SUPPLEMENTARY MATERIAL FOR: TWISTED PARTICLE FILTERS

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Proofs for section 2.2.

Proof of Proposition 1. Under (**H2**) we have $\underline{g} := \inf_{\omega,x} G^{\omega}(x) > 0$. Now consider the sequence of random variables $\{\kappa_n^{\omega}; n \geq 1\}$ defined by

$$\kappa_n^{\omega} := \nu Q_{n-1}^{\omega}(1)\underline{g}\epsilon_-.$$

From (2.8) it is straightforward to establish by induction the following semi-group property: for any $\omega \in \Omega$, and $p, n \geq 0$,

$$Q_{p+n}^{\omega} = Q_p^{\omega} Q_n^{\theta^p \omega}.$$

Combined with (**H2**), this gives

$$\begin{aligned}
\kappa_{p+n}^{\omega} &= \nu Q_p^{\omega} Q_{n-1}^{\theta^p\omega}(1)\underline{g}\epsilon_- \\
&= \nu Q_{p-1}^{\omega} Q^{\theta^{p-1}\omega} Q_{n-1}^{\theta^p\omega}(1)\underline{g}\epsilon_- \\
&\geq \kappa_p^{\omega}\kappa_n^{\theta^p\omega},
\end{aligned}$$

and so

$$-\log \kappa_{p+n}^{\omega} \leq -\log \kappa_p^{\omega} - \log \kappa_n^{\theta^p\omega}.$$

Furthermore, under (**H2**) $\sup_{\omega,x} G^{\omega}(x) < \infty$, so there exists a finite constant c such that

$$\int_{\Omega} -\log \kappa_n^{\omega} \mathbb{P}(d\omega) \geq -cn$$

for any $n \geq 1$. These considerations, combined with (**H1**), allow application of Kingman's sub-additive ergodic theorem to establish that there exists a constant $\Lambda \in (-\infty, \infty)$ such that

$$\frac{1}{n}\log \kappa_n^{\omega} \longrightarrow \Lambda, \quad \text{as} \quad n \to \infty, \quad \mathbb{P} - a.s.$$

Under (**H2**), for any $\omega \in \Omega$ and $\mu \in \mathcal{P}(\mathsf{X})$,

$$\frac{\nu Q_n^{\omega}(1)}{\mu Q_n^{\omega}(1)} = \frac{\nu Q^{\omega} Q_{n-1}^{\theta\omega}(1)}{\mu Q^{\omega} Q_{n-1}^{\theta\omega}(1)} \leq \beta\frac{\epsilon_+}{\epsilon_-},$$

and combining this with a lower bound of a similar form we find

$$\sup_{\omega\in\Omega}\left|\frac{1}{n}\log\kappa_{n+1}^{\omega}-\frac{1}{n}\log\mu Q_n^{\omega}(1)\right|\leq\frac{1}{n}\log\left(\beta\frac{\epsilon_+}{\epsilon_-}\right)+\frac{1}{n}\left|\log(\epsilon_-\underline{g})\right|$$

so

$$\frac{1}{n}\log\mu Q_n^{\omega}(1)\longrightarrow\Lambda,\quad\text{as}\quad n\to\infty,\quad\mathbb{P}-a.s.$$

The proof is complete upon noting (2.9). □

Before presenting the proof of Proposition 2 it is opportune to observe that, for any $n\geq 1$

$$\Phi_n^{\omega}=\Phi_{n-1}^{\theta\omega}\circ\Phi^{\omega}. \tag{S.1}$$

The formula is validated by noticing that $\Phi_0^{\omega}=Id$, $\Phi_1^{\omega}=\Phi^{\omega}$, and when (S.1) holds at rank n, using the definition of Φ_{n+1}^{ω}, composing $\Phi^{\theta^n\omega}$ on the left of the objects appearing in (S.1) and then using the definition of $\Phi_n^{\theta\omega}$ gives the equalities:

$$\Phi_{n+1}^{\omega}=\Phi^{\theta^n\omega}\circ\Phi_n^{\omega}=\Phi^{\theta^n\omega}\circ\Phi_{n-1}^{\theta\omega}\circ\Phi^{\omega}=\Phi_n^{\theta\omega}\circ\Phi^{\omega},$$

so that the formula (S.1) holds at rank $n+1$. A simple inductive argument then shows that for any $n,m\geq 1$ and any $\omega\in\Omega$,

$$\Phi_{n+m}^{\omega}=\Phi_n^{\theta^m\omega}\circ\Phi_m^{\omega}. \tag{S.2}$$

PROOF OF PROPOSITION 2. Throughout the proof C is a finite constant whose value may change on each appearance.

We first address (2.13) and (2.15). Applying (S.2) with $\theta^{-n-m}\omega$ in place of ω gives

$$\Phi_{n+m}^{\theta^{-n-m}\omega}=\Phi_n^{\theta^{-n}\omega}\circ\Phi_m^{\theta^{-n-m}\omega},$$

so for any $\mu\in\mathcal{P}(\mathsf{X})$, taking $\mu'=\Phi_m^{\theta^{-n-m}\omega}(\mu)$ and applying (2.12) we obtain, for any $n,m\geq 1$,

$$\sup_{\omega\in\Omega}\sup_{\mu\in\mathcal{P}(\mathsf{X})}\left|\left[\Phi_n^{\theta^{-n}\omega}(\mu)-\Phi_{n+m}^{\theta^{-n-m}\omega}(\mu)\right](\varphi)\right|\leq\|\varphi\|\,C\rho^n. \tag{S.3}$$

Taking $\varphi=\mathbb{I}_A$ for any $A\in\mathcal{X}$, (S.3) shows that for some fixed μ and ω, $\left\{\Phi_n^{\theta^{-n}\omega}(\mu)(A);n\geq 0\right\}$ is a real-valued Cauchy sequence and together with [5, Theorem 1] this establishes the existence of $\eta^{\omega}\in\mathcal{P}(\mathsf{X})$ such that (2.13) holds. The lack of dependence of η^ω on μ follows by another application of (2.12). Moreover, taking $m \to \infty$ in (S.3) we obtain

$$\sup_{\omega\in\Omega}\sup_{\mu\in\mathcal{P}(\mathsf{X})}\left|\left[\Phi_n^{\theta^{-n}\omega}(\mu)-\eta^\omega\right](\varphi)\right| \leq \|\varphi\|\, C\rho^n, \tag{S.4}$$

and thus $\Phi_n^{\theta^{-n}\omega}(\mu)$ converges in total variation to η^ω, uniformly over μ and ω. This establishes (2.15).

We next address (2.14) and (2.16). We shall establish that

$$\sup_{(\omega,x)\in\Omega\times\mathsf{X}}\sup_{\mu\in\mathcal{P}(\mathsf{X})}\left|\frac{Q_{n+1}^\omega(1)(x)}{\Phi_{n+1}^{\theta^{-n-1}\omega}(\mu)Q_{n+1}^\omega(1)}-\frac{Q_n^\omega(1)(x)}{\Phi_n^{\theta^{-n}\omega}(\mu)Q_n^\omega(1)}\right| \leq C\rho^n. \tag{S.5}$$

First note that under (**H2**),

$$\widetilde{h}_{n,\mu}^\omega(x) := \frac{Q_n^\omega(1)(x)}{\Phi_n^{\theta^{-n}\omega}(\mu)Q_n^\omega(1)}$$

satisfies for any ω,ω',x,x',μ,

$$\sup_{n,m\geq 0}\frac{\widetilde{h}_{n+m,\mu}^\omega(x)}{\widetilde{h}_{n,\mu}^{\omega'}(x')} = \sup_{n,m\geq 0}\frac{Q_{n+m}^\omega(1)(x)}{\Phi_{n+m}^{\theta^{-n-m}\omega}(\mu)Q_{n+m}^\omega(1)}\frac{\Phi_n^{\theta^{-n}\omega'}(\mu)Q_n^{\omega'}(1)}{Q_n^{\omega'}(1)(x')} \leq \left(\beta\frac{\epsilon_+}{\epsilon_-}\right)^2. \tag{S.6}$$

We have

$$\begin{aligned}
&\left|\frac{Q_{n+1}^\omega(1)(x)}{\Phi_{n+1}^{\theta^{-n-1}\omega}(\mu)Q_{n+1}^\omega(1)}-\frac{Q_n^\omega(1)(x)}{\Phi_n^{\theta^{-n}\omega}(\mu)Q_n^\omega(1)}\right|\\
&=\left|\frac{Q_{n+1}^\omega(1)(x)\Phi_n^{\theta^{-n}\omega}(\mu)Q_n^\omega(1)-Q_n^\omega(1)(x)\Phi_{n+1}^{\theta^{-n-1}\omega}(\mu)Q_{n+1}^\omega(1)}{\Phi_{n+1}^{\theta^{-n-1}\omega}(\mu)Q_{n+1}^\omega(1)\Phi_n^{\theta^{-n}\omega}(\mu)Q_n^\omega(1)}\right|\\
&\leq\left|\frac{Q_{n+1}^\omega(1)(x)\Phi_n^{\theta^{-n}\omega}(\mu)Q_n^\omega(1)-Q_n^\omega(1)(x)\Phi_n^{\theta^{-n}\omega}(\mu)Q_{n+1}^\omega(1)}{\Phi_{n+1}^{\theta^{-n-1}\omega}(\mu)Q_{n+1}^\omega(1)\Phi_n^{\theta^{-n}\omega}(\mu)Q_n^\omega(1)}\right|\\
&\quad+\left|\frac{Q_n^\omega(1)(x)\Phi_n^{\theta^{-n}\omega}(\mu)Q_{n+1}^\omega(1)-Q_n^\omega(1)(x)\Phi_{n+1}^{\theta^{-n-1}\omega}(\mu)Q_{n+1}^\omega(1)}{\Phi_{n+1}^{\theta^{-n-1}\omega}(\mu)Q_{n+1}^\omega(1)\Phi_n^{\theta^{-n}\omega}(\mu)Q_n^\omega(1)}\right|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{Q_n^\omega(1)(x)}{\Phi_{n+1}^{\theta^{-n-1}\omega}(\mu)Q_{n+1}^\omega(1)} \\
&\quad \times \left| \frac{Q_{n+1}^\omega(1)(x)\Phi_n^{\theta^{-n}\omega}(\mu)Q_n^\omega(1) - Q_n^\omega(1)(x)\Phi_n^{\theta^{-n}\omega}(\mu)Q_{n+1}^\omega(1)}{Q_n^\omega(1)(x)\Phi_n^{\theta^{-n}\omega}(\mu)Q_n^\omega(1)} \right| \\
&\quad + \frac{Q_n^\omega(1)(x)}{\Phi_n^{\theta^{-n}\omega}(\mu)Q_n^\omega(1)} \left| \frac{\Phi_n^{\theta^{-n}\omega}(\mu)Q_{n+1}^\omega(1) - \Phi_{n+1}^{\theta^{-n-1}\omega}(\mu)Q_{n+1}^\omega(1)}{\Phi_{n+1}^{\theta^{-n-1}\omega}(\mu)Q_{n+1}^\omega(1)} \right| \\
&= \widetilde{h}_{n,\mu}^\omega(x) \frac{\Phi_n^{\theta^{-n}\omega}(\mu)Q_n^\omega(1)}{\Phi_{n+1}^{\theta^{-n-1}\omega}(\mu)Q_{n+1}^\omega(1)} \left| \frac{Q_{n+1}^\omega(1)(x)}{Q_n^\omega(1)(x)} - \frac{\Phi_n^{\theta^{-n}\omega}(\mu)Q_{n+1}^\omega(1)}{\Phi_n^{\theta^{-n}\omega}(\mu)Q_n^\omega(1)} \right| \\
&\quad + \widetilde{h}_{n,\mu}^\omega(x) \left| \frac{\Phi_n^{\theta^{-n}\omega}(\mu)Q_{n+1}^\omega(1)}{\Phi_{n+1}^{\theta^{-n-1}\omega}(\mu)Q_{n+1}^\omega(1)} - \frac{\Phi_{n+1}^{\theta^{-n-1}\omega}(\mu)Q_{n+1}^\omega(1)}{\Phi_{n+1}^{\theta^{-n-1}\omega}(\mu)Q_{n+1}^\omega(1)} \right| \\
&= \widetilde{h}_{n,\mu}^\omega(x) \frac{\Phi_n^{\theta^{-n}\omega}(\mu)Q_n^\omega(1)}{\Phi_{n+1}^{\theta^{-n-1}\omega}(\mu)Q_{n+1}^\omega(1)} \left| \left[\Phi_n^\omega(\delta_x) - \left(\Phi_n^\omega \circ \Phi_n^{\theta^{-n}\omega} \right)(\mu) \right] Q^{\theta^n\omega}(1) \right| \\
&\quad + \widetilde{h}_{n,\mu}^\omega(x) \left| \left[\Phi_n^{\theta^{-n}\omega}(\mu) - \left(\Phi_n^{\theta^{-n}\omega} \circ \Phi^{\theta^{-n-1}\omega} \right)(\mu) \right] (\widetilde{h}_{n+1,\mu}^\omega) \right| \\
&\leq C\rho^n, \qquad \forall x, \omega, \mu,
\end{aligned}
$$

where (S.6), **(H2)** and (2.12) have been applied. Thus (S.5) is proved and then for any $m, n \geq 1$,

$$
\sup_{\omega,x,\mu} \left| \widetilde{h}_{n,\mu}^\omega(x) - \widetilde{h}_{n+m,\mu}^\omega(x) \right| \leq C \sum_{q=0}^{m-1} \rho^{n+q} \leq \rho^n \frac{C}{1-\rho}. \tag{S.7}
$$

Then $\left\{ \widetilde{h}_{n,\mu}^\omega(x); n \geq 1 \right\}$ is Cauchy and real-valued, which is enough to establish the existence of a pointwise limit $h_\mu : \Omega \times \mathsf{X} \to \mathbb{R}_+$. Taking $m \to \infty$ in (S.7) yields

$$
\sup_{\omega,x,\mu} \left| \widetilde{h}_{n,\mu}^\omega(x) - h_\mu^\omega(x) \right| \leq C\rho^n. \tag{S.8}
$$

For the lack of dependence of h_μ on μ, consider some $\mu' \in \mathcal{P}(\mathsf{X})$ possibly different from μ, and let $h_{\mu'}$ be the pointwise limit of $\left\{ \widetilde{h}_{n,\mu'}^\omega(x); n \geq 1 \right\}$. Then for any $n \geq 1$

$$
\begin{aligned}
\left| h_\mu^\omega(x) - h_{\mu'}^\omega(x) \right| &\leq \left| \widetilde{h}_{n,\mu}^\omega(x) - h_\mu^\omega(x) \right| \\
&\quad + \left| \widetilde{h}_{n,\mu'}^\omega(x) - h_{\mu'}^\omega(x) \right| \\
&\quad + \left| \widetilde{h}_{n,\mu}^\omega(x) - \widetilde{h}_{n,\mu'}^\omega(x) \right|. \qquad \text{(S.9)}
\end{aligned}
$$

Due to (S.8), the first two terms on the r.h.s. of (S.9) can be made arbitrarily small by taking n large enough. For the third term,

$$\begin{aligned} &\left|\widetilde{h}^{\omega}_{n,\mu}(x) - \widetilde{h}^{\omega}_{n,\mu'}(x)\right| \\ &= \widetilde{h}^{\omega}_{n,\mu}(x)\left|\left[\Phi^{\theta^{-n}\omega}_{n}(\mu') - \Phi^{\theta^{-n}\omega}_{n}(\mu)\right](\widetilde{h}^{\omega}_{n,\mu'})\right|. \end{aligned} \tag{S.10}$$

Due to (2.12), (S.6) and (S.8), the term in (S.10) can be made arbitrarily small by taking n large enough. Thus the l.h.s. of (S.9) must be zero, hence $h := h_\mu$ is independent of μ, and (2.16) is proved. The measurability of h stated in part 1) holds as it is the point-wise limit of a sequence of $\mathcal{F} \otimes \mathcal{X}$-measurable functions.

