MATH10013 Probability and Statistics (first half) 2023-24

Homepage `https://people.maths.bris.ac.uk/~maotj/prob.html`

Problem Sheet for tutorial in Week 2: Elementary Probability and Counting Arguments

Questions marked with a star are to be handed in and marked by your tutor.
Questions marked with a cross **may** be covered in problems class.

X1. We roll a fair black four-sided die and a fair white four-sided die.

(a) Describe the sample space Ω, and count how many elementary outcomes are in it.

(b) List and count the elementary outcomes that form the four events

$$\begin{aligned} A &= \{ \text{ black die has odd number}\}, \quad B = \{ \text{ white die has even number}\}, \\ C &= \{ \text{ black die has square number }\}, \quad D = \{ \text{ white die has odd number}\}. \end{aligned}$$

(c) Express the events 'A and B', B or C', 'B and D' in set-theoretic notation. For each of these events, list and count the elementary outcomes.

*2. Again, consider rolling a white and a black four-sided die and write
$B = \{$ white die has even number$\}$ and $C = \{$ black die has square number $\}$.

(a) By listing and counting elementary outcomes, show that $|B \cup C| = |B| + |C| - |B \cap C|$.

(b) List the elementary outcomes that form the events B and $B^c \cap C$. Show that these two events are disjoint, and check that their union is equal to $B \cup C$. Verify that the result $|B \cup C| = |B| + |B^c \cap C|$ holds.

(c) List and count the elementary outcomes that form the event that the black die has a strictly lower score than the white die.

X3. (a) Let Ω be the sample space and let A and B be events (i.e. subsets of Ω). Let C denote the event that exactly one of A and B occurs. Draw a Venn diagram for C and write down an expression for C in terms of unions, intersections and complements involving the events A and B.

(b) Now let $\mathbb{P}$ be a probability measure defined on the events of Ω. Use the axioms to prove an expression for $\mathbb{P}(C)$ in terms of $\mathbb{P}(A), \mathbb{P}(B)$ and $\mathbb{P}(A \cap B)$.

(c) The probability that Alice will fail the Probability examination is 0.12. The probability that Bob will fail it is 0.19. The probability that they will both fail is 0.05.

i. Find the probability that at least one of the two will fail.

ii. Find the probability that both pass.

4. Let Ω be a sample space and let A_1, A_2 and A_3 be events of Ω.

(a) For any two events A and B we know that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. By writing $A_1 \cup A_2 \cup A_3$ as $A_1 \cup (A_2 \cup A_3)$ and using this result, prove that

$$\mathbb{P}(A_1 \cup A_2 \cup A_3) = \mathbb{P}(A_1) + \mathbb{P}(A_2 \cup A_3) - \mathbb{P}((A_1 \cap A_2) \cup (A_1 \cap A_3)).$$

(b) Use the result twice more to prove inclusion-exclusion formula for $\mathbb{P}(A_1 \cup A_2 \cup A_3)$.

*5. Let E and F be two events with $\mathbb{P}(E) \geq 0.8$ and $\mathbb{P}(F) \geq 0.6$. Show using inclusion-exclusion that $\mathbb{P}(E \cap F) \geq 0.4$. [Note: you cannot assume that $\mathbb{P}(E \cap F) = \mathbb{P}(E)\mathbb{P}(F)$]

6. Suppose that one card is to be selected from a deck of twelve cards that contains six red cards numbered from 1 to 6 and six blue cards numbered from 1 to 6. Let A be the event that the number on the card is odd, let B be the event that the card is red and let C be the event that the number on the card is less than 4.

   (a) Write down the sample space Ω for this experiment, and give the subset of Ω corresponding to each of the events A, B and C.

   (b) For each of the following events, write down the subset of Ω, mark them on a Venn diagram and give an expression in terms of unions, intersections and complements involving events A, B and C:

      i. Both A and B occur but not C.
      ii. At least one of A, B and C occur.
      iii. Exactly one of the events A, B and C occurs.

X7. A fair die is thrown repeatedly until a six is obtained. Let A_k be the event that the first six occurs on the kth throw. Assume that $\mathbb{P}(A_k) = \frac{1}{6}\left(\frac{5}{6}\right)^{k-1} \qquad k = 1, 2, \ldots.$

   Let B be the event that an even number of throws are required to obtain a six. Express B in terms of $A_1, A_2, \ldots$. Hence find $\mathbb{P}(B)$.

*8. A certain town with a population of $100\,000$ has three newspapers: I, II, and III. The proportions of townspeople who read these papers are as follows:

I: 26%	I and II: 6%	I and II and III: 2%
II: 18%	I and III: 9%	
III: 22%	II and III: 5%	

   Find the number of people who do not read any of these newspapers.

