \sin((M-n)t - \sin((M-1-n)t), \tag{4.15}$$

by multiplying each row by $\sin(\frac{1}{2}t_n)$ we have that

$$V(t_1,\ldots t_M)\cdot 2^M \prod_n \sin(\tfrac{1}{2}t_n) \tag{4.16}$$

equals a determinant whose rows are

$$\big(\sin(Mt_j) - \sin((M-1)t_j) \quad \cdots \quad \sin(2t_j) - \sin(t_j) \quad \sin(t_j)\big). \tag{4.17}$$

By elementary column operations starting with the last column, this equals a determinant with rows

$$\big(\sin(Mt_j) \quad \sin((M-1)t_j) \quad \cdots \quad \sin(2t_j) \quad \sin(t_j)\big). \tag{4.18}$$

Since

$$\sin((n+1)t) = \sin(t)\left(2^n\cos^n(t) + \text{ lower order terms in } \cos(t)\right), \tag{4.19}$$

by elementary column operations we find that the determinant whose rows are (4.18) equals a determinant with rows

$$\sin(t_j)\times\big(2^{M-1}\cos^{M-1}(t_j) \quad 2^{M-2}\cos^{M-2}(t_j) \quad \cdots \quad 2\cos(t_j) \quad 1\big). \tag{4.20}$$

Since this last determinant is a Vandermonde, we have shown

$$\begin{aligned} V(t_1,\ldots t_M) &= 2^{M(M-3)/2}\prod_n \frac{\sin(t_n)}{\sin(\frac{1}{2}t_n)}\Delta(\cos(t_1),\ldots,\cos(t_M)) \\ &= 2^{M(M-1)/2}\prod_n \cos(\tfrac{1}{2}t_n)\Delta(\cos(t_1),\ldots,\cos(t_M)). \end{aligned} \tag{4.21}$$

Combining (4.13), (4.14), and (4.21) we have

$$J_{\mathbb{R}} = \varepsilon 2^{-M(M+1)/2}\frac{\prod_m(\beta_m - \overline{\beta}_m)\prod_m f_m(\beta_m)}{\prod_m \cos(\frac{1}{2}t_m)\Delta(\cos(t_1),\ldots,\cos(t_M))}. \tag{4.22}$$

Since,

$$\prod_m f_m(\beta_m) = \prod_m (\beta_m + 1) \prod_{k\neq m} (\beta_m - \beta_k)(\beta_m - \overline{\beta}_k)$$
$$= 2^{2M^2-M} \prod_m \cos(\tfrac{1}{2}t_m) \prod_{k\neq m} \sin\left(\frac{t_m - t_k}{2}\right) \sin\left(\frac{t_m + t_k}{2}\right), \tag{4.23}$$

and

$$\Delta(\cos t_1, \ldots, \cos t_M) = \prod_{j<k} (\cos t_k - \cos t_j)$$
$$= 2^{M(M-1)/2} \prod_{j<k} \sin\left(\frac{t_j - t_k}{2}\right) \sin\left(\frac{t_j + t_k}{2}\right), \tag{4.24}$$

and $\beta_m - \overline{\beta}_m = 2i\sin(t_m)$, we find that,

$$J_{\mathbb{R}} = \varepsilon\, 2^{M^2} \prod_m \sin(t_m) \prod_{j<k} \sin\left(\frac{t_k - t_j}{2}\right) \sin\left(\frac{t_k + t_j}{2}\right)$$
$$= \varepsilon \prod_m (\beta_m - \overline{\beta}_m) \prod_{j<k} (\beta_k - \beta_j)(\beta_k - \overline{\beta}_j), \tag{4.25}$$

as claimed. □

In the case of even degree, only a few modifications are needed. The polynomials (4.10) are replaced by

$$f_\ell(x) = \prod_{\substack{m=1\\ m\neq \ell}}^{M} (x - \beta_m)(x - \frac{1}{\beta_m})$$
$$= x^{M-1} \left(\sum_{n=0}^{M-2} (-1)^n e'_{n,\ell}\ \left(x^{M-1-n} + x^{-M+1+n}\right) + (-1)^{M-1} e'_{M-1,\ell} \right), \tag{4.26}$$

so

$$f_\ell(e^{it}) = \varepsilon \left(2 \sum_{n=0}^{M-2} (-1)^n e'_{n,\ell} \cos((M-1-n)t) + (-1)^{M-1} e'_{M-1,\ell} \right). \tag{4.27}$$

The determinant $V(t_1, \ldots, t_M)$ has entries $\cos((M-1-n)t_j)$, except for the last column, where $n = M-1$, which is multiplied by a factor of $\frac{1}{2}$. Since

$$\cos(nt) = 2^{n-1} \cos^n(t) + \text{ lower order terms in } \cos(t), \tag{4.28}$$

we recognize $V(t_1, \ldots, t_M)$ as a Vandermonde, which is 2^M times smaller than the Vandermonde which appeared in the odd degree case. The only other differences in the calculation is to omit the factor $2^M \prod \sin(t_n/2)$ from (4.16) and $\sin(t_j)$ from (4.20). The overall effect of those factors is to multiply the Jacobian by

$$\prod_n \frac{\sin(t_n)}{2\sin(\frac{t_n}{2})} = \prod_n \cos\left(\frac{t_n}{2}\right). \tag{4.29}$$

That factor replaces $\prod_m(\beta_m+1) = 2^M \prod_m \cos(\frac{1}{2}t_m)$ which is now omitted from $\prod_m f_m(\beta_m)$, the power of 2 making up for the factor of 2^M missing from $V(t_1,\ldots,t_M)$. So the end result is the exact same formula for the Jacobian.