Turning now to prove part 3) of the Proposition, note that by (S.4), $\lambda_\omega = \eta^{\omega}(G^{\omega}) = \lim_{n\to\infty} \Phi^{\theta^{-n}\omega}_{n}(\mu)(G^{\omega})$, i.e. λ is the point-wise limit of a sequence of measurable functions, and is therefore measurable. The λ part of (2.17) holds immediately under **(H2),** and the h part holds due to (S.6), a similar lower bound, and (S.8).

We now turn to the proof of part 4), firstly establishing that the triple (η, h, λ) does indeed satisfy (2.18). For the measure equation, we have

$$\frac{\Phi^{\theta^{-n}\omega}_{n}(\mu)Q^{\omega}(A)}{\Phi^{\theta^{-n}\omega}_{n}(\mu)(G^{\omega})} = \Phi^{\omega}(\Phi^{\theta^{-n}\omega}_{n}(\mu))(A) = \Phi^{\theta^{-n-1}\theta\omega}_{n+1}(\mu)(A). \tag{S.11}$$

By the strong convergence in (S.4), and the fact that under **(H2)**, we have the bound $\sup_x Q^{\omega}(A)(x) \leq \sup_x G^{\omega}(x) < \infty$, the left hand side of (S.11) converges to $\eta^{\omega}Q^{\omega}(A)/\lambda_\omega$ and the right hand side of (S.11) converges to $\eta^{\theta\omega}(A)$. Thus η satisfies the first equation in (2.18).

For the second equation in (2.18), choose $n \geq 1$ arbitrarily and notice that

$$
\begin{aligned}
(\text{S.12}) \quad & Q^{\omega}\left(\widetilde{h}^{\theta\omega}_{n,\mu}\right)(x) \\
&= \frac{Q^{\omega}Q^{\theta\omega}_{n}(1)(x)}{\Phi^{\theta^{-n+1}\omega}_{n}(\mu)Q^{\theta\omega}_{n}(1)} \\
&= \frac{Q^{\omega}_{n+1}(1)(x)}{\Phi^{\theta^{-n+1}\omega}_{n}(\mu)Q^{\theta\omega}_{n}(1)} \\
&= \widetilde{h}^{\omega}_{n+1,\mu}(x)\frac{\Phi^{\theta^{-n-1}\omega}_{n+1}(\mu)Q^{\omega}_{n+1}(1)}{\Phi^{\theta^{-n+1}\omega}_{n}(\mu)Q^{\theta\omega}_{n}(1)} \\
&= \widetilde{h}^{\omega}_{n+1,\mu}(x)\frac{\Phi^{\theta^{-n-1}\omega}_{n+1}(\mu)Q^{\omega}_{n+1}(1)}{\Phi^{\theta^{-n+1}\omega}_{n-1}(\mu)Q^{\omega}Q^{\theta\omega}_{n}(1)}.\Phi^{\theta^{-n+1}\omega}_{n-1}(\mu)Q^{\omega}(1) \\
&= \widetilde{h}^{\omega}_{n+1,\mu}(x)\frac{\Phi^{\theta^{-n+1}\omega}_{n-1}\left[\Phi^{\theta^{-n-1}\omega}_{2}(\mu)\right]Q^{\omega}_{n+1}(1)}{\Phi^{\theta^{-n+1}\omega}_{n-1}(\mu)Q^{\omega}_{n+1}(1)}.\Phi^{\theta^{-n+1}\omega}_{n-1}(\mu)Q^{\omega}(1) \\
(\text{S.13}) \quad & \longrightarrow h^{\omega}(x)\cdot 1\cdot\lambda_{\omega}, \quad \text{as} \quad n\to\infty.
\end{aligned}
$$

where the convergence in (S.13) is due to (S.8); (2.12) applied in conjunction with the property

$$\sup_{n\geq 1}\sup_{x,x'}\frac{Q^{\omega}_{n+1}(1)(x)}{Q^{\omega}_{n+1}(1)(x')} < \infty,$$

which holds under (**H2**); and (2.15). Furthermore the l.h.s. of (S.12) converges to $Q^{\omega}(h^{\theta\omega})(x)$, because by (S.8), for any $x \in \mathsf{X}$,

$$\left|Q^{\omega}(h^{\theta\omega} - \widetilde{h}^{\theta\omega}_{n,\mu})(x)\right| \leq G^{\omega}(x)\sup_{z}\left|h^{\theta\omega}(z) - \widetilde{h}^{\theta\omega}_{n,\mu}(z)\right| \to 0, \quad \text{as} \quad n\to\infty.$$

This verifies that the second equation of (2.18) is satisfied. For the third equation, we have, for any $n \geq 1$,

$$
\begin{aligned}
|\eta^{\omega}(h^{\omega}) - 1| &= \left|\eta^{\omega}(h^{\omega}) - \Phi^{\theta^{-n}\omega}_{n}(\mu)(\widetilde{h}^{\omega}_{n,\mu})\right| \\
&\leq \left|\left[\eta^{\omega} - \Phi^{\theta^{-n}\omega}_{n}(\mu)\right](h^{\omega})\right| + \sup_{x}\left|h^{\omega}(x) - \widetilde{h}^{\omega}_{n,\mu}(x)\right|
\end{aligned}
$$

and since we have already proved part 3), which implies $\sup_x h^{\omega}(x) < \infty$, the convergence in (S.4) and (S.8) show it must be the case that $\eta^{\omega}(h^{\omega}) = 1$.

Now for the uniqueness element of part 4). Suppose that there exists a triple $\left(\bar{\eta}, \bar{h}, \bar{\lambda}\right)$ of the desired nature and such that

$$(\text{S.14}) \quad \bar{\eta}^{\omega}Q^{\omega} = \bar{\lambda}_{\omega}\bar{\eta}^{\theta\omega}, \quad Q^{\omega}(\bar{h}^{\theta\omega}) = \bar{\lambda}_{\omega}\bar{h}^{\omega}, \quad \bar{\eta}^{\omega}(\bar{h}^{\omega}) = 1, \quad \text{for all } \omega\in\Omega.$$

Then integrating the first equation in (S.14), we have $\bar{\lambda}_\omega = \bar{\eta}_\omega(G^\omega)$, because $\bar{\eta}_\omega$ is, by hypothesis, a probability measure. Thus $\Phi^\omega(\bar{\eta}^\omega) = \bar{\eta}^{\theta\omega}$ for any ω, so via iteration we find $\Phi_n^{\theta^{-n}\omega}(\bar{\eta}^{\theta^{-n}\omega}) = \bar{\eta}^\omega$, and strong convergence of (S.4) then demands that $\bar{\eta}^\omega = \eta^\omega$. Thus $\bar{\eta} = \eta$ and therefore $\bar{\lambda} = \lambda$. It remains to show that $\bar{h} = h$. To this end, first note that for any $\varphi_n \in \mathcal{L}(\mathsf{X})$, with $\sup_n \|\varphi_n\| < \infty$,

$$
\begin{aligned}
\left|\frac{Q_n^\omega(\varphi_n)(x)}{\prod_{p=0}^{n-1}\lambda_{\theta^p\omega}} - h^\omega(x)\eta^{\theta^n\omega}(\varphi_n)\right| &\leq \left|\frac{Q_n^\omega(\varphi_n)(x)}{Q_n^\omega(1)(x)} - \eta^{\theta^n\omega}(\varphi_n)\right|\beta\frac{\epsilon_+}{\epsilon_-} \\
&\quad + \|\varphi_n\|\left|\frac{Q_n^\omega(1)(x)}{\Phi_n^{\theta^{-n}\omega}(\eta^{\theta^{-n}\omega})Q_n^\omega(1)} - h_\omega(x)\right| \\
&\rightarrow 0, \quad \text{as} \quad n \rightarrow \infty
\end{aligned} \tag{S.15}
$$

where the identity $\prod_{p=0}^{n-1}\lambda_{\theta^p\omega} = \eta^\omega Q_n^\omega(1) = \Phi_n^{\theta^{-n}\omega}(\eta^{\theta^{-n}\omega})Q_n^\omega(1)$, which holds due to the measure equation in (2.18), then **(H2)**, and then (S.4) and (S.8) have been applied. But, under the hypotheses that $Q^\omega(\bar{h}^{\theta\omega}) = \bar{\lambda}_\omega\bar{h}^\omega$ and $\bar{\eta}^\omega(\bar{h}^\omega) = 1$ for all ω, using the already proved $\bar{\eta} = \eta$, we have the equality

$$
\frac{Q_n^\omega(\bar{h}^{\theta^n\omega})(x)}{\prod_{p=0}^{n-1}\lambda_{\theta^p\omega}} - h^\omega(x)\eta^{\theta^n\omega}(\bar{h}^{\theta^n\omega}) = \bar{h}^\omega(x) - h^\omega(x),
$$

so taking $\varphi_n = \bar{h}^{\theta^n\omega}/\bar{\eta}^{\theta^n\omega}(\bar{h}^{\theta^n\omega})$ in (S.15), and noting that under **(H2)**,

$$
\frac{\|\bar{h}^\omega\|}{\bar{\eta}^\omega(\bar{h}^\omega)} \leq \sup_{\omega,x,x'}\frac{\bar{h}^\omega(x)}{\bar{h}^\omega(x')} = \sup_{\omega,x,x'}\frac{Q\left(\bar{h}^{\theta^{-1}\omega}\right)(x)}{Q\left(\bar{h}^{\theta^{-1}\omega}\right)(x')} \leq \beta\frac{\epsilon_+}{\epsilon_-},
$$

we find $\bar{h}^\omega(x) = h^\omega(x)$. This completes the proof of part 4), and therefore the proposition. $\square$

Proof of Proposition 3.

$$
\begin{aligned}
\mu Q_n^\omega(1) &= \mu Q_0^\omega(1)\prod_{p=1}^{n}\frac{\mu Q_p^\omega(1)}{\mu Q_{p-1}^\omega(1)} \\
&= 1 \cdot \prod_{p=1}^{n}\frac{\mu Q_{p-1}^\omega Q^{\theta^{p-1}\omega}(1)}{\mu Q_{p-1}^\omega(1)} \\
&= \prod_{p=0}^{n-1}\Phi_p^\omega(\mu)Q^{\theta^p\omega}(1) = \prod_{p=0}^{n-1}\Phi_p^\omega(\mu)(G^{\theta^p\omega}),
\end{aligned}
$$

and so

$$\begin{aligned}\frac{1}{n}\log \mu Q_n^{\omega}(1) &= \frac{1}{n}\sum_{p=0}^{n-1}\log \lambda_{\theta^p\omega} \\ &+\frac{1}{n}\sum_{p=0}^{n-1}\left[\log\left[\Phi_p^{\omega}(\mu)(G^{\theta^p\omega})\right]-\log\lambda_{\theta^p\omega}\right].\end{aligned} \tag{S.16}$$

Now the λ part of (2.17) ensures that $\mathbb{E}\left[|\log\lambda|\right]<\infty$, so by the ergodic theorem, the first term on the right of (S.16) converges:

$$\frac{1}{n}\sum_{p=0}^{n-1}\log\lambda_{\theta^p\omega}\to\mathbb{E}\left[\log\lambda\right],\quad \mathbb{P}-a.s.$$

For the other term, replacing ω with $\theta^n\omega$ in (2.15) gives

$$\left|\Phi_n^{\omega}(\mu)(G^{\theta^n\omega})-\lambda_{\theta^n\omega}\right|\to 0,\quad \text{as}\quad n\to\infty,$$

and under **(H2),** $0<\inf_{\omega,x}G^{\omega}(x)\leq\sup_{\omega,x}G^{\omega}(x)<\infty$, so

$$\left|\log\left[\Phi_n^{\omega}(\mu)(G^{\theta^n\omega})\right]-\log\lambda_{\theta^n\omega}\right|\to 0,\quad \text{as}\quad n\to\infty.$$

The second term on the right of (S.16) converges to zero by Cesaro averaging. The proof is complete upon recalling from Proposition 1 that $n^{-1}\log\mu Q_n^{\omega}(1)\to\Lambda$, $\mathbb{P}-a.s.$, and noting that by Proposition 2, $Q^{\omega}(h^{\theta\omega})=\lambda_{\omega}h^{\omega}$. □

REMARK 8. In the case that Y consists of a single point, $|\mathsf{Y}|=1$, then $|\Omega|=1$, and (**H1**) holds automatically. In this situation, dropping ω from the notation, (η,h,λ) are the Perron-Frobenius eigen-measure/function/value of Q, and $\lambda=e^{\Lambda}$. The twisted Markov kernel $Q(x,dx')h(x')/\lambda h(x)=M(x,dx')h(x')/M(h)(x)$ is of interest when importance sampling for certain Markov chain rare events [1]. A time-homogeneous Markov chain with this transition is known as the *h-process* associated with *Q:* for a discussion of this and related matters, see [3, Section 2.7.1], references therein and Remark 10, below.

Proofs for section 2.4. Throughout this section we assume $N\geq 1$ is fixed arbitrarily.

For $\widetilde{\mathbf{M}}\in\mathbb{M}$, define

$$\widetilde{\mathbf{R}}(\omega,x,dx'):=\mathbf{G}(\omega,x)^2\phi^{\omega}(x,x')^2\widetilde{\mathbf{M}}(\omega,x,dx'). \tag{S.17}$$

The proofs of Proposition 4 and Theorem 1 involve a generalized eigenvalue and eigen-function for $\widetilde{\mathbf{R}}$, and our first objective is to verify that such quantities exist. In order to do so, we now check that when (**H2**) holds, $\widetilde{\mathbf{R}}$ satisfies a regularity condition of a similar form. Define

$$\mathbf{J}(\omega,x) := \int_{\mathsf{X}^N} \mathbf{G}(\omega,x)^2 \phi^\omega(x,x')^2 \widetilde{\mathbf{M}}(\omega,x,dx')$$

and

$$\mathbf{L}(\omega,x,dx') := \frac{\mathbf{G}(\omega,x)^2 \phi^\omega(x,x')^2 \widetilde{\mathbf{M}}(\omega,x,dx')}{\mathbf{J}(\omega,x)},$$

so clearly

$$\widetilde{\mathbf{R}}(\omega,x,dx') = \mathbf{J}(\omega,x)\mathbf{L}(\omega,x,dx').$$

LEMMA 5. *Assume (**H2**), let $\widetilde{\mathbf{M}}$ be any member of $\mathbb{M}$ and let $\widetilde{\nu}$ be the accompanying measure in (2.26). Then there exist constants $\alpha \in [1,\infty)$, $(\delta_-,\delta_+) \in (0,\infty)^2$ and $\mu \in \mathcal{P}(\mathsf{X}^N)$, such that*

$$\frac{\mathbf{J}(\omega,x)}{\mathbf{J}(\omega',x')} \le \alpha, \quad \forall \left(\omega,\omega',x,x'\right) \in \Omega^2 \times \mathsf{X}^{2N}, \tag{S.18}$$

$$\mu(dx) \propto \left[\frac{d\nu^{\otimes N}}{d\widetilde{\nu}}(x)\right]^2 \widetilde{\nu}(dx) \tag{S.19}$$

and

$$\delta_-\mu(\cdot) \le \mathbf{L}(\omega,x,\cdot) \le \delta_+\mu(\cdot), \quad \forall\,(\omega,x) \in \Omega \times \mathsf{X}^N. \tag{S.20}$$

PROOF. For any $A \in \mathcal{X}^{\otimes N}$,

$$\begin{aligned}
&\int_A \phi^\omega(x,x')^2 \widetilde{\mathbf{M}}^\omega(x,dx') \\
&= \int_A \left[\frac{d\mathbf{M}^\omega(x,\cdot)}{d\nu^{\otimes N}}(x')\frac{d\nu^{\otimes N}}{d\widetilde{\nu}}(x')\frac{d\widetilde{\nu}}{d\widetilde{\mathbf{M}}^\omega(x,\cdot)}(x')\right]^2 \widetilde{\mathbf{M}}^\omega(x,dx') \\
&\le \frac{\epsilon_+^{2N}\widetilde{\epsilon}_+}{\widetilde{\epsilon}_-^2}\int_A \left[\frac{d\nu^{\otimes N}}{d\widetilde{\nu}}(x')\right]^2 \widetilde{\nu}(dx') < \infty, \qquad \text{(S.21)}
\end{aligned}$$

where Lemma1, (**H2**) and the definition of $\mathbb{M}$ have been used. By a similar argument,

$$\begin{aligned}
&\int_A \phi^\omega(x,x')^2 \widetilde{\mathbf{M}}^\omega(x,dx') \\
&\ge \frac{\epsilon_-^{2N}\widetilde{\epsilon}_-}{\widetilde{\epsilon}_+^2}\int_A \left[\frac{d\nu^{\otimes N}}{d\widetilde{\nu}}(x')\right]^2 \widetilde{\nu}(dx'). \qquad \text{(S.22)}
\end{aligned}$$