9. A box (or urn) initially contains one red and one white ball. An infinite sequence of balls is drawn from the box. Each time a ball is drawn this ball is replaced together with another white ball. Thus when the nth ball is drawn there is the one red ball and n white balls present. Let A_n be the event that the nth ball drawn is the red ball.

   (a) Let B be the event that the red ball is drawn at least once. Write down an expression for B in terms of the A_n.

   (b) For $m \geq 1$, let C_m be the event that the red ball is drawn at least once after the mth drawn. Write down an expression for C_m in terms of A_n.

(c) Let D be the event that the red ball is drawn infinitely often. Write down an expression for D in terms of the C_m.

10. (a) In how many ways can six people sit in a row?
    (b) In how many ways can three girls and three boys sit in a row if girls sit next to each other and boys sit next to each other?
    (c) In how many ways can three girls and three boys sit in a row if only boys are required to sit next to each other?
    (d) In how many ways can three girls and three boys sit in a row if no two children of the same sex are allowed to sit next to each other?

X11. Out of 5 women and 7 men we wish to form a committee of 2 women and 3 men.

(a) In how many ways can we do that?

(b) And how many ways if two of the 7 men refuse to serve together?

*12. Out of 8 women and 6 men we wish to form a committee of 3 women and 3 men.

(a) In how many ways can we do that?

(b) How many ways if two of the 6 men refuse to serve together?

(c) How many ways if two of the 8 women refuse to serve together?

(d) How many ways if one of the women and one of the men refuse to serve together?

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Problem Sheet for tutorial in Week 4: Counting arguments and conditional probability

Questions marked with a star are to be handed in and marked by your tutor.
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1. Alice and Bob step in the lift on the ground floor of a 11-storey building. We assume that they both choose a floor 1 through 10 each with equal chance, independently of each other. What is the probability that Alice goes higher than Bob?

*2. I have 4 Maths, 3 Chemistry, 2 History and one French language book (no two are identical). In how many ways can I order them on my bookshelf if I want books of the same topics to stay next to each other?

X3. On a horse race of 7 horses, let S denote the event that *Star* is among the first three, and M the event that *Magic* has finished in an even position. Assuming each outcome to be equally likely, find the probability of $S \cap M$ and of $S \cup M$.

4. Six people, named A, B, C, D, E, F sit in line randomly. What is the probability that exactly i people sit between A and B, $i = 0,\ 1,\ 2,\ 3,\ 4$?

X5. 6 women and 6 men are randomly divided into two groups of sizes 6 each. What is the probability that each of the groups will have exactly 3 women and 3 men?

*6. 19 deer live in a forest, out of which 6 are tagged. Later on, 5 random deer of the 19 are captured. What is the probability that among those captured, 2 will be tagged? What assumptions are you making?

7. 12 ladies and 13 gentlemen attend a dance class. In how many ways can the instructor select 6 ladies, 6 gentlemen, and then pair them?

*8. Out of n people we want to form a committee of k, one of whom will be the president of the committee.

   (a) By picking the committee first, then its president, argue that there are $\binom{n}{k}\,k$ ways to do this.

   (b) By choosing the non-president members of the committee first, then picking the president from the remaining people, argue that there are $\binom{n}{k-1}\,(n-k+1)$ many ways.

   (c) Pick the president first, then select the non-president members, and argue that the number of ways is $n\,\binom{n-1}{k-1}$.

   (d) Draw the conclusion:

   $$\binom{n}{k}\,k = \binom{n}{k-1}\,(n-k+1) = n\,\binom{n-1}{k-1}.$$

   Prove this also analytically (by writing out all the factorials).

X9. An urn contains 5 red, 7 green and 8 yellow balls. Drawing 5 without replacement, what is the probability that we have each of the three colours at least once?

10. (a) I am from a family of two children. What is the probability the other child is a girl?

    (b) The King of Elbonia comes from a family of two children. What is the probability the other child is a girl? (Note: In Elbonia, the oldest boy in the family is made King).

*11. We roll three fair dice. Write the events $A = \{\text{at least one die shows a six}\}$ and $B = \{\text{all three dice are different}\}$. Calculate the probabilities $\mathbb{P}(B)$ and $\mathbb{P}(A^c \cap B)$. Deduce the values of $\mathbb{P}(A \cap B)$ and of $\mathbb{P}(A|B)$.

12. We repeatedly roll two four-sided (tetrahedron) dice at the same time, and stop when at least one of them shows a four. What is the probability that the other one also shows a four? *HINT: it is not $\frac{1}{4}$. List the relevant reduced sample space.* .

13. We roll a red, a blue and a yellow die (each fair). Denote the numbers they show by R, B and Y, respectively.

    (a) What is the probability that the three numbers are all different?