References

[1] M.V. Berry and J.P. Keating, The Riemann zeros and eigenvalue asymptotics, *SIAM Rev.*, **41**(2):236–266, 1999.

[2] E. Bogomolny, O. Bohigas and P. Leboeuf, Quantum chaotic dynamics and random polynomials, *J. Stat. Phys.*, **85**:639–679, 1996

[3] J.B. Conrey and D.W. Farmer, Mean values of L-functions and symmetry, *Int. Math. Res. Notices*, **17**:883–908, 2000, arXiv:math.nt/9912107.

[4] J.B. Conrey, D.W. Farmer, J.P. Keating, M.O. Rubinstein, and N.C. Snaith, Integral moments of L-functions, *Proc. Lond. Math. Soc.*, **91**(1):33–104, 2005, arXiv:math.nt/0206018.

[5] J.E.A. Dunnage, The number of real zeros of a random trigonometric polynomial, *Proc. Lond. Math. Soc.* **16**(3):53–84, 1966

[6] K. Farahmand, *Topics in Random Polynomials*, Addison Wesley, London 1998

[7] D.W. Farmer and S. Lemurell, Maass forms and their L-functions, *preprint*, 2005, arXiv:math.NT/0506102.

[8] D.W. Farmer, E. Howe, and S. Koutsoliotas, Random polynomials, random matrices, and L-functions, I *in preparation.*

[9] D.A. Hejhal, On the triple correlation of zeros of the zeta function, *Inter. Math. Res. Notices*, **7**:293–302, 1994.

[10] C.P. Hughes, J.P. Keating, and N. O'Connell, Random matrix theory and the derivative of the Riemann zeta function, *Proc. R. Soc. Lond. A*, **456**:2611–2627, 2000.

[11] H. Iwaniec, W. Luo, and P. Sarnak, Low lying zeros of families of L-functions, *Inst. Hautes tudes Sci. Publ. Math.* No. 91 (2000), 55–131.

[12] N.M. Katz and P. Sarnak, *Random Matrices, Frobenius Eigenvalues and Monodromy*, American Mathematical Society Colloquium Publications, 45. American Mathematical Society, Providence, Rhode Island, 1999.

[13] N.M. Katz and P. Sarnak, Zeros of zeta functions and symmetry, *Bull. Amer. Math. Soc.*, **36**:1–26, 1999.

[14] J.P. Keating, The Riemann zeta function and quantum chaology, In *Quantum Chaos*; editors, G. Casati, I Guarneri, and U. Smilansky, pages 145–85. North-Holland, Amsterdam, 1993.

[15] J.P. Keating and N.C. Snaith, Random matrix theory and L-functions at $s = 1/2$, *Commun. Math. Phys*, **214**:91–110, 2000.

[16] J.P. Keating and N.C. Snaith, Random matrix theory and $\zeta(1/2+it)$, *Commun. Math. Phys.*, **214**:57–89, 2000.

[17] M.L. Mehta, *Random Matrices*, Academic Press, London, second edition, 1991.

[18] H.L. Montgomery, The pair correlation of the zeta function, *Proc. Symp. Pure Math*, **24**:181–93, 1973.

[19] A.M. Odlyzko, The 10^{20}th zero of the Riemann zeta function and 70 million of its neighbors, *Preprint*, 1989, http://www.dtc.umn.edu/ odlyzko/unpublished/index.html.

[20] K. Petersen and C. Sinclair *Conjugate reciprocal polynomials with all roots on the unit circle*, preprint, 2005.

[21] M. Rubinstein, *Evidence for a Spectral Interpretation of Zeros of L-functions*, PhD thesis, Princeton University, 1998.

[22] Z. Rudnick and P. Sarnak, Zeros of principal L-functions and random matrix theory, *Duke Mathematical Journal*, **81**(2):269–322, 1996.

[23] C. Sinclair, "Multiplicative Distance Functions", PhD thesis, The University of Texas at Austin, 2005.

[24] H. Weyl, *Classical Groups*, Princeton University Press, 1946.

American Institute of Mathematics
360 Portage Ave.
Palo Alto, CA 94306

USA
farmer@aimath.org

School of Mathematics
University of Bristol
Bristol BS8 1TW
United Kingdom
f.mezzadri@bristol.ac.uk
n.c.snaith@bristol.ac.uk