By Lemma 1 and taking $A = \mathsf{X}^N$ in (S.21) and (S.22), the bound of (S.18) holds with

$$\alpha = \beta^2 \frac{\epsilon_+^{2N}}{\epsilon_-^{2N}} \frac{\widetilde{\epsilon}_+^3}{\widetilde{\epsilon}_-^3}.$$

The bound of (S.20) holds with $\delta_- = \left(\epsilon_-^{2N}\widetilde{\epsilon}_-^3\right) / \left(\epsilon_+^{2N}\widetilde{\epsilon}_+^3\right)$ and $\delta_+ = 1/\delta_-$. ☐

REMARK 9. Having established the regularity properties (S.18)-(S.20), we notice that $\widetilde{\mathbf{R}}$ will have properties which are exactly similar to those properties of Q established in Propositions 1, 2 and 3, which we shall now summarize (we will not write a proof explicitly, since the arguments follow precisely the same programme as in the proofs of the afore-mentioned Propositions). We shall write

$$\widetilde{\mathbf{R}}_0^\omega := Id, \qquad \widetilde{\mathbf{R}}_n^\omega := \widetilde{\mathbf{R}}_{n-1}^\omega \widetilde{\mathbf{R}}^{\theta^{n-1}\omega}, \quad n \geq 1.$$

For any $N \geq 1$ and $\widetilde{\mathbf{M}} \in \mathbb{M}$,

- there exists a constant $\Xi_N \in (-\infty, \infty)$ such that for any $\mu \in \mathcal{P}(\mathsf{X}^N)$,

$$\frac{1}{n} \log \mu \widetilde{\mathbf{R}}_n^\omega(1) \to \Xi_N, \quad \text{as } n \to \infty, \quad \text{for} \quad \mathbb{P} - a.a.\ \omega. \tag{S.23}$$

- there exists a random variable $\xi : \Omega \to \mathbb{R}_+$ (depending on N) and a function $\ell : \Omega \times \mathsf{X}^N \to \mathbb{R}_+$, measurable w.r.t. $\mathcal{F} \otimes \mathcal{X}^{\otimes N}$, such that

$$\sup_{\omega,\omega'} \frac{\xi_\omega}{\xi_{\omega'}} < \infty, \qquad \sup_{\omega,\omega',x,x'} \frac{\ell(\omega,x)}{\ell(\omega,x')} < \infty, \tag{S.24}$$

  and

$$\widetilde{\mathbf{R}}^\omega \left(\ell^{\theta\omega}\right) = \xi_\omega \ell^\omega \tag{S.25}$$

- for any $x \in \mathsf{X}^N$,

$$\Xi_N = \mathbb{E}\left[\log \xi\right] = \int_\Omega \log \frac{\widetilde{\mathbf{R}}^\omega \left(\ell^{\theta\omega}\right)(x)}{\ell^\omega(x)} \mathbb{P}(d\omega). \tag{S.26}$$

Note that in the above displays, the dependence of various quantities on $\widetilde{\mathbf{M}}$ is suppressed from the notation.

We can now deal with the proof of Proposition 4 and then a collection of Lemmas which prove Theorem 1.

Proof of Proposition 4. By Proposition 1, for any $\mu_0 \in \mathcal{P}(\mathsf{X})$,

$$\frac{2}{n}\log \mu_0 Q_n^{\omega}(1) \quad \rightarrow \quad 2\Lambda, \quad \text{for } \mathbb{P}-a.a.\ \omega,$$

and by (S.23),

$$\frac{1}{n}\log \mu_0^{\otimes N}\widetilde{\mathbf{R}}_n^{\omega}(1) \rightarrow \Xi_N, \quad \text{for } \mathbb{P}-a.a.\ \omega. \tag{S.27}$$

Then as

$$\widetilde{\mathcal{V}}_{n,N}^{\omega} = \frac{\mu_0^{\otimes N}\widetilde{\mathbf{R}}_n^{\omega}(1)}{\left[\mu_0 Q_n^{\omega}(1)\right]^2},$$

the proof is complete, with $\Upsilon_N(\widetilde{\mathbf{M}}) = \Xi_N - 2\Lambda$. $\square$

Remark 10. Further to Remark 8, if $|\mathsf{Y}| = 1$ and additionally $N = 1$, then we have (again dropping ω from the notation)

$$\frac{\mathbf{M}(x,dx')\mathbf{h}(x')}{\mathbf{M}\left(\mathbf{h}\right)(x)} = \frac{M(x,dx')h(x')}{M\left(h\right)(x)},$$

so that the Markov kernel addressed in Theorem 1 is exactly that of the h-process associated with Q.

The proof of Theorem 1 is now given in Lemmas 6-8. The first can be viewed as generalizing the necessity part of the proof of [1, Theorem 3] to the case of non-negative kernels driven by an ergodic shift.

Lemma 6. *[1)⇒2)] If $\Upsilon_N(\widetilde{\mathbf{M}}) = 0$, then for $\mathbb{P}$-almost all $\omega \in \Omega$ there exists $A_\omega \in \mathcal{X}^{\otimes N}$ such that $\nu^{\otimes N}(A_\omega^c) = 0$ and for any $x \in A_\omega$,*

$$\widetilde{\mathbf{M}}^{\omega}(x,B) = \frac{\int_B \mathbf{M}^{\omega}(x,dx')\mathbf{h}^{\theta\omega}(x')}{\int_{\mathsf{X}^N}\mathbf{M}^{\omega}(x,dz)\mathbf{h}^{\theta\omega}(z)}, \quad \textit{for all } B \in \mathcal{X}^{\otimes N}.$$

Proof. We need to introduce a notational convention before proceeding with the main body of the proof. For any $\varphi : \Omega \times \mathsf{X}^N \rightarrow \mathbb{R}$ a function measurable w.r.t. $\mathcal{F} \otimes \mathcal{X}^{\otimes N}$, let the $\mathbb{P}$-essential supremum of the collection of functions $\left\{\varphi(\cdot,x); x \in \mathsf{X}^N\right\}$ (in the sense of [4, V, 18.]) be χ, i.e. χ is a random variable on $(\Omega,\mathcal{F},\mathbb{P})$. In a slight abuse of our ω-section notation we shall write, for any ω in Ω,

$$\text{ess sup}_x \varphi^{\omega}(x) := \chi_\omega. \tag{S.28}$$

Let $\gamma(\omega,x) = \mathbf{h}(\omega,x)^2/\ell(\omega,x)$, and ρ be the $\mathbb{P}$-essential supremum of the collection of functions $\left\{\gamma(\cdot,x)\,;\,x\in\mathsf{X}^N\right\}$. Then in accordance with the convention (S.28) we shall write:

$$\operatorname{ess\,sup}_x \frac{\mathbf{h}^\omega(x)^2}{\ell^\omega(x)} := \rho_\omega.$$

From (S.24), the definition of $\mathbf{h}$ in (S.24) and (2.17),

$$\sup_{\omega,\omega',x,x'} \frac{\gamma(\omega,x)}{\gamma(\omega',x')} = \sup_{\omega,\omega',x,x'} \frac{\mathbf{h}^\omega(x)^2}{\ell^\omega(x)} \frac{\ell^{\omega'}(x')}{\mathbf{h}^{\omega'}(x')^2} < \infty \tag{S.29}$$

and therefore, at least up to a set of $\mathbb{P}$-measure zero, ρ_ω is uniformly bounded above and below away from zero in ω. This observation, the bound (S.29) and Lemma 5 will ensure that various expectations appearing below are finite.

We now proceed with the proof. Since by Proposition 2 and (S.24), λ and ξ are uniformly bounded above and below away from zero in ω, we may write $\Upsilon_N(\widetilde{\mathbf{M}}) = \Xi_N - 2\Lambda = \mathbb{E}\left[\log\frac{\xi}{\lambda^2}\right]$ (where the first equality is as in the above proof of Proposition *4*). We are going to prove that the condition $\mathbb{E}\left[\log\frac{\xi}{\lambda^2}\right] = 0$ implies that for $\mathbb{P}$-almost all $\omega\in\Omega$, there exists $A_\omega \in \mathcal{X}^{\otimes N}$ such that $\nu^{\otimes N}(A_\omega^c) = 0$ and for any $x\in A_\omega$,

$$\widetilde{\mathbf{M}}^\omega(x,B) = \frac{\int_B \mathbf{M}^\omega(x,dx')\mathbf{h}^{\theta\omega}(x')}{\int_{\mathsf{X}^N} \mathbf{M}^\omega(x,dz)\mathbf{h}^{\theta\omega}(z)}, \quad \text{for all } B\in\mathcal{X}^{\otimes N}. \tag{S.30}$$

For any $x\in\mathsf{X}^N$,

$$\begin{aligned}
\xi_\omega &= \frac{\widetilde{\mathbf{R}}^\omega(\ell^{\theta\omega})(x)}{\ell^\omega(x)}\\
&\geq \frac{1}{\rho_{\theta\omega}\ell^\omega(x)}\int \mathbf{G}^\omega(x)^2\phi^\omega(x,x')^2\widetilde{\mathbf{M}}^\omega(x,dx')\mathbf{h}^{\theta\omega}(x')^2\\
&\geq \frac{1}{\rho_{\theta\omega}\ell^\omega(x)}\left[\int \mathbf{G}^\omega(x)\phi^\omega(x,x')\widetilde{\mathbf{M}}^\omega(x,dx')\mathbf{h}^{\theta\omega}(x')\right]^2 &&\text{(S.31)}\\
&= \frac{1}{\rho_{\theta\omega}\ell^\omega(x)}\left[\int \mathbf{Q}^\omega(x,dx')\mathbf{h}^{\theta\omega}(x')\right]^2\\
&= \frac{\mathbf{h}^\omega(x)^2}{\rho_{\theta\omega}\ell^\omega(x)}\lambda_\omega^2, \qquad \mathbb{P}-a.s., &&\text{(S.32)}
\end{aligned}$$

where the first equality is just (S.25), the inequality (S.31) is due to Jensen's inequality and the equality in (S.32) holds due to Lemma 2. Here and similarly elsewhere in the proof, the existence of the set of full $\mathbb{P}$-measure on which the equations hold is deduced using the definition of the essential supremum of a collection of functions, Tonelli's theorem and the fact that $\widetilde{\mathbf{R}}^{\omega}(x,\cdot)$ is equivalent to the probability measure μ in Lemma 5. The following inequalities then hold for $\mathbb{P}$-almost all ω:

$$
\begin{aligned}
&\xi_\omega \\
&\geq \frac{1}{\rho_{\theta\omega}} \operatorname{ess\,sup}_x \frac{1}{\ell^\omega(x)} \int \mathbf{G}^\omega(x)^2 \phi^\omega(x,x')^2 \widetilde{\mathbf{M}}^\omega(x,dx') \mathbf{h}^{\theta\omega}(x')^2 && \text{(S.33)}\\
&\geq \frac{\lambda_\omega^2}{\rho_{\theta\omega}} \operatorname{ess\,sup}_x \frac{\mathbf{h}^\omega(x)^2}{\ell^\omega(x)}. && \text{(S.34)}
\end{aligned}
$$

We then have

$$
\begin{aligned}
&\mathbb{E}\left[\log \xi\right] \\
&\geq \int_\Omega \log\left[\frac{1}{\rho_{\theta\omega}} \operatorname{ess\,sup}_x \frac{1}{\ell^\omega(x)} \int \widetilde{\mathbf{R}}^\omega(x,dx')\mathbf{h}^{\theta\omega}(x')^2\right]\mathbb{P}(d\omega) \\
&= \int_\Omega \left[\log\left[\operatorname{ess\,sup}_x \frac{1}{\ell^\omega(x)} \int \widetilde{\mathbf{R}}^\omega(x,dx')\mathbf{h}^{\theta\omega}(x')^2\right] - \log \rho_{\theta\omega}\right]\mathbb{P}(d\omega) \\
&= \int_\Omega \log\left[\operatorname{ess\,sup}_x \frac{1}{\ell^\omega(x)} \int \widetilde{\mathbf{R}}^\omega(x,dx')\mathbf{h}^{\theta\omega}(x')^2\right]\mathbb{P}(d\omega) - \mathbb{E}\left[\log \rho\right] && \text{(S.35)}\\
&= \int_\Omega \log\left[\frac{1}{\rho_{\omega}} \operatorname{ess\,sup}_x \frac{1}{\ell^\omega(x)} \int \widetilde{\mathbf{R}}^\omega(x,dx')\mathbf{h}^{\theta\omega}(x')^2\right]\mathbb{P}(d\omega) \\
&\geq \int_\Omega \log\left[\frac{\lambda_\omega^2}{\rho_\omega} \operatorname{ess\,sup}_x \frac{\mathbf{h}^\omega(x)^2}{\ell^\omega(x)}\right]\mathbb{P}(d\omega) && \text{(S.36)}\\
&= \mathbb{E}\left[\log \lambda^2\right] = \mathbb{E}\left[\log \xi\right], && \text{(S.37)}
\end{aligned}
$$

where (S.33) has been applied; (S.35) holds because θ preserves $\mathbb{P}$; (S.36) holds due to (S.34) and similarly because θ preserves $\mathbb{P}$; and the final equality in (S.37) holds by hypothesis (since we are trying to prove (S.30)). Thus we conclude

$$
\begin{aligned}
&\mathbb{E}\left[\log \xi\right] \\
&= \int_\Omega \log\left[\frac{1}{\rho_\omega} \operatorname{ess\,sup}_x \frac{\widetilde{\mathbf{R}}^\omega\left(\left[\mathbf{h}^{\theta\omega}\right]^2\right)(x)}{\ell^\omega(x)}\right]\mathbb{P}(d\omega). && \text{(S.38)}
\end{aligned}
$$

Now for $\epsilon > 0$ and $\omega \in \Omega$, introduce

$$A_{\omega,\epsilon} := \left\{x : \ell^\omega(x) < \frac{(1+\epsilon)}{\rho_\omega}\mathbf{h}^\omega(x)^2\right\}.$$

Then for any $x \in \mathsf{X}^N$,

$$\ell^{\omega}(x) \geq \frac{\mathbf{h}^{\omega}(x)^2}{\rho_{\omega}} + \epsilon \frac{\mathbf{h}^{\omega}(x)^2}{\rho_{\omega}} \mathbb{I}_{A^c_{\omega,\epsilon}}(x), \quad \mathbb{P}-a.s.$$

Now by Proposition 2 and the definition of $\mathbf{h}$, we know $\inf_{\omega,x} \mathbf{h}(\omega,x) > 0$, and similarly, by (S.24), $\inf_{\omega,x} \ell(\omega,x) > 0$. Combined with Lemma 5 this ensures that there is a strictly positive constant k independent of ϵ and ω such that for any $x \in \mathsf{X}^N$,

$$\begin{aligned}
\xi_{\omega} &= \frac{\widetilde{\mathbf{R}}^{\omega}\left(\ell^{\theta\omega}\right)(x)}{\ell^{\omega}(x)} \\
&\geq \frac{\widetilde{\mathbf{R}}^{\omega}\left(\left[\mathbf{h}^{\theta\omega}\right]^2\right)(x)}{\rho_{\theta\omega}\ell^{\omega}(x)} + \frac{\epsilon}{\rho_{\theta\omega}\ell^{\omega}(x)} \int_{A^c_{\theta\omega,\epsilon}} \widetilde{\mathbf{R}}^{\omega}(x,dx')\mathbf{h}^{\theta\omega}(x')^2 \\
&\geq \frac{\widetilde{\mathbf{R}}^{\omega}\left(\left[\mathbf{h}^{\theta\omega}\right]^2\right)(x)}{\rho_{\theta\omega}\ell^{\omega}(x)} + \frac{\epsilon}{\rho_{\theta\omega}} k\mu(A^c_{\theta\omega,\epsilon}), \quad \mathbb{P}-a.s.,
\end{aligned}$$

where μ is as in Lemma 5. Thus we find, once again using the fact that θ preserves $\mathbb{P}$,

$$\mathbb{E}\left[\log \xi\right] \geq \int_{\Omega} \log \left[\frac{1}{\rho_{\omega}} \operatorname{ess\,sup}_x \frac{\widetilde{\mathbf{R}}^{\omega}\left(\left[\mathbf{h}^{\theta\omega}\right]^2\right)(x)}{\ell^{\omega}(x)} + \frac{\epsilon}{\rho_{\omega}} k\mu(A^c_{\theta\omega,\epsilon}) \right] \mathbb{P}(d\omega),$$

but we have already proved (S.38), and since ϵ, k and ρ_{ω} are strictly positive and finite we deduce that

$$\mathbb{P}\left(\left\{\omega : \mu(A^c_{\theta\omega,\epsilon}) = 0\right\}\right) = 1. \tag{S.39}$$

Now for any $\omega \in \Omega$, if $\mu(A^c_{\theta\omega,\epsilon}) = 0$, then by (S.19) $\left(d\nu^{\otimes N}/d\widetilde{\nu}\right)(x)$ is zero for $\widetilde{\nu} - a.a.$ $x \in A^c_{\theta\omega,\epsilon}$, and then $\nu^{\otimes N}(A^c_{\theta\omega,\epsilon}) = 0$. Therefore, from (S.39) we find

$$\begin{aligned}
1 &= \mathbb{P}\left(\left\{\omega : \nu^{\otimes N}(A^c_{\theta\omega,\epsilon}) = 0\right\}\right) \\
&= \mathbb{P}\left(\left\{\omega : \nu^{\otimes N}(A_{\theta\omega,\epsilon}) = 1\right\}\right) = \mathbb{P}\left(\left\{\omega : \nu^{\otimes N}(A_{\omega,\epsilon}) = 1\right\}\right),
\end{aligned} \tag{S.40}$$

where the final equality holds since θ preserves $\mathbb{P}$.