    (b) Given that all three numbers are different, what is the probability that $R < B < Y$?

    (c) Determine $\mathbb{P}(R < B < Y)$.

14. Each of three balls, independently, was painted gold or black with equal probability. The balls were then put in an urn.

    (a) Suppose we see that the black paint was used, that is, at least one of the three balls is black. Then what is the probability that all of the other balls are also painted black?

    (b) Instead, suppose now that the urn tilts, one of the three balls rolls out, and we see it is black. Now what is the probability that all of the other balls are also painted black?

    Explain your answer.

15. Once upon a time Odysseus met a junction of three roads. One of them lead to Athens, the other lead to Mycenae, and the third lead to Sparta, but he didn't know which route goes to which city. He chose one of the routes by rolling a die, giving equal chance to each of these choices. Athenians only tell the truth in one case out of three, Mycenae citizens lie every second time, but people of Sparta are always honest. In the city he arrived, he asked the first person he met how much two times two was, and got four as the answer. What is the probability that Odysseus was in Athens?

*16. An insurance company classifies people as: good risks, average risks or bad risks. Their records indicate that the probabilities that good, average, and bad risk persons will be involved in an accident over a 1-year span are, respectively, 0.05, 0.15, and 0.3. We know that 20% of the population are good risks, 50% are average risks, and 30% are bad risks.

Hint: Let F_G, F_A, F_B respectively be the events that a randomly selected person is good, average or bad risk. These form a partition of the sample space. Let also A be the event that the random person gets involved in an accident in a given year.

(a) What proportion of people have accidents in a particular year?

(b) If a policyholder had no accidents last year, what is the probability that (s)he is a good, average, or bad risk?

17. A student knows the correct answer for a question with probability p. To test this, an m-choice test is given to the student (one option is correct, the other $m - 1$ are false). If the student knows the correct answer they give it, otherwise the student guesses and answers each option with equal chance. Given they got the question right, what is the probability that the student knew the answer? What happens when $m = 1$ and when m is very large?

18. Alice applied to university, and her result will be announced when she is on holiday. She doesn't want bad news during her trip, so instructs her mum not to call her with bad news. However, if she got no call Alice would know that she has been rejected. To avoid this she also tells her mum that in case of good news a coin should be flipped, and she only wants a call if it is heads. Thus, even without the call, Alice cannot be sure she has been rejected.

    Let α be the probability that she is rejected, and β the probability that she is rejected given her mum doesn't call her. Calculate β in terms of α and confirm that $\beta \geq \alpha$.

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Problem Sheet for tutorial in Week 6: Discrete random variables, expectation and variance

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X1. Consider a multiple-choice exam with 4 possible answers for each of the 6 questions, where a student hasn't revised and just guesses at random. Argue that the number of correct answers they get will be binomially distributed. What is the probability that a student would get 5 or more correct answers just by guessing?

*2. Steve has not prepared for his exam, where he has to answer 10 yes or no questions and the pass mark is 8. He can give the correct answer to each question with probability 60%. What is the probability he will pass?

*3. 1 lottery scratchcard in 200 000 wins. In Bristol, 600,000 scratchcards are sold per week.

(a) Assuming that the total number of winning tickets is approximately Poisson, find the probability there will be at least 6 winning scratchcards in a particular week.

(b) Argue that the 'number of weeks in the next 12 with at least 6 winning scratchcards in Bristol' will be binomially distributed, with parameters you should specify. What is the probability that at least 2 out of the next 12 weeks will have at least 6 winning scratchcards?

(c) Counting the present week as number 1, what is the probability that the first week to have at least 6 winning scratchcards will be week number i, $i \geq 1$? (Hint: argue that the geometric distribution is relevant).

X4. Approximately 80 000 marriages took place in a country last year. Estimate the probability that for at least one of these couples

(a) both partners were born on April 30;

(b) both partners celebrate their birthday on the same day of the year.

*5. In the muffin factory each raisin has a small chance to end up in a particular muffin, and these raisins behave pretty much independently. Hence, we assume the number X of raisins in a muffin is well modelled by a Poisson distribution:

(a) How many raisins should there be in a muffin on average if we want at least one raisin in any given muffin with probability at least 0.95?

(b) We find that 4% of a certain type of muffins have no raisins in it. What is the probability that a muffin from this series has more than two raisins in it?

6. Each student on a test has to answer 20 yes or no questions. Assume that independently for each question, a student knows the correct answer with probability 0.7, believes that she knows the correct answer but is wrong with probability 0.1, and doesn't know the answer with probability 0.2. In this latter case she tosses a coin to determin his answer. What is the probability that she will answer at least 19 questions correctly?

X7. Five men and five women are ranked according to their scores on an examination. Assume that no two scores are alike and all $10!$ possible rankings are equally likely. Let X denote the highest ranking achieved by a woman (for instance, $X = 1$ if the top-ranked person is female). Find the probability mass function $\mathbb{P}(X = i)$, $i = 1, 2, \ldots, 10$ of X.

8. A man has n keys on his keyring, out of which only one opens a door. How many times is he expected to try keys if he tries them completely randomly, without excluding unsuccessful keys from his further trials?

9. Married couples in a community only have children until the first boy is born (we neglect cases of twins). What is the ratio of sexes in this community? Are sexes of children in a particular family independent?

X10. Rolling two four-sided (tetrahedron) dice, by writing down the mass function, what is the expected value of the higher and of the smaller of the two numbers shown?

*11. Let X be the number of sixes obtained when a fair die is thrown 3 times.

(a) What is the distribution of X? (Including all the relevant parameters)

(b) Write down a table giving the values of $\mathbb{P}(X = x)$ for $x = 0, 1, 2, 3$ (give each answer as a fraction with denominator 216), and show that these values sum to 1.

(c) Use the table derived in part (b) to numerically evaluate $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$, and find the variance $\mathrm{Var}\,(X)$. All these answers should be given as fractions in their lowest terms.

12. We draw randomly without replacement 2 balls from an urn that contains 4 red and 6 white balls. Denote by X the number of white balls drawn.

(a) Find the probability mass function $p_X(x) = \mathbb{P}(X = x)$ (for $x = 0, 1, 2$) of X and show that this adds to 1.

(b) Calculate the expectation and variance of X (as fractions in their lowest terms).

(c) What would the mean and variance of the number of white balls be if we sampled **with** replacement? (Hint: what would the distribution of the number of white balls be in this case?).

13. In a game a player bets on a number out of $1, 2, \ldots, 6$. Then three dice are rolled. If the number bet occurs i times, $i = 1, 2, 3$, then the player receives £i. If the number does not show up, then the player loses £1. Is this game fair (i.e. is $\mathbb{E}X = 0$)?

14. Let $X \sim \text{Poi}(\lambda)$. Show that

$$\mathbb{P}(X \text{ is even}) = \frac{1}{2}(1 + e^{-2\lambda}).$$

HINT: Use the fact that $\mathbb{E}(-1)^X = 2\mathbb{P}(X \text{ is even}) - 1$? (WHY?)

X15. There are two questions on a quiz show, one of them pays £A for the correct answer, the other pays £B. I can choose which question to answer first, but get knocked out of the game when I get a wrong answer. I know the correct answer for these questions with respective probabilities p_A and p_B. Find a condition on A, B, p_A and p_B giving the order I should answer them in to maximise my expected payoff?

*16. A fair coin is flipped 120 times, X denotes the number of Heads.

(a) Give an upper bound on the probability $\mathbb{P}(|X - 60| > 20)$ using Chebyshev's inequality, and hence deduce a lower bound on $\mathbb{P}(40 \le X \le 80)$.

(b) Use Chebyshev's inequality to calculate an upper bound on the probability that $\mathbb{P}(X = 0 \text{ or } 120)$. Give an exact expression for this probability, and comment on the usefulness of Chebyshev's inequality in this case.

17. **St. Petersburg paradox.** A coin is flipped until it first lands on heads. If this happens at the n^{th} flip then we win £2^n. Show that the expected value of our winnings is infinite.

(a) Would you pay 1 million Pounds to pay this game once?

(b) Would you pay 1 million Pounds to play this game as many times as you wish if you only have to settle after the last game?

18. Every night several meteorologists predict the probability of rain next day. To rate them, we score their predictions as follows. If a meteorologist reports probability p of rain next day, then score

- $1 - (1 - p)^2$ is given if it indeed rains,
- $1 - p^2$ is given if it does not rain

that day. Suppose a meteorologist thinks probability p^* of rain next day, what value of p should she report for maximising her expected score on this rating?

19. A man has n keys on his keyring, out of which only one opens a door. How many times is he expected to try keys if he tries them completely randomly, but excludes unsuccessful keys from his further trials?

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Problem Sheet for tutorial in Week 8: Joint distributions, properties of mean and variance

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X1. Suppose that two cards are drawn at random from a deck of cards without replacement. Let X be the number of aces obtained and let Y be the number of queens obtained.

(a) Obtain a table giving the joint probability mass function for X and Y. (For simplicity leave everything as a fraction with 52×51 in the denominator.)

(b) Obtain the marginal distribution of X.

(c) Obtain the conditional distribution of X given $Y = 0$.

(d) Are X and Y independent?