From the definition of $A_{\omega,\epsilon}$ it is clear that for any ω,

$$\bigcap_{m \geq 1} A_{\omega,1/m} = \left\{ x : \ell^{\omega}(x) = \frac{1}{\rho_{\omega}} \mathbf{h}^{\omega}(x)^2 \right\} =: A^{\star}_{\omega}, \tag{S.41}$$

and

$$A_{\omega,1/(m+1)} \subseteq A_{\omega,1/m}, \quad \forall m \geq 1. \tag{S.42}$$

Now suppose that for some $\omega \in \Omega$, $\nu(A_{\omega,1/m}) = 1$ for all $m \geq 1$. Then in light of (S.41) and (S.42) , continuity of probability under $\nu^{\otimes N}$ dictates $\nu^{\otimes N}(A^{\star}_{\omega}) = 1$. On the other hand, if $\nu^{\otimes N}(A^{\star}_{\omega}) = 1$, then it must be that $\nu^{\otimes N}(A_{\omega,1/m}) = 1$ for all $m \geq 1$. Therefore

$$\left\{\omega : \nu^{\otimes N}(A^{\star}_{\omega}) = 1\right\} = \bigcap_{m\geq 1} \left\{\omega : \nu^{\otimes N}(A_{\omega,1/m}) = 1\right\}. \tag{S.43}$$

Evaluating $\nu^{\otimes N}$ on the sets in (S.42) we find

$$\left\{\omega : \nu^{\otimes N}(A_{\omega,1/(m+1)}) = 1\right\} \subseteq \left\{\omega : \nu^{\otimes N}(A_{\omega,1/m}) = 1\right\}, \quad \forall m \geq 1. \tag{S.44}$$

Then by (S.40), (S.43), (S.44) and continuity of probability under $\mathbb{P}$,

$$\mathbb{P}\left(\left\{\omega : \nu^{\otimes N}(A^{\star}_{\omega}) = 1\right\}\right) = 1. \tag{S.45}$$

Now (S.45) ensures that, for $\mathbb{P}-a.a.\ \omega$, $A^{\star}_{\omega}$ is non-empty, so it is legitimate to write, for any $x \in A^{\star}_{\omega}$,

$$\begin{aligned}
\frac{\xi_\omega}{\lambda_\omega^2} &= \frac{\widetilde{\mathbf{R}}^{\omega}\left(\ell^{\theta\omega}\right)(x)}{\ell^{\omega}(x)} \frac{\mathbf{h}^{\omega}(x)^2}{\left[\mathbf{Q}^{\omega}(\mathbf{h}^{\theta\omega})(x)\right]^2} \\
&= \frac{\rho_\omega}{\rho_{\theta\omega}} \frac{\widetilde{\mathbf{R}}^{\omega}\left(\left[\mathbf{h}^{\theta\omega}\right]^2\right)(x)}{\left[\mathbf{Q}^{\omega}(\mathbf{h}^{\theta\omega})(x)\right]^2} \\
&= \frac{\rho_\omega}{\rho_{\theta\omega}} \frac{\mathbf{G}^{\omega}(x)^2 \int \phi^{\omega}(x,x')^2 \widetilde{\mathbf{M}}^{\omega}(x,dx')\mathbf{h}^{\theta\omega}(x')^2}{\mathbf{G}^{\omega}(x)^2 \left(\int \phi^{\omega}(x,x')\widetilde{\mathbf{M}}^{\omega}(x,dx')\mathbf{h}^{\theta\omega}(x')\right)^2} \\
&\geq \frac{\rho_\omega}{\rho_{\theta\omega}},
\end{aligned} \tag{S.46}$$

where the final inequality is due to Jensen's inequality. Therefore we have proved

$$\frac{\xi_\omega}{\lambda_\omega^2}\frac{\rho_{\theta\omega}}{\rho_\omega} \geq 1, \quad \mathbb{P}-a.s.$$

However, by hypothesis, $\mathbb{E}\left[\log \frac{\xi}{\lambda^2}\right] = 0$, and combined with the fact that θ preserves $\mathbb{P}$ we find

$$\int_\Omega \log\left(\frac{\xi_\omega}{\lambda_\omega^2}\frac{\rho_{\theta\omega}}{\rho_\omega}\right)\mathbb{P}(d\omega) = \mathbb{E}\left[\log\frac{\xi}{\lambda^2}\right] + \int_\Omega \log\rho_{\theta\omega}\mathbb{P}(d\omega) - \int_\Omega \log\rho_\omega \mathbb{P}(d\omega) = 0,$$

therefore it must be the case that in fact

$$\frac{\xi_\omega}{\lambda_\omega^2}\frac{\rho_{\theta\omega}}{\rho_\omega} = 1, \quad \mathbb{P} - a.s. \tag{S.47}$$

Equation (S.47) implies equality must hold in the instance of Jensen's inequality (S.46), i.e. for $\mathbb{P}$-almost all ω, and all $x \in A_\omega^\star$,

$$\phi^\omega(x, x')\mathbf{h}^{\theta\omega}(x') = c^\omega(x), \qquad \widetilde{\mathbf{M}}^\omega(x, \cdot) - a.s.,$$

and therefore

$$\widetilde{\mathbf{M}}^\omega(x, B) = \frac{1}{c^\omega(x)}\int_B \mathbf{M}^\omega(x, dx')\mathbf{h}^{\theta\omega}(x'), \quad \text{for any } B \in \mathcal{X}^{\otimes N}.$$

Normalization then dictates

$$\widetilde{\mathbf{M}}^\omega(x, B) = \frac{\int_B \mathbf{M}^\omega(x, dx')\mathbf{h}^{\theta\omega}(x')}{\mathbf{M}^\omega(\mathbf{h}^{\theta\omega})(x)},$$

and this completes the proof. □

LEMMA 7. *[2)⇒3)] If for $\mathbb{P}$-almost all $\omega \in \Omega$ there exists $A_\omega \in \mathcal{X}^{\otimes N}$ such that $\nu^{\otimes N}(A_\omega^c) = 0$ and for any $x \in A_\omega$,*

$$\widetilde{\mathbf{M}}^\omega(x, B) = \frac{\int_B \mathbf{M}^\omega(x, dx')\mathbf{h}^{\theta\omega}(x')}{\int_{\mathsf{X}^N} \mathbf{M}^\omega(x, dz)\mathbf{h}^{\theta\omega}(z)}, \quad \textit{for all } B \in \mathcal{X}^{\otimes N},$$

then $\sup_n \widetilde{\mathcal{V}}_{n,N}^\omega < \infty$ *for $\mathbb{P}$-almost all $\omega \in \Omega$.*

PROOF. Let $C \in \mathcal{F}$ with $\mathbb{P}(C) = 1$ be a set of ω's for which the hypothesis of the Lemma holds. Then for each $n \geq 0$, let $C_n := \bigcap_{p=0}^n \theta^{-p}C$, and $C^\star := \bigcap_{n=0}^\infty C_n$. Since θ preserves $\mathbb{P}$, it follows that $\mathbb{P}(C_n) = 1$ for all $n \geq 1$, and since $C_n \searrow C^\star$, we have by continuity of probability $\mathbb{P}(C^\star) = 1$. Furthermore, by construction, we have that for any $\omega \in C^\star$ and any $n \geq 0$ there exists $A_{\theta^n\omega} \in \mathcal{X}^{\otimes N}$ such that $\nu^{\otimes N}(A_{\theta^n\omega}^c) = 0$ and for any $x \in A_{\theta^n\omega}$,

$$\widetilde{\mathbf{M}}^{\theta^n\omega}(x, B) = \frac{\int_B \mathbf{M}^{\theta^n\omega}(x, dx')\mathbf{h}^{\theta^{n+1}\omega}(x')}{\int_{\mathsf{X}^N} \mathbf{M}^{\theta^n\omega}(x, dz)\mathbf{h}^{\theta^{n+1}\omega}(z)}, \quad \text{for all } B \in \mathcal{X}^{\otimes N}. \tag{S.48}$$

Now pick any $\omega \in C^\star$. This ω remains fixed throughout the remainder of the proof. Observe firstly that for any $n \geq 0$ and $x \in A_{\theta^n\omega}$, using Lemma 1 and the fact that by Proposition 2 $\mathbf{h}$ is strictly positive, the measures $\nu^{\otimes N}$,

$\mathbf{M}^{\theta^n\omega}(x,\cdot)$ and $\widetilde{\mathbf{M}}^{\theta^n\omega}(x,\cdot)$, are mutually absolutely continuous. Assume now that $n \geq 2$ and observe secondly that since $\mathbf{G}^{\omega}(x) = \lambda_\omega \mathbf{h}^{\omega}(x)/\mathbf{M}^{\omega}(\mathbf{h}^{\theta\omega})(x)$, we have from (S.48) that, for any $(x_1,\ldots,x_n) \in \left[\prod_{p=1}^{n} A_{\theta^p\omega}\right]$,

$$\prod_{p=1}^{n-1} \mathbf{G}^{\theta^p\omega}(x_p)\phi^{\theta^p\omega}(x_p,x_{p+1}) = \frac{\mathbf{h}^{\theta\omega}(x_1)}{\mathbf{h}^{\theta^n\omega}(x_n)} \prod_{p=1}^{n-1} \lambda_{\theta^p\omega}.$$

Combining these two observations with $\nu^{\otimes N}(A^c_{\theta^n\omega}) = 0$ and again using $\mathbf{G}^{\omega}(x) = \lambda_\omega \mathbf{h}^{\omega}(x)/\mathbf{M}^{\omega}(\mathbf{h}^{\theta\omega})(x)$, we obtain:

$$\begin{aligned}
&\mu_0^{\otimes N}\widetilde{\mathbf{R}}_n^{\omega}(1)\\
&= \widetilde{\mathsf{E}}_N^{\omega}\left[\prod_{p=0}^{n-1} \mathbf{G}^{\theta^p\omega}(\zeta_p)^2\phi^{\theta^p\omega}(\zeta_p,\zeta_{p+1})^2\right]\\
&= \widetilde{\mathsf{E}}_N^{\omega}\left[\prod_{p=0}^{n-1} \mathbb{I}_{A_{\theta^{p+1}\omega}}(\zeta_p)\mathbf{G}^{\theta^p\omega}(\zeta_p)^2\phi^{\theta^p\omega}(\zeta_p,\zeta_{p+1})^2\right]\\
&= \left(\prod_{p=1}^{n-1}\lambda_{\theta^p\omega}\right)^2 \widetilde{\mathsf{E}}_N^{\omega}\left[\mathbf{G}^{\omega}(\zeta_0)^2\phi^{\omega}(\zeta_0,\zeta_1)^2\left(\frac{\mathbf{h}^{\theta\omega}(\zeta_1)}{\mathbf{h}^{\theta^n\omega}(\zeta_n)}\right)^2\right]\\
&= \left(\prod_{p=0}^{n-1}\lambda_{\theta^p\omega}\right)^2 \widetilde{\mathsf{E}}_N^{\omega}\left[\frac{\mathbf{h}^{\omega}(\zeta_0)^2}{\mathbf{M}^{\omega}\left(\mathbf{h}^{\theta\omega}\right)^2(\zeta_0)}\phi^{\omega}(\zeta_0,\zeta_1)^2\left(\frac{\mathbf{h}^{\theta\omega}(\zeta_1)}{\mathbf{h}^{\theta^n\omega}(\zeta_n)}\right)^2\right]\\
&\leq \left(\prod_{p=0}^{n-1}\lambda_{\theta^p\omega}\right)^2\left(\sup_{(\omega,\omega',x,x')}\frac{h^{\omega}(x)}{h^{\omega'}(x')}\right)^4 \widetilde{\mathsf{E}}_N^{\omega}\left[\phi^{\omega}(\zeta_0,\zeta_1)^2\right]\\
&\leq \left(\prod_{p=0}^{n-1}\lambda_{\theta^p\omega}\right)^2\left(\sup_{(\omega,\omega',x,x')}\frac{h^{\omega}(x)}{h^{\omega'}(x')}\right)^4 \frac{\epsilon_+^{2N}\widetilde{\epsilon}_+}{\widetilde{\epsilon}_-^2}\,\widetilde{\nu}\left(\left[\frac{d\nu^{\otimes N}}{d\widetilde{\nu}}\right]^2\right), \qquad \text{(S.49)}
\end{aligned}$$

where the final inequality is an application of (S.21). In the case $n \leq 1$ a similar upper bound can be obtained. Now using $G^\omega(x) = \lambda_\omega h^\omega(x)/M^\omega(h^{\theta\omega})(x)$,

$$\begin{aligned}
[\mu_0 Q_n^\omega(1)]^2 &= \mathsf{E}^\omega\left[\prod_{p=0}^{n-1} G^{\theta^p\omega}(X_p)\right]^2 \\
&= \left(\prod_{p=0}^{n-1}\lambda_{\theta^p\omega}\right)^2 \mathsf{E}^\omega\left[\prod_{p=0}^{n-1}\frac{h^{\theta^p\omega}(X_p)}{M^{\theta^p\omega}(h^{\theta^{p+1}\omega})(X_p)}\right]^2 \\
&= \left(\prod_{p=0}^{n-1}\lambda_{\theta^p\omega}\right)^2 \mathsf{E}^\omega\left[\frac{h^\omega(X_0)}{h^{\theta^n\omega}(X_n)}\prod_{p=1}^{n}\frac{h^{\theta^p\omega}(X_p)}{M^{\theta^{p-1}\omega}(h^{\theta^p\omega})(X_{p-1})}\right]^2 \\
&\geq \left(\prod_{p=0}^{n-1}\lambda_{\theta^p\omega}\right)^2 \left(\inf_{(\omega,\omega',x,x')}\frac{h^\omega(x)}{h^{\omega'}(x')}\right)^2, \qquad \text{(S.50)}
\end{aligned}$$

where $\mathsf{E}^\omega\left[h^{\theta^p\omega}(X_p)\middle| X_0,\ldots,X_{p-1}\right] = M^{\theta^{p-1}\omega}(h^{\theta^p\omega})(X_{p-1})$ has been applied. Combining (S.49) with (S.50), we arrive at

$$\sup_{n\geq 1}\frac{\mu_0^{\otimes N}\widetilde{\mathbf{R}}_n^\omega(1)}{[\mu_0 Q_n^\omega(1)]^2} \leq \left(\sup_{(\omega,\omega',x,x')}\frac{h^\omega(x)}{h^{\omega'}(x')}\right)^6 < \infty,$$

where the final inequality is due to Proposition 2. $\square$

LEMMA 8. *[3) $\Rightarrow$ 1)] If $\sup_n \widetilde{\mathcal{V}}_{n,N}^\omega < \infty$ for $\mathbb{P}-a.a.\ \omega$, then $\Upsilon_N(\widetilde{\mathbf{M}}) = 0$.*