*2. A bag contains four dice labelled $1, \ldots, 4$. The die labelled x has x white faces and $(6-x)$ black faces, $x = 1, \ldots, 4$. A die is chosen at random from the bag and rolled. Define $X =$ the number labelling the chosen die and

$$Y = \begin{cases} 0 & \text{if the face showing on the die is black} \\ 1 & \text{if the face showing on the die is white} \end{cases}$$

(a) Construct a table displaying the values of the marginal probability mass function for X and a separate table displaying the values of the conditional probability mass function for Y given $X = x$, for general x.

(b) Write down the relationship between the marginal pmf for x, the conditional pmf for Y and the joint pmf. Hence construct a table showing the joint probability mass function for X and Y.

(c) Write down the relationship between the joint pmf, the marginal for Y and the conditional for X. Hence find the conditional probability mass function for X given $Y = 1$.

*3. We roll two fair dice, Determine the joint probability mass function and marginals of X and Y in the two cases where:

(a) X be the maximum of the two numbers and Y be the sum of the two numbers;

(b) X be the first number and Y the maximum of the two numbers;

X4. Consider an urn containing 4 red, 3 green and 6 blue balls. We draw two balls randomly (without replacement) from the urn, and write X for the number of red balls chosen and Y for the number of blue balls chosen. Find the joint probability mass function of X and Y, and calculate the marginal distribution of X.

Explain how you can find the marginal distribution of X more directly.

5. Find the probability mass function of the sum of $X \sim$ Bernoulli(p_1) and an independent $Y \sim$ Bernoulli(p_2) variable.

6. Let $p_1 \neq p_2$, $X \sim$ Geom(p_1) and $Y \sim$ Geom(p_2) be independent. Calculate the probability mass function of $X + Y$.

7. On a quiz show a team of n people has to answer a yes-or-no question, where n is odd. Each of the team members can give the correct answer with probability p, independently of each other. They have to decide upon the strategy:

   (a) either they pick one team member who will answer the question,

   (b) or they take a majority vote to come up with the team's answer.

   Show that 7a is preferable when $p < \frac{1}{2}$, it doesn't matter when $p = \frac{1}{2}$, and 7b is preferable when $p > \frac{1}{2}$. *HINT: Add two people to the team, thus increasing the number from n to $n+2$. Does the probability of a correct answer increase, stay, or decrease?*

8. A newsboy purchases papers at £1.00 and sells them at £1.50. However, he is not allowed to return unsold papers. If his daily demand is a binomial random variable with $n = 10$ and $p = 1/3$, approximately how many papers should he purchase so as to maximise his expected profit?

X9. If $\mathbb{E}(X) = 1$ and $\mathrm{Var}\ (X) = 5$, find (a) $\mathbb{E}[(2+X)^2]$ (b) $\mathrm{Var}\ (4+3X)$.

10. From a random variable X, subtract its mean and then divide the result by the standard deviation. What is the expected value and variance of the new random variable?

11. If X and Y are independent and identically distributed with variance σ^2, find $\mathbb{E}[(X-Y)^2]$.

*12. Roll two fair dice, and write U and V for the score on the first and second die respectively. Define $X = U - V$ and $Y = U + V$ and find $\mathrm{Cov}\ (X, Y)$. Are X and Y independent?

X13. We roll a fair die n times. Let X_i be the indicator of a 5 on the i^{th} roll, and Y_i be be indicator of a 6, $i = 1, \ldots, n$. Let X be the number of times we see 5 occurring and Y the number of times we see 6. Compute the correlation coefficient of X and Y.

X14. 5 students enter the lift on the ground floor of the Maths Building, and they each choose one of the floors 1...4 independently and randomly. Find the expectation and variance of the number of stops the lift makes.

   Hint: Use indicators for floors, and consider the number Y of floors where it doesn't stop.

15. Suppose $Y = aX + b$ and show $\varrho(X, Y) = \begin{cases} +1 & \text{if } a > 0, \\ -1 & \text{if } a < 0. \end{cases}$

*16. A fair coin is tossed n times. Define the random variables X and Y by

$$\begin{aligned} X &= \text{number of heads in the first } k \text{ tosses} \quad (k < n) \\ Y &= \text{number of heads in all } n \text{ tosses.} \end{aligned}$$

Define the random variable Z by $Z = Y - X$.

(a) Calculate $\mathbb{E}(X)$, $\mathbb{E}(Y)$ and $\mathbb{E}(Z)$.

(b) State the value of $\mathrm{Cov}\ (X, Z)$ and hence find $\mathrm{Cov}\ (X, Y)$ and the correlation coefficient $\varrho(X, Y)$.

(c) Why is it not the case that $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$?

*17. Suppose the random variable X can only take three values $-1, 0$ and 1 and that each of these three values has the same probability. Let the random variable Y be defined by $Y = X^2$. Show that

(a) X and Y are not independent,

(b) but X and Y are uncorrelated (i.e. $\mathrm{Cov}\ (X, Y) = 0$).

18. In a game of Roulette, a player bets £1, and loses his bet with probability 19/37, but is given his bet and an extra pound back with probability 18/37. Explain using the Weak Law of Large Numbers whether the casino loses money with this game in the long run.

19. **Monte Carlo integration.** Let $f\ :\ [0,\,1] \to \mathbb{R}$ be a function with $J\ := \int_0^1 |f(x)|^2 dx$ finite. We will numerically estimate integral $I\ := \int_0^1 f(x)dx$. Let $U_1,\ U_2,\ \ldots$ be i.i.d. $\mathrm{U}(0,\ 1)$ random variables and $I_n := \big(f(U_1) + f(U_2) + \cdots + f(U_n)\big)/n$.

(a) Show that $I_n \to I$ in the sense of the WLLN (this is called *in probability*).

(b) Use Chebyshev's inequality to estimate the probability $\mathbb{P}\left(|I_n - I| > \frac{a}{\sqrt{n}}\right)$ of error.

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Problem Sheet for tutorial in Week 10: Continuous random variables

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X1. A filling station is supplied with petrol once a week. If its weekly volume of sales in thousands of litres is a random variable X with probability density function

$$f_X(x) = \begin{cases} 5(1-x)^4, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$$

what should the capacity of the tank be, to ensure that the probability of the supply's being exhausted in a given week is 0.01?

*2. Compute the expectation and variance of a random variable X with density

$$f_X(x) = \begin{cases} 2x, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

3. You arrive at a bus stop at 10 o'clock, knowing that the bus will arrive at some time uniformly distributed between 10 and 10:30.

   (a) What is the probability that you will have to wait longer than 10 minutes?

   (b) If at 10:15 the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?

X4. The lifetime of the *probabilium* radioactive particle is exponentially distributed with mean value 3 years. What is its half-life (the amount of time required for half of the number of atoms of a sample to decay)?

*5. A certain type of radio has a lifetime which is exponential with mean 6 years. My friend and I both buy a used (but working) radio of this type. It turns out that mine was made 9 years ago while my friend's set is only 2 years old.

   (a) What is the probability that both my and my friend's radio will work 6 years from now?

   (b) What is the chance that my radio stops working before my friend's does?

X6. Which of the following functions can be a distribution function?

   (a) $F(x) = \begin{cases} 1 + e^{1-x}, & \text{if } x > -1, \\ 0, & \text{otherwise} \end{cases}$

(b) $F(x) = \begin{cases} 2 - \dfrac{2}{x+1}, & \text{if } x \geq 0, \\ 0, & \text{otherwise} \end{cases}$

(c) $F(x) = \begin{cases} 1 - e^{-x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise} \end{cases}$

(d) $F(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ \dfrac{x}{4} \cdot (4 - x), & \text{if } 0 < x \leq 2, \\ 1, & \text{if } x > 2 \end{cases}$

X7. Which of the following functions can be a probability density function?

(a) $f(x) = \begin{cases} \dfrac{2}{x} & \text{, if } x > 1, \\ 0 & \text{, otherwise} \end{cases}$

(b) $f(x) = \begin{cases} \dfrac{\sin(x)}{2} & \text{, if } 0 < x < 2, \\ 0 & \text{, otherwise} \end{cases}$

(c) $f(x) = \begin{cases} 2e^{-2x} & \text{, if } x > 0, \\ 0 & \text{, otherwise} \end{cases}$

8. Let $X \geq 0$ be a continuous random variable. By writing a double integral and exchanging the order of integration, prove $\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)dt$.

9. My friend from IT wants to generate a discrete uniform random variable on the set $\{1, 2, \ldots, n\}$ (that is, one taking on each of the numbers $1, 2, \ldots, n$ with equal chance). To this order he first takes a $U \sim \text{Uniform}(0, 1)$ number as these are generally available from random generators. Then he does

$$X = \lfloor n \cdot U + 1 \rfloor$$

where $\lfloor \cdot \rfloor$ denotes the lower integer part. Is X the random variable he wants? Explain.

10. 2% of electric components of a given type break down within 1000 hours of operation. Assume that time before breakdown has an exponential distribution. What is the probability that such a component will work for longer than the average?

11. For a memoryless light bulb, the probability that it operates for more than 2000 hours is 2/3. In a city 20 of these light bulbs are installed.

    (a) What is the probability that after 1000 hours exactly 15 bulbs are operational?

    (b) What is the probability that after 1000 hours exactly 15 bulbs are operational and after an additional 500 hours exactly 13 are operational?