PROOF. Obvious. $\square$

PROOF OF LEMMA 3. We first address 1) $\Rightarrow$ 2). By Theorem 1, if $\Upsilon_N(\mathbf{M}) = 0$, it must be the case that up to ω and x being in null sets,

$$\mathbf{M}^\omega(x,B) = \frac{\int_B \mathbf{M}^\omega(x,dx')\mathbf{h}^{\theta\omega}(x')}{\int_{\mathsf{X}^N}\mathbf{M}^\omega(x,dz)\mathbf{h}^{\theta\omega}(z)}, \quad \text{for all } B\in\mathcal{X}^{\otimes N}.$$

and therefore using **(H2)**, we find there must exist a random variable, say χ, such that

$$\text{for } \mathbb{P}-a.a.\ \omega, \quad \mathbf{h}^\omega(x) = \chi_\omega, \quad \text{for } \nu^{\otimes N}-a.a.\ x.$$

Under **(H2)**, the first equation in (2.18) shows that the probability measure η^ω is equivalent to ν and therefore

$$\text{for } \mathbb{P}-a.a.\ \omega, \quad [\eta^\omega]^{\otimes N}(\mathbf{h}^\omega) = \chi_\omega.$$

But by the third equation in (2.18), we have $[\eta^\omega]^{\otimes N}(\mathbf{h}^\omega) = \eta^\omega(h^\omega) = 1$, and it follows that

$$\text{for } \mathbb{P} - a.a.\ \omega, \quad h^\omega(x) = 1, \quad \text{for } \nu - a.a.\ x. \tag{S.51}$$

Now we show 2) $\Rightarrow$ 3). Suppose that (S.51) holds. Then on appropriate sets, $Q^\omega(h^{\theta\omega}) = \lambda_\omega h^\omega$ reduces to

$$G^\omega(x) = \lambda_\omega =: C_\omega.$$

Finally, we show 3) $\Rightarrow$ 1). Suppose there exists a random variable $C : \Omega \to \mathbb{R}_+$ such that

$$\text{for } \mathbb{P} - a.a.\ \omega, \quad G^\omega(x) = C_\omega, \quad \text{for } \nu - a.a.\ x.$$

Then since by hypothesis $\widetilde{\mathbf{M}}^\omega = \mathbf{M}^\omega$ we have $\widetilde{\mathcal{V}}_n^{\omega,N} = 1$ for all $n \geq 1$ and $\mathbb{P} - a.a.\ \omega$. Therefore $\Upsilon_N(\mathbf{M}) = \lim_{n\to\infty} n^{-1} \log \widetilde{\mathcal{V}}_n^{\omega,N} = 0$ with $\mathbb{P}$ probability 1. This completes the proof. □

Proofs for section 3. The following operators will enter into our $N \to \infty$ analysis of the twisted particle system. Define

$$\widehat{\Phi}^\omega : \mathcal{P}(\mathsf{X}) \to \mathcal{P}(\mathsf{X}), \qquad \widehat{\Phi}^\omega(\mu)(dx) := \frac{\mu Q^\omega(dx)\psi^\omega(x)}{\mu Q^\omega(\psi^\omega)}, \quad \mu \in \mathcal{P}(\mathsf{X}).$$

and also for each $N \geq 1$,

$$\Gamma_N^\omega : \mathcal{P}(\mathsf{X}) \to \mathcal{P}(\mathsf{X}), \quad \Gamma_N^\omega(\mu) := \frac{1}{N}\widehat{\Phi}^\omega(\mu) + \left(1 - \frac{1}{N}\right)\Phi^\omega(\mu), \quad \mu \in \mathcal{P}(\mathsf{X}).$$

$$\Gamma_{0,N}^\omega := Id, \qquad \Gamma_{n,N}^\omega := \Gamma_N^{\theta^{n-1}\omega} \circ \cdots \circ \Gamma_N^{\theta\omega} \circ \Gamma_N^\omega, \quad n \geq 1.$$

For purposes of developing limits and fluctuation studies in the regime $N \to \infty$, it is convenient to construct the particle system of interest as follows.

Let K_N be set of all bijections between $\{1, \ldots, N\}$ and itself, and let its power set be $\mathcal{K}_N$. Let $\mathsf{Z}_N := \left(\mathsf{X}^N \times \mathsf{K}_N\right)^{\mathbb{N}}$ be the set of infinite sequences valued in $\mathsf{X}^N \times \mathsf{K}_N$, endow it with the σ-algebra $\mathcal{Z}_N := \left(\mathcal{X}^N \otimes \mathcal{K}_N\right)^{\otimes\mathbb{N}}$, so as to form a measurable space $(\mathsf{Z}_N, \mathcal{Z}_N)$. Let $\left\{\left(\widehat{\zeta}_n, \kappa_n\right); n \geq 0\right\}$ be the coordinate process on Z_N.

$$\widehat{\eta}_n^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{\widehat{\zeta}_n^i} \quad n \geq 0.$$

Now for some fixed $\omega \in \Omega$, let us introduce a probability measure, P_N^ω, on $(\mathsf{Z}_N, \mathcal{Z}_N)$ according to the following prescription. Under P_N^ω, the sequences $\left\{\widehat{\zeta}_n; n \geq 0\right\}$ and $\{\kappa_n; n \geq 0\}$ are independent. The sequence $\left\{\widehat{\zeta}_n; n \geq 0\right\}$ is a Markov chain with the following law. At time $n = 0$, the random variables $\left\{\widehat{\zeta}_0^i\right\}_{i=1}^N$ are independent and identically distributed according to μ_0. At time $n \geq 1$, the random variables $\left\{\widehat{\zeta}_n^i\right\}_{i=1}^N$ are conditionally independent given $\widehat{\zeta}_{n-1}$, and

$$\widehat{\zeta}_n^1 | \widehat{\zeta}_{n-1} \sim \widehat{\Phi}^{\theta^{n-1}\omega}(\widehat{\eta}_{n-1}^N), \quad \widehat{\zeta}_n^i | \widehat{\zeta}_{n-1} \sim \Phi^{\theta^{n-1}\omega}(\widehat{\eta}_{n-1}^N), \quad i = 2, \ldots, N. \tag{S.52}$$

Finally, the components of the sequence $\{\kappa_n; n \geq 0\}$ are independent and identically distributed according to the uniform distribution on $(\mathsf{K}_N, \mathcal{K}_N)$. Expectation under P_N^ω will be denoted by E_N^ω.

Remark 11. Our interest in this construction is that if we define, for each $n \geq 0$ and $i = 1, \ldots, N$, the random variables

$$\zeta_n^i := \widehat{\zeta}_n^{\kappa_n(i)}$$

we obviously have the identity of empirical measures

$$\widehat{\eta}_n^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\widehat{\zeta}_n^i} = \frac{1}{N}\sum_{i=1}^{N} \delta_{\zeta_n^i} = \eta_n^N, \quad n \geq 0. \tag{S.53}$$

Furthermore, using the fact that with $\widetilde{\mathbf{M}}$ as in (3.1), for any $(\omega, A) \in \Omega \times \mathcal{X}^{\otimes N}$, $\widetilde{\mathbf{M}}^\omega(x, A)$ is invariant to permutations of the coordinates $x = (x^1, \ldots, x^N)$, it is straightforward to check that under P_N^ω, the process $\{\zeta_n; n \geq 0\}$ is Markov, with

$$\zeta_n | \zeta_{n-1} \quad \sim \quad \widetilde{\mathbf{M}}^{\theta^{n-1}\omega}(\zeta_{n-1}, \cdot), \quad n \geq 1, \tag{S.54}$$

which is exactly the transition law of interest. The important thing here is that the identity of random measures (S.53) permits us, by construction, to perform asymptotic analysis of functionals of the empirical measures $\left\{N^{-1}\sum_{i=1}^N \delta_{\zeta_n^i}; n \geq 0\right\}$ through study of the random variables $\left\{\widehat{\zeta}_n; n \geq 0\right\}$ rather than $\{\zeta_n; n \geq 0\}$: the conditional independence structure of the former as per (S.52) is easier to work with than that of the latter under (S.54).

REMARK 12. To see that Algorithm 3 does indeed implement (S.54), notice that for $A \in \mathcal{X}^{\otimes N}$, $x = (x^1, \ldots, x^N) \in \mathsf{X}^N$, $z = (z^1, \ldots, z^N) \in \mathsf{X}^N$, and

$$\begin{aligned}&\int_A \mathbf{M}^\omega(x, dz)\boldsymbol{\psi}^{\theta\omega}(z)\\&= \frac{1}{N}\sum_{k=1}^N \int_A \left(\prod_{j=1}^N \frac{\sum_{i=1}^N Q^\omega(x^i, dz^j)}{\sum_{i=1}^N G^\omega(x^i)}\right)\psi^{\theta\omega}(z^k),\end{aligned}$$

where $\prod$ indicates tensor product of measures and we have abusively written $\psi^{\theta\omega}(z^k)$ to represent the function which maps $z = (z^1, \ldots, z^N) \in \mathsf{X}^N \mapsto \psi^{\theta\omega}(z^k) \in \mathbb{R}_+$. In particular then,

$$\int_{\mathsf{X}^N} \mathbf{M}^\omega(x, dz)\boldsymbol{\psi}^{\theta\omega}(z) = \left[\sum_{i=1}^N Q^\omega(\psi^{\theta\omega})(x^i)\right] / \left[\sum_{i=1}^N G^\omega(x^i)\right],$$

and

$$\begin{aligned}&\widetilde{\mathbf{M}}^\omega(x, A) =\\&\frac{1}{N}\sum_{k=1}^N \int_A \left(\prod_{j\neq k} \frac{\sum_{i=1}^N Q^\omega(x^i, dz^j)}{\sum_{i=1}^N G^\omega(x^i)}\right)\left(\frac{\sum_{i=1}^N Q^\omega(x^i, dz^k)\psi^{\theta\omega}(z^k)}{\sum_{i=1}^N Q^\omega(\psi^{\theta\omega})(x^i)}\right).\end{aligned} \tag{S.55}$$

We proceed with a crude but simply proved L_p error estimate.

LEMMA 9. *Assume **(H3)**. Then for each $\omega \in \Omega$, $n \geq 1$, $\mu \in \mathcal{P}(\mathsf{X})$ and $p \geq 1$ there exist finite constants B_n^ω and $C_{n,p}^\omega$ such that for any $\varphi \in \mathcal{L}(\mathsf{X})$,*

$$\left|\left[\Gamma_{N,n}^\omega(\mu) - \Phi_n^\omega(\mu)\right](\varphi)\right| \leq \|\varphi\| \frac{B_n^\omega}{N} \tag{S.56}$$

$$\mathsf{E}_N^\omega\left[\left|\left[\widehat{\eta}_n^N - \eta_n^\omega\right](\varphi)\right|^p\right]^{1/p} \leq \|\varphi\| \frac{C_{n,p}^\omega}{\sqrt{N}}. \tag{S.57}$$

PROOF. The proof of (S.56) is by induction. At rank $n = 0$ the inequality holds trivially since $\widehat{\Phi}_0^\omega = \Phi_0^\omega = Id$. Suppose the inequality holds at rank $n-1$. Then at rank n,

$$\begin{aligned}&\left|\left[\Gamma_{N,n}^\omega(\mu) - \Phi_n^\omega(\mu)\right](\varphi)\right|\\&\leq \left|\left[\Gamma_N^{\theta^{n-1}\omega}(\Gamma_{N,n-1}^\omega(\mu)) - \Phi^{\theta^{n-1}\omega}(\Gamma_{N,n-1}^\omega(\mu))\right](\varphi)\right|\\&\quad + \left|\left[\Phi^{\theta^{n-1}\omega}(\Gamma_{N,n-1}^\omega(\mu)) - \Phi^{\theta^{n-1}\omega}(\Phi_{n-1}^\omega(\mu))\right](\varphi)\right|\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{N}\left|\left[\widehat{\Phi}^{\theta^{n-1}\omega}(\Gamma^{\omega}_{N,n-1}(\mu)) - \Phi^{\theta^{n-1}\omega}(\Gamma^{\omega}_{N,n-1}(\mu))\right](\varphi)\right| \\
&\quad + \left|\frac{\Gamma^{\omega}_{N,n-1}(\mu)Q^{\theta^{n-1}\omega}(\varphi)}{\Gamma^{\omega}_{N,n-1}(\mu)Q^{\theta^{n-1}\omega}(1)}\left[1 - \frac{\Gamma^{\omega}_{N,n-1}(\mu)Q^{\theta^{n-1}\omega}(1)}{\Phi^{\omega}_{n-1}(\mu)Q^{\theta^{n-1}\omega}(1)}\right]\right| \\
&\quad + \left|\frac{\left[\Gamma^{\omega}_{N,n-1}(\mu) - \Phi^{\omega}_{n-1}(\mu)\right]Q^{\theta^{n-1}\omega}(\varphi)}{\Phi^{\omega}_{n-1}(\mu)Q^{\theta^{n-1}\omega}(1)}\right|,
\end{aligned}
$$

and the result holds by the induction hypothesis and some simple manipulations.

The proof of (S.57) is also by induction. At time $n = 0$, the random variables $\left\{\widehat{\zeta}_0^i\right\}_{i=1}^N$ are iid according to μ_0, so by the Marcinkiewicz–Zygmund (MZ) inequality,

$$
\mathsf{E}^{\omega}_N\left[\left|\left[\widehat{\eta}^N_0 - \mu_0\right](\varphi)\right|^p\right]^{1/p} \le \|\varphi\|\frac{C^{\omega}_{0,p}}{\sqrt{N}}.
$$

Suppose the inequality holds at rank $n-1$. Then at rank n,

$$
\begin{aligned}
&\mathsf{E}^{\omega}_N\left[\left|\left[\widehat{\eta}^N_n - \eta^{\omega}_n\right](\varphi)\right|^p\right]^{1/p} \\
&\le \mathsf{E}^{\omega}_N\left[\left|\left[\widehat{\eta}^N_n - \Gamma^{\theta^{n-1}\omega}_N(\widehat{\eta}^N_{n-1})\right](\varphi)\right|^p\right]^{1/p} \quad \text{(S.58)} \\
&\quad + \mathsf{E}^{\omega}_N\left[\left|\left[\Gamma^{\theta^{n-1}\omega}_N(\widehat{\eta}^N_{n-1}) - \Gamma^{\theta^{n-1}\omega}_N(\eta^{\omega}_{n-1})\right](\varphi)\right|^p\right]^{1/p} \quad \text{(S.59)} \\
&\quad + \left|\left[\eta^{\omega}_n - \Gamma^{\theta^{n-1}\omega}_N(\eta^{\omega}_{n-1})\right](\varphi)\right|. \quad \text{(S.60)}
\end{aligned}
$$

The term in (S.58) is dealt with by application of the MZ inequality and the term in (S.60) is dealt with using (S.56). For the term in (S.59),

$$
\begin{aligned}
&\left[\Gamma^{\theta^{n-1}\omega}_N(\widehat{\eta}^N_{n-1}) - \Gamma^{\theta^{n-1}\omega}_N(\eta^{\omega}_{n-1})\right](\varphi) \\
&= \frac{1}{N}\left[\widehat{\Phi}^{\theta^{n-1}\omega}(\widehat{\eta}^N_{n-1}) - \widehat{\Phi}^{\theta^{n-1}\omega}(\eta^{\omega}_{n-1})\right](\varphi) \\
&\quad + \left(1 - \frac{1}{N}\right)\frac{\widehat{\eta}^N_{n-1}Q^{\theta^{n-1}\omega}(\varphi)}{\widehat{\eta}^N_{n-1}Q^{\theta^{n-1}\omega}(1)}\left(1 - \frac{\widehat{\eta}^N_{n-1}Q^{\theta^{n-1}\omega}(1)}{\eta^{\omega}_{n-1}Q^{\theta^{n-1}\omega}(1)}\right) \\
&\quad + \left(1 - \frac{1}{N}\right)\left[\widehat{\eta}^N_{n-1} - \eta^{\omega}_{n-1}\right]\frac{Q^{\theta^{n-1}\omega}(\varphi)}{\eta^{\omega}_{n-1}Q^{\theta^{n-1}\omega}(1)},
\end{aligned}
$$

and the result follows by elementary manipulations involving the MZ inequality and the induction hypothesis. □

Proof of Lemma 4. The result follows from the L_p estimate (S.57), a standard Borel-Cantelli argument and the identity (S.53). □

Lemma 10 is an extension of the CLT in [2, Chapter 9, Theorem 9.3.1] to the case of certain random test functions and the setting of the twisted particle system. It is established by an application of [6, Theorem 3.33, p. 478].