X12. Let $X \sim \text{Exp}(\lambda)$, and $c > 0$. By considering the distribution function or otherwise, show that $cX \sim \text{Exp}(\frac{\lambda}{c})$.

*13. Consider a standard normal random variable Z with density

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right).$$

Given real numbers a and b, define the random variable $X = g(Z) = aZ + b$. Calculate an explicit expression for the density f_X of X, and use this to identify the distribution of X (including its parameters).

14. Bob plays roulette in the casino. Every round he bets 10 tokens on 'red'. After 100 rounds he has 300 tokens fewer than when he started. Is it reasonable to think that the croupier is cheating? (On the game roulette one has 37 fields, numbered from 0 to 36. Out of these, '0' has colour green, and 18 fields are red, and 18 are black. Betting on 'red' pays 10 extra tokens when the roll is red, and loses the 10 tokens bet when it's not red.)

X15. We randomly select a point on the $[0, 1]$ interval of the x axis. Let D denote the distance between this point and the point at coordinate $(0, 1)$ of the plane. Determine the density function of the distribution of the random variable D.

16. Prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Here is a way to do it: substitute $y = \sqrt{2z}$ into the definition $\Gamma(\frac{1}{2}) = \int_0^\infty z^{-1/2} e^{-z}\, dz$, and compare with the standard normal density.

17. Compute the k^{th} moment of the Gamma(α, λ) distribution for any integer $k \geq 1$ and $\alpha, \lambda > 0$ reals. In particular, verify that $\mathbb{E}X = \alpha/\lambda$, $\text{Var } X = \alpha/\lambda^2$.

X18. If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, compute

(a) $\mathbb{P}(X < 8)$; (b) $\mathbb{P}(X > 5)$; (c) $\mathbb{P}(4 < X < 16)$.

19. Suppose that X is a normal random variable with mean (that is, expected value) 5. If $\mathbb{P}(X > 9) = 0.2$, approximately what is $\text{Var }(X)$?

*20. The time to do a problem sheet is well approximated by a normal random variable of mean $\mu = 280$ minutes and standard deviation $\sigma = 10$ minutes.

(a) What proportion of sheets take more than 290 minutes?

(b) Given that a sheet has taken more than 290 minutes, what is the probability it will take less than 300 minutes?

*21. Use the DeMoivre–Laplace theorem to estimate (to 2 decimal places) the probability that the number of 6s rolled is between 1460 and 1530, if we roll a fair die 9000 times.

22. How many times should a coin be tossed to make the probability of 'proportion of heads being between 47% and 53%' be at least 0.95? (Assume the normal approximation is valid)

MATH10013 Probability and Statistics (first half) 2023-24

Homepage `https://people.maths.bris.ac.uk/~maotj/prob.html`

Problem Sheet for tutorial in Week 12: Conditional expectation and moment generating functions

Questions marked with a star are to be handed in and marked by your tutor.
Questions marked with a cross **may** be covered in problems class.

X1. Let $X \sim \text{Poi}(\lambda)$ and $Y \sim \text{Poi}(\mu)$ be independent. Identify the distribution of $(X \mid X+Y)$, i.e. find and recognise conditional mass function $P_{X|X+Y}(i \mid n) = \mathbb{P}(X = i \mid X+Y = n)$.

Give the expectation of $\mathbb{E}(X|X+Y)$ and use the tower law to confirm the value of $\mathbb{E}X$.

2. Let X and Y be i.i.d. Geom(p) random variables. Calculate $\mathbb{P}(X = i \mid X+Y = n)$ for a fixed $n \geq 2$, and confirm that $\sum_i \mathbb{P}(X = i \mid X+Y = n) = 1$.

   Hint: recall from previous sheet that $\mathbb{P}(X+Y = n) = (n-1)p^2(1-p)^{n-2}$.

3. Let $Y \sim \text{Bin}(4, p)$ and assume that the conditional distribution of X given that $Y = y$ is Poi(y). Write down $\mathbb{E}(X|Y = y)$. Use the tower law $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|Y))$ to find $\mathbb{E}(X)$.

*4. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables. Calculate $\mathbb{E}(X_1 \mid X_1+X_2+\cdots+X_n = x)$. Hint: consider the quantity $\mathbb{E}(X_1+X_2+\cdots+X_n \mid X_1+X_2+\cdots+X_n = x)$.