For $d \geq 1$ and $m \geq 1$, let $\varphi : (p, q, x) \in \mathbb{N} \times \{1, \ldots, m\} \times \mathsf{X} \mapsto \varphi_{p,q}(x) \in \mathbb{R}^d$ be a bounded measurable function and for each $N \geq 1$ let $\beta^N := \left\{\beta^N_{p,q}; (p, q) \in \mathbb{N} \times \{1, \ldots, m\}\right\}$ be a collection of $\mathbb{R}^d$-valued random variables on the probability space $(\mathsf{Z}_N, \mathcal{Z}_N, \mathsf{P}^\omega_N)$ such that for any $p \geq 1$ and $1 \leq q \leq m$, $\beta^N_{p,q}$ is measurable w.r.t. $\sigma(\widehat{\zeta}_0, \ldots, \widehat{\zeta}_{p-1})$ and $\left\{\beta^N_{0,q}; q \in \{1, \ldots, m\}\right\}$ are constants. Then let the random function $f^N : (p, x) \in \mathbb{N} \times \mathsf{X} \mapsto f^N_p(x) \in \mathbb{R}^d$ be defined by

$$f^{N,i}_p(x) := \sum_{q=1}^{m} \beta^{N,i}_{p,q} \varphi^i_{p,q}(x), \quad 1 \leq i \leq d, \tag{S.61}$$

where $f^{N,i}_p$ is the ith coordinate of f^N_p, and for $n \geq 0$ define

$$M^{\omega,N}_n(f^N) \;\; := \;\; \sum_{p=0}^{n} \left[\widehat{\eta}^N_p(f^N_p) - \Gamma^{\theta^{p-1}\omega}_N \left(\widehat{\eta}^N_{p-1}\right)(f^N_p)\right],$$

with the convention that $\Gamma^{\theta^{-1}\omega}_N \left(\widehat{\eta}^N_{-1}\right)(\varphi) = \mu_0$. Clearly $\left\{M^{\omega,N}_n(f^N); n \geq 0\right\}$ is an $\mathbb{R}^d$-valued martingale w.r.t. to the natural filtration of $\left\{\widehat{\zeta}_n; n \geq 0\right\}$.

Lemma 10. *Assume* ***(H3)*** *and that there exist deterministic and finite constants* $(\beta_{p,q}; (p, q) \in \mathbb{N} \times \{1, \ldots, m\})$ *each valued in* $\mathbb{R}^d$ *such that for each* p, q, i,

$$\beta^{N,i}_{p,q} \to \beta^i_{p,q} \tag{S.62}$$

in probability as $N \to \infty$. *Then with*

$$f^i_p(x) := \sum_{q=1}^{m} \beta^i_{p,q} \varphi^i_{p,q}(x),$$

for any fixed $\omega \in \Omega$, *the* $\mathbb{R}^d$*-valued martingale* $\left\{\sqrt{N} M^{\omega,N}_n(f^N); n \geq 0\right\}$ *converges in law to an* $\mathbb{R}^d$*-valued Gaussian martingale* $\{M^\omega_n(f); n \geq 0\}$ *such that* *for any* $1 \leq i, j \leq d$,

$$\forall n \geq 0, \quad \left\langle M^{\omega}(f^i), M^{\omega}(f^j) \right\rangle_n = \sum_{p=0}^{n} \eta_p^{\omega} \left[(f_p^i - \eta_p^{\omega}(f_p^i))(f_p^j - \eta_p^{\omega}(f_p^j)) \right],$$

where $M^{\omega}(f^i)$ *is the* i*-th coordinate of* $M^{\omega}(f)$.

PROOF. For $a \in \mathbb{R}$ let $[a]$ be the integer part of a and let $\{a\} = a - [a]$. Consider the decomposition

$$\sqrt{N} M_n^{\omega,N}(f^N) = \sum_{k=1}^{(n+1)N} U_k^N(f^N)$$

where for any $1 \leq k \leq (n+1)N$ and (p, i) satisfying $1 \leq i \leq N$ and $k = pN + i$,

$$U_k^N(f^N) = \begin{cases} \dfrac{1}{\sqrt{N}} \left[f_p^N(\widehat{\zeta}_p^1) - \widehat{\Phi}^{\theta^{p-1}\omega}(\widehat{\eta}_{p-1}^N)(f_p^N) \right], & k - pN = 1, \\ \dfrac{1}{\sqrt{N}} \left[f_p^N(\widehat{\zeta}_p^i) - \Phi^{\theta^{p-1}\omega}(\widehat{\eta}_{p-1}^N)(f_p^N) \right], & k - pN = i > 1. \end{cases}$$

We will establish distributional convergence of the process

$$X_t^N(f^N) := \sum_{k=1}^{[Nt]+N} U_k^N(f^N),$$

to a continuous Gaussian martingale by application of [6, Theorem 3.33, p. 478]. The dependence on ω of $U_k^N(f^N)$, $X_t^N(f^N)$ and various other quantities is suppressed from the notation in the remainder of the proof.

For fixed N, let $\mathcal{H}_k^N$ be the σ-algebra generated by the random variables $\widehat{\zeta}_p^i$ for any (p, i) such that $pN + i \leq k$. Then

$$\sum_{k=1}^{[Nt]+N} \mathsf{E}_N^{\omega} \left[U_k^N(f^{N,i}) U_k^N(f^{N,j}) | \mathcal{H}_{k-1}^N \right] = \mathbb{C}_t^N(f^{N,i}, f^{N,j}),$$

where

$$\mathbb{C}_t^N(f^{N,i}, f^{N,j}) = C_{[t]}^N(f^{N,i}, f^{N,j}) + \delta C_t^N(f^{N,i}, f^{N,j}),$$

$$C_n^N(f^{N,i}, f^{N,j}) = \frac{1}{N} \widehat{d}_n^N(i,j) + \left(1 - \frac{1}{N}\right) d_n^N(i,j),$$

$$\begin{aligned}
\widehat{d}_n^N(i,j) &= \sum_{p=0}^{n} \widehat{\Phi}^{\theta^{p-1}\omega}(\widehat{\eta}_{p-1}^N)\left[\widehat{f}_p^{N,i}\widehat{f}_p^{N,j}\right],\\
d_n^N(i,j) &= \sum_{p=0}^{n} \Phi^{\theta^{p-1}\omega}(\widehat{\eta}_{p-1}^N)\left[\bar{f}_p^{N,i}\bar{f}_p^{N,j}\right],
\end{aligned}$$

$$\begin{aligned}
\widehat{f}_p^{N,i} &= f_p^{N,i} - \widehat{\Phi}^{\theta^{p-1}\omega}(\widehat{\eta}_{p-1}^N)(f_p^{N,i}),\\
\bar{f}_p^{N,i} &= f_p^{N,i} - \Phi^{\theta^{p-1}\omega}(\widehat{\eta}_{p-1}^N)(f_p^{N,i}),
\end{aligned}$$

and

$$\begin{aligned}
&\delta C_t^N(f^{N,i}, f^{N,j})\\
&= \mathbb{I}\left[[Nt]-N[t]\geq 1\right]\frac{1}{N}\widehat{\Phi}^{\theta^{[t]}\omega}(\widehat{\eta}_{[t]}^N)\left[\widehat{f}_{[t]+1}^{N,i}\widehat{f}_{[t]+1}^{N,j}\right]\\
&+\mathbb{I}\left[[Nt]-N[t]\geq 2\right]\frac{([Nt]-N[t]-1)}{N}\Phi^{\theta^{[t]}\omega}(\widehat{\eta}_{[t]}^N)\left[\bar{f}_{[t]+1}^{N,i}\bar{f}_{[t]+1}^{N,j}\right]
\end{aligned}$$

with the convention $\widehat{\Phi}^{\theta^{-1}\omega}(\widehat{\eta}_{-1}^N) = \Phi^{\theta^{-1}\omega}(\widehat{\eta}_{-1}^N) = \mu_0$.

Now for $n \geq 0$ consider

$$\mathbb{C}_n(f^i, f^j) := \sum_{p=0}^{n} \eta_p^\omega\left[(f_p^i - \eta_p^\omega(f_p^i))(f_p^j - \eta_p^\omega(f_p^j))\right].$$

By Lemma 4 and (S.62), for any $n \geq 0$ and $t \in \mathbb{R}_+$, (and using $[Nt]/N \to t$)

$$C_n^N(f^{N,i}, f^{N,j}) \to \mathbb{C}_n(f^i, f^j),$$

and

$$\begin{aligned}
&\delta C_t^N(f^{N,i}, f^{N,j})\\
&\to \{t\}\eta_{[t]+1}^\omega\left[\left(f_{[t]+1}^i - \eta_{[t]+1}^\omega(f_{[t]+1}^i)\right)\left(f_{[t]+1}^j - \eta_{[t]+1}^\omega(f_{[t]+1}^j)\right)\right],
\end{aligned}$$

both in probability as $N \to \infty$. Defining

$$\mathbb{C}_t(f^i, f^j) := \mathbb{C}_{[t]}(f^i, f^j) + \{t\}\left(\mathbb{C}_{[t]+1}(f^i, f^j) - \mathbb{C}_{[t]}(f^i, f^j)\right)$$

we have proved that for any $t \in \mathbb{R}_+$,

$$\sum_{k=1}^{[Nt]+N} \mathsf{E}_N^\omega\left[U_k^N(f^{N,i})U_k^N(f^{N,j})|\mathcal{H}_{k-1}^N\right] \longrightarrow \mathbb{C}_t(f^i, f^j).$$

in probability as $N \to \infty$.

We now need to verify the conditional Lindeberg condition is satisfied: we shall prove that for any $t \in \mathbb{R}_+$ and $\epsilon > 0$,

$$\sum_{k=1}^{[Nt]+N} \mathsf{E}_N^{\omega}\left[\left|U_k^N(f^N)\right|^2 \mathbb{I}[\left|U_k^N(f^N)\right| > \epsilon] | \mathcal{H}_{k-1}^N\right] \to 0, \tag{S.63}$$

in probability as $N \to \infty$. To this end, fix $t \in \mathbb{R}_+$, set $n = [t]+1$ and define $\overline{\|\varphi\|}_n := \vee_{0 \le p \le n, 1 \le q \le m, 1 \le i \le d} \left\|\varphi_{p,q}^i\right\|$. Now for any $1 \le k \le [Nt]+N$, setting $p = [(k-1)/N]$, we have by hypothesis of the lemma that $\beta_{p,q}^N$ is measurable w.r.t. $\mathcal{H}_{k-1}^N$ for any $q \in \{1, \dots, m\}$, and therefore

$$\begin{aligned}
&\mathsf{E}_N^{\omega}\left[\left|U_k^N(f^N)\right|^2 \mathbb{I}[\left|U_k^N(f^N)\right| > \epsilon] | \mathcal{H}_{k-1}^N\right] \\
&\le \frac{4}{N} \sum_{i=1}^d \sum_{q=1}^m \left|\beta_{p,q}^{N,i}\right|^2 \left\|\varphi_{p,q}^i\right\|^2 \mathbb{I}\left[2\left(\sum_{i=1}^d \sum_{q=1}^m \left|\beta_{p,q}^{N,i}\right|^2 \left\|\varphi_{p,q}^i\right\|^2\right)^{1/2} > \epsilon\sqrt{N}\right] \\
&\le \frac{4}{N} \overline{\|\varphi\|}_n^2 \sum_{i=1}^d \sum_{q=1}^m \left|\beta_{p,q}^{N,i}\right|^2 \mathbb{I}\left[\epsilon^{-1} N^{-1/2} 2 \overline{\|\varphi\|}_n \left(\sum_{i=1}^d \sum_{q=1}^m \left|\beta_{p,q}^{N,i}\right|^2\right)^{1/2} > 1\right] \\
&\le \frac{8}{\epsilon N^{3/2}} \overline{\|\varphi\|}_n^{3/2} \left(\sum_{i=1}^d \sum_{q=1}^m \left|\beta_{p,q}^{N,i}\right|^2\right)^{3/2},
\end{aligned}$$

so in turn,

$$\begin{aligned}
&\sum_{k=1}^{[Nt]+N} \mathsf{E}_N^{\omega}\left[\left|U_k^N(f^N)\right|^2 \mathbb{I}[\left|U_k^N(f^N)\right| > \epsilon] | \mathcal{H}_{k-1}^N\right] \\
&\le \frac{8}{\epsilon\sqrt{N}} \overline{\|\varphi\|}_n^{3/2} \sum_{p=0}^n \left(\sum_{i=1}^d \sum_{q=1}^m \left|\beta_{p,q}^{N,i}\right|^2\right)^{3/2}.
\end{aligned} \tag{S.64}$$

The right hand side of (S.64) converges to zero in probability as $N \to \infty$ due to (S.62) and the continuous mapping theorem. This establishes (S.63), as required.

Therefore $\{X_t^N(f^N); t \in \mathbb{R}_+\}$ converges in law to a continuous Gaussian martingale $\{X_t(f); t \in \mathbb{R}_+\}$ such that for any $t \in \mathbb{R}_+$ and $1 \le i, j \le d$,

$$\left\langle X(f^i), X(f^j)\right\rangle_t = \mathbb{C}_t(f^i, f^j).$$

The proof is complete upon noting for $n \in \mathbb{N}$, $X_n^N(f^N) = \sqrt{N} M_n^{\omega,N}(f^N)$. $\square$

PROOF OF THEOREM 2. We first address convergence of the unnormalized measures. Keeping in mind (S.53), consider the decomposition:

$$\begin{aligned}
&\gamma_n^{\omega,N}(\varphi) - \gamma_n^{\omega}(\varphi) \\
&= \sum_{p=0}^{n} \gamma_p^{\omega,N} Q_{n-p}^{\theta^p\omega}(\varphi) - \gamma_{p-1}^{\omega,N} Q_{n-p+1}^{\theta^{p-1}\omega}(\varphi) \\
&= \sum_{p=0}^{n} \gamma_p^{\omega,N}(1) \left[\widehat{\eta}_p^N - \frac{\widehat{\eta}_p^N(\psi^{\theta^p\omega})}{\Phi^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right)(\psi^{\theta^p\omega})} \Phi^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right) \right] Q_{n-p}^{\theta^p\omega}(\varphi) \\
&= \sum_{p=0}^{n} \gamma_p^{\omega,N}(1) \left[\widehat{\eta}_p^N - \Phi^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right) \right] (g_{p,n}^{\omega,N}) \\
\text{(S.65)} &= \mathcal{R}_n^{\omega,N} + \mathcal{M}_n^{\omega,N},
\end{aligned}$$

with the conventions that $\gamma_{-1}^{\omega,N} Q^{\theta^{-1}\omega} = \Phi^{\theta^{p-1}\omega}\left(\widehat{\eta}_{-1}^N\right) = \Gamma_N^{\theta^{p-1}\omega}\left(\widehat{\eta}_{-1}^N\right) = \mu_0$; and where

$$g_{p,n}^{\omega,N} := Q_{n-p}^{\theta^p\omega}(\varphi) - \frac{\psi^{\theta^p\omega}}{\Phi^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right)(\psi^{\theta^p\omega})} \Phi^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right) Q_{n-p}^{\theta^p\omega}(\varphi),$$

with the convention $\psi^{\omega}/\Phi^{\omega}\left(\widehat{\eta}_{-1}^N\right)(\psi^{\omega}) = 1$; and

$$\mathcal{M}_n^{\omega,N} := \sum_{p=0}^{n} \gamma_p^{\omega,N}(1) \left[\widehat{\eta}_p^N - \Gamma_N^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right) \right] (g_{p,n}^{\omega,N}),$$

$$\begin{aligned}
\mathcal{R}_n^{\omega,N} &:= \sum_{p=0}^{n} \gamma_p^{\omega,N}(1) \left[\Gamma_N^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right) - \Phi^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right) \right] (g_{p,n}^{\omega,N}) \\
&= \frac{1}{N} \sum_{p=0}^{n} \gamma_p^{\omega,N}(1) \left[\widehat{\Phi}^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right) - \Phi^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right) \right] (g_{p,n}^{\omega,N}) \\
&= \frac{1}{N} \sum_{p=0}^{n} \gamma_p^{\omega,N}(1) \widehat{\Phi}^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right) (g_{p,n}^{\omega,N}),
\end{aligned}$$

where $\Phi^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right)(g_{p,n}^{\omega,N}) = 0$ has been used. Lemma 4 and the continuous mapping theorem ensure that $\gamma_n^{\omega,N}(1) \to \gamma_n^{\omega}(1)$ and $\sqrt{N}\mathcal{R}_n^{\omega,N} \to 0$ in probability as $N \to \infty$. By Slutsky's lemma it remains to prove the convergence in distribution of $\sqrt{N}\mathcal{M}_n^{\omega,N}$. This is achieved using Lemma 10 and a programme of arguments similar to [2, Proof of Proposition 9.4.1.]; therefore some steps are only summarised.

For $0 \le p \le n$, define

$$V_p^N := \sqrt{N}\left[\widehat{\eta}_p^N - \Gamma_N^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right)\right]\left(g_{p,n}^{\omega,N}\right).$$

We shall first apply Lemma 10 to address the convergence in law of the random vector $\left(V_0^N,\ldots,V_n^N\right)$. For the quantities in (S.61) let $m=2$, $d=n+1$ and for $i=0,\ldots,n$, set

$$\beta_{p.1}^{N,i+1} := \mathbb{I}[p=i],$$
$$\beta_{p.2}^{N,i+1} := \mathbb{I}[p=i]\frac{\Phi^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right)Q_{n-p}^{\theta^p\omega}(\varphi)}{\Phi^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right)\left(\psi^{\theta^p\omega}\right)},$$

$$\varphi_{p,1}^{i+1} := \mathbb{I}[p=i]Q_{n-p}^{\theta^p\omega}(\varphi),\ 0\le p\le n,$$
$$\varphi_{p,2}^{i+1} := \mathbb{I}[p=i]\psi^{\theta^p\omega},\ 1\le p\le n,$$

and $\beta_{0.2}^{N,i+1}=\varphi_{0,2}^{i+1}=\mathbb{I}[i=0]$. Then by construction, for $p=0,\ldots,n$,

$$\begin{aligned}
&M_n^{\omega,N}(f^{N,p+1})\\
&=\left[\widehat{\eta}_p^N - \Gamma_N^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right)\right]\left(\sum_{q=1}^m \beta_{p,q}^{N,p+1}\varphi_{p,q}^{p+1}\right)\\
&=\left[\widehat{\eta}_p^N - \Gamma_N^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right)\right]\left(Q_{n-p}^{\theta^p\omega}(\varphi)+\psi^{\theta^p\omega}\frac{\Phi^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right)Q_{n-p}^{\theta^p\omega}(\varphi)}{\Phi^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right)\left(\psi^{\theta^p\omega}\right)}\right)\\
&=\left[\widehat{\eta}_p^N - \Gamma_N^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right)\right]\left(g_{p,n}^{\omega,N}\right)\\
&=V_p^N/\sqrt{N},
\end{aligned}$$

where $M_n^{\omega,N}(f^{N,p+1})$ is the $p+1$th coordinate of $M_n^{\omega,N}(f^N)$ appearing in Lemma 10. Thus by Lemma 10, the random vector $\left(V_0^N,\ldots,V_n^N\right)$ converges in law to a centered Gaussian random vector, say $(V_0,\ldots,V_n)$. Then using the again the fact that for each n, $\gamma_n^N(1)$ converges to $\gamma_n(1)$ in probability, Slutsky's lemma together with the continuous mapping theorem ensure that $\sqrt{N}\mathcal{M}_n^{\omega,N}$ converges in law to $\sum_{p=0}^n \gamma_p(1)V_p$, and by exactly similar arguments to [2, Proof of Corollary 9.4.1.], the components of $(V_0,\ldots,V_n)$ are independent, so $\sum_{p=0}^{n}\gamma_p(1)V_p$ is a Gaussian random variable with variance as given in the statement of the theorem.

Now for the normalised measures. Consider the decomposition:

$$\begin{aligned}\widehat{\eta}_n^N\left(\varphi-\eta_n^\omega(\varphi)\right) &= \widehat{\eta}_0^N\overline{Q}_{0,n}^\omega(\bar{\varphi})+\sum_{p=1}^{n}\widehat{\eta}_p^N\overline{Q}_{p,n}^\omega(\bar{\varphi})-\widehat{\eta}_{p-1}^N\overline{Q}_{p-1,n}^\omega(\bar{\varphi})\\ &= \mathcal{R}_n^{\omega,N}+\mathcal{M}_n^{\omega,N},\end{aligned}$$

where $\bar{\varphi}=\varphi-\eta_n^\omega(\varphi)$ and

$$\begin{aligned}\mathcal{M}_n^{\omega,N} &:= \sum_{p=0}^{n}\left[\widehat{\eta}_p^N-\Gamma_N^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right)\right]\overline{Q}_{p,n}^\omega(\bar{\varphi})\\ \mathcal{R}_n^{\omega,N} &:= \sum_{p=1}^{n}\Gamma_N^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right)\overline{Q}_{p,n}^\omega(\bar{\varphi})-\widehat{\eta}_{p-1}^N\overline{Q}_{p-1,n}^\omega(\bar{\varphi})\\ &= \sum_{p=1}^{n}\frac{1}{N}\left[\widehat{\Phi}^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right)-\Phi^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right)\right]\overline{Q}_{p,n}^\omega(\bar{\varphi})\\ &\quad+\sum_{p=1}^{n}\left(1-\frac{\widehat{\eta}_{p-1}^N(G^{\theta^{p-1}\omega})}{\eta_{p-1}^\omega(G^{\theta^{p-1}\omega})}\right)\Phi^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right)\overline{Q}_{p,n}^\omega(\bar{\varphi}),\end{aligned}$$

with the convention $\Gamma_N^{\theta^{-1}\omega}\left(\widehat{\eta}_{-1}^N\right)=\mu_0$. Lemma 10 provides the desired convergence in distribution of $\sqrt{N}\mathcal{M}_n^{N,\omega}(\bar{\varphi})$, so by Slutsky's lemma, it remains only to prove that $\sqrt{N}\mathcal{R}_n^{\omega,N}\to 0$ in probability. To this end, noticing that $\eta_p^\omega\overline{Q}_{p,n}^\omega(\bar{\varphi})=0$, we have

$$\begin{aligned}&\mathsf{E}_N^\omega\left[\left|\mathcal{R}_n^{\omega,N}\right|\right]\\ &\leq\sum_{p=0}^{n}\frac{2}{N}\left\|\overline{Q}_{p,n}^\omega(\bar{\varphi})\right\|\\ &+\sum_{p=0}^{n}\left(\eta_{p-1}^\omega(G^{\theta^{p-1}\omega})^{-1}\mathsf{E}_N^\omega\left[\left|\left[\eta_{p-1}^\omega-\widehat{\eta}_{p-1}^N\right](G^{\theta^{p-1}\omega})\right|^2\right]^{1/2}\right.\\ &\qquad\left.\cdot\,\mathsf{E}_N^\omega\left[\left|\left[\Phi^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right)-\eta_p^\omega\right]\overline{Q}_{p,n}^\omega(\bar{\varphi})\right|^2\right]^{1/2}\right),\end{aligned}$$

and

$$\begin{aligned}
&\mathsf{E}_N^{\omega}\left[\left|\left[\Phi^{\theta^{p-1}\omega}\left(\widehat{\eta}_{p-1}^N\right)-\eta_p^{\omega}\right]\overline{Q}_{p,n}^{\omega}(\bar{\varphi})\right|^2\right]^{1/2}\\
\leq\ &\mathsf{E}_N^{\omega}\left[\left|\frac{\widehat{\eta}_{p-1}^N\overline{Q}_{p-1,n}^{\omega}(\bar{\varphi})}{\widehat{\eta}_{p-1}^N\left(G^{\theta^{p-1}\omega}\right)}\right|^2\left|1-\frac{\widehat{\eta}_{p-1}^N\left(G^{\theta^{p-1}\omega}\right)}{\eta_{p-1}^{\omega}\left(G^{\theta^{p-1}\omega}\right)}\right|^2\right]^{1/2}\\
&+\frac{1}{\eta_{p-1}^{\omega}\left(G^{\theta^{p-1}\omega}\right)}\mathsf{E}_N^{\omega}\left[\left|\left[\widehat{\eta}_{p-1}^N-\eta_{p-1}^{\omega}\right]\overline{Q}_{p-1,n}^{\omega}(\bar{\varphi})\right|^2\right]^{1/2}\\
\leq\ &\frac{\left\|\overline{Q}_{p,n}^{\omega}(\bar{\varphi})\right\|}{\eta_{p-1}^{\omega}\left(G^{\theta^{p-1}\omega}\right)}\mathsf{E}_N^{\omega}\left[\left|\left[\eta_{p-1}^{\omega}-\widehat{\eta}_{p-1}^N\right]\left(G^{\theta^{p-1}\omega}\right)\right|^2\right]^{1/2}\\
&+\frac{1}{\eta_{p-1}^{\omega}\left(G^{\theta^{p-1}\omega}\right)}\mathsf{E}_N^{\omega}\left[\left|\left[\widehat{\eta}_{p-1}^N-\eta_{p-1}^{\omega}\right]\overline{Q}_{p-1,n}^{\omega}(\bar{\varphi})\right|^2\right]^{1/2},
\end{aligned}$$

where for the final inequality

$$\left|\widehat{\eta}_{p-1}^N\overline{Q}_{p-1,n}^{\omega}(\bar{\varphi})/\widehat{\eta}_{p-1}^N\left(G^{\theta^{p-1}\omega}\right)\right|\leq\widehat{\eta}_{p-1}^N\left(G^{\theta^{p-1}\omega}\right)\left\|\overline{Q}_{p,n}^{\omega}(\bar{\varphi})\right\|/\widehat{\eta}_{p-1}^N\left(G^{\theta^{p-1}\omega}\right)$$

has been used. Then by application of Lemma 9 we conclude that there exists a constant $b_{\omega}(n)$ such that

$$N\mathsf{E}_N^{\omega}\left[\left|\mathcal{R}_n^{\omega,N}\right|\right]\leq\|\varphi\|\,b_{\omega}(n),$$

which implies, via Markov's inequality, that $\sqrt{N}\mathcal{R}_n^{\omega,N}\to 0$ in probability, as required. $\square$

We now turn to the proof of Proposition 5. As per (S.17), we will work with the non-negative kernel

$$\widetilde{\mathbf{R}}(\omega,x,dx')=\mathbf{G}(\omega,x)^2\phi^{\omega}(x,x')^2\widetilde{\mathbf{M}}(\omega,x,dx'),$$

In order to prove Proposition 5 we shall introduce the kernel

$$\mathbf{S}(\omega,x,dx'):=\frac{1}{\left[\mathbf{h}^{\omega}(x)\right]^2}\widetilde{\mathbf{R}}(\omega,x,dx')\left[\mathbf{h}^{\theta\omega}(x')\right]^2,$$

which may be regarded as a randomized similarity transform of $\widetilde{\mathbf{R}}$. Then writing, in the usual fashion,

$$\mathbf{S}_n^{\omega}:=\mathbf{S}^{\omega}\cdots\mathbf{S}^{\theta^{n-1}\omega},\quad n\geq 1,\tag{S.66}$$

it is clear that

$$\mathbf{S}_n^{\omega}(1)(x) = \frac{1}{[\mathbf{h}^{\omega}(x)]^2}\widetilde{\mathbf{R}}_n^{\omega}\left(\left[\mathbf{h}^{\theta^n\omega}\right]^2\right)(x),$$

and we have

LEMMA 11. *Assume* ***(H1)***, ***(H2)*** *and*

$$\sup_{\omega,\omega' x,x'} \frac{\psi^{\omega}(x)}{\psi^{\omega'}(x')} < \infty.$$

Then for any $x \in \mathsf{X}^N$

$$\lim_{n\to\infty} \frac{1}{n}\log \mathbf{S}_n^{\omega}(1)(x) = \lim_{n\to\infty} \frac{1}{n}\log \widetilde{\mathbf{R}}_n^{\omega}(1)(x), \quad \mathbb{P}-a.s.$$

PROOF. Using the bound $\sup_{\omega,\omega' x,x'} h^{\omega}(x)/h^{\omega'}(x')$ of Proposition 2, we have

$$\begin{aligned}\frac{1}{n}\log \mathbf{S}_n^{\omega}(1)(x) &= \frac{1}{n}\log \widetilde{\mathbf{R}}_n^{\omega}\left(\left[\mathbf{h}^{\theta^n\omega}\right]^2\right)(x) - \frac{2}{n}\log\left[\mathbf{h}^{\omega}(x)\right]\\ &\le \frac{1}{n}\log \widetilde{\mathbf{R}}_n^{\omega}(1)(x) + \frac{c}{n},\end{aligned}$$

for some finite constant c which does not depend on ω or n. This bound, combined with a similar lower one, complete the proof. ☐

The second component in the proof of Proposition 5 is the following Lemma, whose proof we briefly postpone.