5. The number of accidents that a person has in a given year is a Poisson random variable with parameter λ. However, suppose that the value of λ changes from person to person, being equal to 2 for 60 percent of the population, and 3 for the other 40 percent. A person is chosen at random. What is the probability that this person

   (a) has no accidents this year;

   (b) has exactly 3 accidents this year;

X6. Z students enter the elevator on the ground floor of the Maths Building, where $Z \sim \text{Poi}(\lambda)$. They each choose one of the floors 1,..., 4 independently and randomly. Let X be the number of stops the elevator makes and find $\mathbb{E}(X)$. What happens to $\mathbb{E}(X)$ as $\lambda \to \infty$?

*7. Rolling two dice, let X be the first number rolled and Y the larger number shown. Determine the conditional mass function $P_{X|Y}(x \mid y)$ for all relevant x, y values (hint: see week 8 sheet for the joint pmf).

For each $y = 1, \ldots, 6$ calculate $\mathbb{E}(X|Y = y)$, and use these values to calculate

$$\mathbb{E}(\mathbb{E}(X|Y)) = \sum_{y=1}^{6} \mathbb{P}(Y = y)\mathbb{E}(X|Y = y),$$

and confirm that the tower law holds in this case.

X8. Let X be a standard normal variable, and let I be independent of X with $\mathbb{P}(I = 1) = \mathbb{P}(I = 0) = 1/2$. Define

$$Y := \begin{cases} X, & \text{if } I = 1, \\ -X, & \text{if } I = 0. \end{cases}$$

(a) Show that Y is also standard normal.

(b) Are I and Y independent?

(c) Are X and Y independent?

(d) Show that $\mathrm{Cov}\,(X, Y) = 0$.

9. **Conditional covariance.** The *conditional covariance of X and Y, conditioned on Z* is defined as

$$\mathrm{Cov}\,(X, Y \mid Z) = \mathbb{E}\big[\big(X - \mathbb{E}(X \mid Z)\big) \cdot \big(Y - \mathbb{E}(Y \mid Z)\big) \mid Z\big].$$

(a) Show that

$$\mathrm{Cov}\,(X, Y \mid Z) = \mathbb{E}(XY \mid Z) - \mathbb{E}(X \mid Z) \cdot \mathbb{E}(Y \mid Z).$$

(b) Prove the conditional covariance formula

$$\mathrm{Cov}\,(X, Y) = \mathbb{E}[\mathrm{Cov}\,(X, Y \mid Z)] + \mathrm{Cov}\,[\mathbb{E}(X \mid Z), \mathbb{E}(Y \mid Z)].$$

X10. Calculate the moment generating function of the Binomial(n, p) distribution by direct computation.

*11. (a) Let $X \sim U(\alpha, \beta)$. Determine its moment generating function $M_X(t)$.

(b) Let Y be the number shown after rolling a fair die. Determine its moment generating function $M_Y(t)$.

(c) Now let $Z \sim U(0, 1)$ independent of the above Y and, using $M_{Y+Z}(t) = M_Y(t) \cdot M_Z(t)$, conclude that $Y + Z \sim U(1, 7)$.

*12. Let $X \sim \mathrm{Geom}(p)$, the number of trials until the first success in a sequence of independent experiments with success probability p. Let I be the indicator of the success of the first trial.

(a) What is the distribution of $(X \mid I = 1)$?

(b) What is the distribution of $(X \mid I = 0)$? (*Remember the memoryless property.*)

(c) By the Tower rule,

$$M(t) = \mathbb{E}e^{tX} = \mathbb{E}\mathbb{E}\big(e^{tX} \mid I\big).$$

Expand the right hand-side using your previous answers, then solve this equation for $M(t)$, thus determining the moment generating function of the Geom(p) distribution.

13. Let X have moment generating function $M(t)$, and let $\Psi(t) = \ln M(t)$. Show that

$$\Psi(t)\big|_{t=0} = 0, \qquad \Psi'(t)\big|_{t=0} = \mathbb{E}(X), \qquad \Psi''(t)\big|_{t=0} = \mathrm{Var}\ (X).$$

X14. Use the Central Limit Theorem to approximate the probability that the sum of 10 000 rolls of a fair die falls between 34 800 and 35 200.

15. A die is rolled until the total sum of all rolls exceeds 300. Approximate the probability that at least 80 rolls are necessary.

*16. What is the approximate probability that the sum of 50 independent and identically distributed random variables X_i falls in the interval $[0,\ 30]$ if the distribution of X_i

   (a) is uniform on $[0, 1]$

   (b) has density function $f(x) = 2x$ on $[0, 1]$

17. We have 100 lightbulbs whose lifetimes are independent exponentials with mean 5 hours. If the bulbs are used one at a time, with a failed bulb being replaced immediately by a new one, approximate the probability that there is still a working bulb after 525 hours.

18. The mean mark of a student on an exam is 74, and the standard deviation is 14 marks. 100 students take this exam. Estimate the probability that the average mark on this exam exceeds 75.