LEMMA 12. *Assume* ***(H2)***,

$$\sup_{\omega,\omega' x,x'} \frac{\psi^{\omega}(x)}{\psi^{\omega'}(x')} < \infty,$$

and fix $\omega \in \Omega$. *Then for any* $N \ge 2$ *and* $x \in \mathsf{X}^N$,

$$\begin{aligned}&\left|\frac{\mathbf{S}^{\omega}(1)(x)}{\lambda_{\omega}^2} - 1\right|\\ &\le \frac{1}{N-1}\left[2\sup_{(z,z')\in\mathsf{X}^2}\frac{\psi^{\theta\omega}(z)}{\psi^{\theta\omega}(z')} - 1\right]\\ &\quad\cdot \sup_{z\in\mathsf{X}^N}\left(\frac{\mathbf{M}^{\omega}\left(\boldsymbol{\psi}^{\theta\omega}\right)(z)}{\mathbf{M}^{\omega}(\mathbf{h}^{\theta w})(z)}\right)\sup_{(z,z')\in\mathsf{X}^2}\left(\frac{h^{\theta\omega}(z)}{\psi^{\theta\omega}(z)} - \frac{h^{\theta\omega}(z')}{\psi^{\theta\omega}(z')}\right)\end{aligned}$$

where λ_{ω} *is as in Proposition 2.*

Proof of Proposition 5. Let us choose the initial distribution μ_0 to be the eigen-measure η^ω defined in Proposition 2. Iterative application of the equation $\eta^\omega Q^\omega = \lambda_\omega \eta^{\theta\omega}$ shows that

$$\mathsf{E}^\omega\left[\prod_{p=0}^{n-1} G^{\theta^p\omega}(X_p)\right] = \prod_{p=0}^{n-1}\lambda_{\theta^p\omega},$$

and then by Lemma 11, Proposition 4, (S.66) and the property that θ preserves $\mathbb{P}$, we have

$$\begin{aligned}
\Upsilon_N(\widetilde{\mathbf{M}}) &= \lim_{n\to\infty} \quad \frac{1}{n}\log \mathbf{S}_n^\omega(1)(x) - \frac{2}{n}\sum_{p=0}^{n-1}\log\lambda_{\theta^p\omega}, \\
&\leq \log\left[\mathbb{P} - ess\ \sup_\omega\left(\sup_{x\in\mathsf{X}}\frac{\mathbf{S}^\omega(1)(x)}{\lambda_\omega^2}\right)\right],
\end{aligned}$$

where the limit holds for $\mathbb{P}$ almost all ω. Applying Lemma 12, and noticing $\sup_{z\in\mathsf{X}^N}\mathbf{M}^\omega(\boldsymbol{\psi}^{\theta\omega})(z)/\mathbf{M}^\omega(\mathbf{h}^{\theta w})(z) \leq \sup_{x\in\mathsf{X}}\psi^{\theta\omega}(x)/h^{\theta\omega}(x)$, the proof is complete. □

Proof of Lemma 12. At various places in the proof we shall write, for some suitable function φ, $\mathrm{osc}(\varphi) := \sup_{x,y}|\varphi(x)-\varphi(y)|$.

The starting point is the expression:

$$\begin{aligned}
\frac{\mathbf{S}^\omega(1)(x)}{\lambda_\omega^2} - 1 &= \frac{\mathbf{G}^\omega(x)^2}{\lambda_\omega^2\mathbf{h}^\omega(x)^2}\mathbf{M}^\omega(\boldsymbol{\psi}^{\theta\omega})(x)^2\int_{\mathsf{X}^N}\widetilde{\mathbf{M}}^\omega(x,dz)\frac{\mathbf{h}^{\theta\omega}(z)^2}{\boldsymbol{\psi}^{\theta\omega}(z)^2} - 1 \\
&= \frac{\mathbf{M}^\omega(\boldsymbol{\psi}^{\theta\omega})(x)\mathbf{G}^\omega(x)^2}{\mathbf{h}^\omega(x)^2\lambda_\omega^2}\int_{\mathsf{X}^N}\mathbf{M}^\omega(x,dz)\frac{\mathbf{h}^{\theta\omega}(z)^2}{\boldsymbol{\psi}^{\theta\omega}(z)} - 1 \\
&= \frac{\mathbf{M}^\omega(\boldsymbol{\psi}^{\theta\omega})(x)}{\mathbf{M}^\omega(\mathbf{h}^{\theta w})(x)^2}\int_{\mathsf{X}^N}\mathbf{M}^\omega(x,dz)\frac{\mathbf{h}^{\theta\omega}(z)^2}{\boldsymbol{\psi}^{\theta\omega}(z)} - 1. \\
&= \int_{\mathsf{X}^N}\frac{\mathbf{M}^\omega(x,dz)\mathbf{h}^{\theta\omega}(z)}{\mathbf{M}^\omega(\mathbf{h}^{\theta w})(x)}\left[\frac{\mathbf{M}^\omega(\boldsymbol{\psi}^{\theta\omega})(x)}{\mathbf{M}^\omega(\mathbf{h}^{\theta w})(x)}\frac{\mathbf{h}^{\theta\omega}(z)}{\boldsymbol{\psi}^{\theta\omega}(z)} - 1\right] \qquad \text{(S.67)} \\
&= \int_{\mathsf{X}^N}\widetilde{\mathbf{M}}^\omega(x,dz)\left[\frac{d\widetilde{\mathbf{M}}^\omega_{\mathrm{opt}}(x,\cdot)}{d\widetilde{\mathbf{M}}^\omega(x,\cdot)}(z) - 1\right]^2.
\end{aligned}$$

where $\mathbf{M}^\omega(\mathbf{h}^{\theta w}) = \lambda_\omega\mathbf{h}^w/\mathbf{G}^\omega$ has been used, and the final equality, included only for purposes of exposition, is valid with the Markov kernel $\widetilde{\mathbf{M}}^\omega_{\mathrm{opt}}(x,dz) = \mathbf{M}^\omega(x,dz)\mathbf{h}^{\theta\omega}(z)/\mathbf{M}^\omega(\mathbf{h}^{\theta w})(x)$.

The main strategy of the proof is to introduce, for each $\omega \in \Omega$, two judiciously chosen Markov kernels, $K^\omega : \mathsf{X}^N \times \mathcal{X}^{\otimes 2} \to [0,1]$ and $L^\omega : \mathsf{X}^N \times \mathcal{X}^{\otimes 2} \to [0,1]$ such that we can write

$$\begin{aligned}
&\int_{\mathsf{X}^N} \frac{\mathbf{M}^\omega(x,dz)\mathbf{h}^{\theta\omega}(z)}{\mathbf{M}^\omega(\mathbf{h}^{\theta w})(x)} \left[\frac{\mathbf{M}^\omega(\boldsymbol{\psi}^{\theta\omega})(x)}{\mathbf{M}^\omega(\mathbf{h}^{\theta w})(x)} \frac{\mathbf{h}^{\theta\omega}(z)}{\boldsymbol{\psi}^{\theta\omega}(z)} - 1 \right] \\
&= \int_{\mathsf{X}^2} \left[K^\omega(x,dz^{1:2}) - L^\omega(x,dz^{1:2}) \right] f^\omega(x,z^2), \qquad \text{(S.68)}
\end{aligned}$$

where

$$f^\omega : (x,z) \in \mathsf{X}^N \times \mathsf{X} \quad \longmapsto \quad \frac{\mathbf{M}^\omega(\boldsymbol{\psi}^{\theta\omega})(x)}{\mathbf{M}^\omega(\mathbf{h}^{\theta w})(x)} \frac{h^{\theta\omega}(z)}{\psi^{\theta\omega}(z)} - 1 \ \in \ \mathbb{R}, \qquad \text{(S.69)}$$

and such that we can control the magnitude of (S.68) using an estimate of $\|K^\omega(x,\cdot) - L^\omega(x,\cdot)\|_{tv}$. Now for any ω and x, by definition $\mathbf{M}^\omega(x,\cdot)$ is a symmetric measure. For $1 \le q \le N$ and $x \in \mathsf{X}^N$, we shall write $\mathbf{M}^\omega_{(q)}(x,\cdot)$ for the marginal of $\mathbf{M}^\omega(x,\cdot)$ over the first $q \le N$ coordinates.

The first Markov kernel we introduce is:

$$K^\omega(x,dz^{1:2}) := \sum_{k=1}^N \sum_{l=1}^N \int_{\mathsf{X}^\mathsf{N}} \frac{1}{N} \frac{\mathbf{M}^\omega(x,d\mathfrak{z}) h^{\theta\omega}(\mathfrak{z}^k)}{\mathbf{M}^\omega(\mathbf{h}^{\theta w})(x)} \frac{\psi^{\theta\omega}(\mathfrak{z}^l)}{\sum_{j=1}^N \psi^{\theta\omega}(\mathfrak{z}^j)} \delta_{(\mathfrak{z}^k,\mathfrak{z}^l)}(dz^{1:2}),$$

where here, and henceforth, $\mathfrak{z} = (\mathfrak{z}^1,\ldots,\mathfrak{z}^N) \in \mathsf{X}^\mathsf{N}$. Elementary manipulations then yield

$$\begin{aligned}
&\int_{\mathsf{X}^2} K^\omega(x,dz^{1:2}) f^\omega(x,z^2) \\
&= \int_{\mathsf{X}^\mathsf{N}} \frac{\mathbf{M}^\omega(x,dz)\mathbf{h}^{\theta\omega}(z)}{\mathbf{M}^\omega(\mathbf{h}^{\theta w})(x)} \left[\frac{\mathbf{M}^\omega(\boldsymbol{\psi}^{\theta\omega})(x)}{\mathbf{M}^\omega(\mathbf{h}^{\theta w})(x)} \frac{\mathbf{h}^{\theta\omega}(z)}{\boldsymbol{\psi}^{\theta\omega}(z)} - 1 \right],
\end{aligned}$$

where f^ω is as in (S.69).

The second Markov kernel is:

$$L^\omega(x,dz^{1:2}) := \frac{\mathbf{M}^\omega_{(2)}(x,dz^{1:2}) h^{\theta\omega}(z^1) \psi^{\theta\omega}(z^2)}{\mathbf{M}^\omega(\mathbf{h}^{\theta w})(x) \mathbf{M}^\omega(\boldsymbol{\psi}^{\theta\omega})(x)},$$

so that then

$$\int_{\mathsf{X}^2} L^\omega(x,dz^{1:2}) f^\omega(x,z^2) = 0.$$

Furthermore, we have:

$$\begin{aligned}
&K^{\omega}(x, dz^{1:2}) \\
&= \frac{1}{N}\sum_{k=1}^{N}\int_{\mathsf{X}^N} \frac{\mathbf{M}^{\omega}(x, d\mathfrak{z})h^{\theta\omega}(\mathfrak{z}^k)}{\mathbf{M}^{\omega}(\mathbf{h}^{\theta w})(x)} \frac{\psi^{\theta\omega}(\mathfrak{z}^k)}{\sum_{j=1}^{N}\psi^{\theta\omega}(\mathfrak{z}^j)}\delta_{(\mathfrak{z}^k,\mathfrak{z}^k)}(dz^{1:2}) \\
&\quad + \left[\left(1-\frac{1}{N}\right)\frac{\mathbf{M}^{\omega}_{(2)}(x, dz^{1:2})h^{\theta\omega}(z^1)\psi^{\theta\omega}(z^2)}{\mathbf{M}^{\omega}(\mathbf{h}^{\theta w})(x)}\right. \\
&\qquad \left.\cdot\int_{\mathsf{X}^{N-2}} \frac{\mathbf{M}^{\omega}_{(N-2)}(x, d\mathfrak{z}^{1:N-2})}{N^{-1}\left[\psi^{\theta\omega}(z^1)+\psi^{\theta\omega}(z^2)+\sum_{j=1}^{N-2}\psi^{\theta\omega}(\mathfrak{z}^j)\right]}\right] \\
&\geq \left(1-\frac{1}{N}\right)\frac{\mathbf{M}^{\omega}_{(2)}(x, dz^{1:2})h^{\theta\omega}(z^1)\psi^{\theta\omega}(z^2)}{\mathbf{M}^{\omega}(\mathbf{h}^{\theta w})(x)} \\
&\qquad \cdot\int_{\mathsf{X}^{N-2}} \mathbf{M}^{\omega}_{(N-2)}(x, d\mathfrak{z}^{1:N-2})\left[N^{-1}\bar{\psi}(z^1, z^2, \mathfrak{z}^{1:N-2})\right]^{-1}
\end{aligned} \tag{S.70}$$

where in the final display $\bar{\psi}(z^1, z^2, \mathfrak{z}^{1:N-2}) = \psi^{\theta\omega}(z^1)+\psi^{\theta\omega}(z^2)+\sum_{j=1}^{N-2}\psi^{\theta\omega}(\mathfrak{z}^j)$.

Then, noting that $L^{\omega}(x,\cdot) \ll K^{\omega}(x,\cdot)$, for $\mathbf{M}^{\omega}_{(2)}(x,\cdot)$ – almost all $z^{1:2} \in \mathsf{X}^2$,

$$\begin{aligned}
&\frac{dL^{\omega}(x,\cdot)}{dK^{\omega}(x,\cdot)}(z^{1:2}) \\
&\leq \left(1-\frac{1}{N}\right)^{-1}\frac{1}{\mathbf{M}^{\omega}(\boldsymbol{\psi}^{\theta\omega})(x)}\frac{1}{N} \\
&\quad \cdot\left(\int \mathbf{M}^{\omega}_{(N-2)}(x, d\mathfrak{z}^{1:N-2})\left[\psi^{\theta\omega}(z^1)+\psi^{\theta\omega}(z^2)+\sum_{j=1}^{N-2}\psi^{\theta\omega}(\mathfrak{z}^j)\right]^{-1}\right)^{-1} \\
&\leq \left(1-\frac{1}{N}\right)^{-1}\frac{1}{\mathbf{M}^{\omega}(\boldsymbol{\psi}^{\theta\omega})(x)}\frac{1}{N} \\
&\quad \cdot\left[\psi^{\theta\omega}(z^1)+\psi^{\theta\omega}(z^2)+\int\sum_{j=1}^{N-2}\psi^{\theta\omega}(\mathfrak{z}^j)\mathbf{M}^{\omega}_{(N-2)}(x, d\mathfrak{z}^{1:N-2})\right] \\
&= 1+\frac{1}{N-1}\left[\frac{\psi^{\theta\omega}(z^1)+\psi^{\theta\omega}(z^2)}{\mathbf{M}^{\omega}(\boldsymbol{\psi}^{\theta\omega})(x)}-1\right],
\end{aligned}$$

where the first inequality uses (S.70) and the second is due to Jensen's inequality.

Since $L^\omega(x,\cdot) \ll K^\omega(x,\cdot)$, we then have, with the set

$$A = \{z^{1:2} \in \mathsf{X}^2 : \frac{dL^\omega(x,\cdot)}{dK^\omega(x,\cdot)}(z^{1:2}) \geq 1\},$$

$$\begin{aligned}
\|L^\omega(x,\cdot) - K^\omega(x,\cdot)\|_{tv} &= \int_A \left[\frac{dL^\omega(x,\cdot)}{dK^\omega(x,\cdot)}(z^{1:2}) - 1\right] K^\omega(x, dz^{1:2}) \\
&\leq \int_A \frac{1}{N-1}\left[\frac{2\sup_{\mathfrak{z}} \psi^{\theta\omega}(\mathfrak{z})}{\mathbf{M}^\omega(\boldsymbol{\psi}^{\theta\omega})(x)} - 1\right] K^\omega(x, dz^{1:2}). \\
&\leq \frac{1}{N-1}\left[\frac{2\sup_{\mathfrak{z}} \psi^{\theta\omega}(\mathfrak{z})}{\mathbf{M}^\omega(\boldsymbol{\psi}^{\theta\omega})(x)} - 1\right]. \qquad \text{(S.71)}
\end{aligned}$$

The tv-norm can also be expressed as

$$\begin{aligned}
&\|K^\omega(x,\cdot) - L^\omega(x,\cdot)\|_{tv} \\
&= \sup_{\{\varphi:\mathrm{osc}(\varphi)\leq 1\}} \left|\int \left[K^\omega(x, dz^{1:2}) - L^\omega(x, dz^{1:2})\right] \varphi(z^{1:2})\right|
\end{aligned}$$

and combining this with (S.67), (S.68) and (S.71) we obtain

$$\begin{aligned}
&\left|\frac{\mathbf{S}^\omega(1)(x)}{\lambda_\omega^2} - 1\right| \\
&= \left|\int \left[K^\omega(x, dz^{1:2}) - L^\omega(x, dz^{1:2})\right] f^\omega(x, z^2)\right| \\
&\leq \frac{1}{N-1}\left[2 \sup_{z,z'\in\mathsf{X}} \frac{\psi^{\theta\omega}(z)}{\psi^{\theta\omega}(z')} - 1\right] \mathrm{osc}\left(f^\omega(x,\cdot)\right) \\
&\leq \frac{1}{N-1}\left[2 \sup_{z,z'\in\mathsf{X}} \frac{\psi^{\theta\omega}(z)}{\psi^{\theta\omega}(z')} - 1\right] \sup_x \left(\frac{\mathbf{M}^\omega(\boldsymbol{\psi}^{\theta\omega})(x)}{\mathbf{M}^\omega(\mathbf{h}^{\theta w})(x)}\right) \mathrm{osc}\left(\frac{h^{\theta\omega}}{\psi^{\theta\omega}}\right).
\end{aligned}$$

□

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