1. (a) (**9 marks**)

   i. Give an example of a closed and unbounded set $C \subseteq \omega_3$ with $C \neq \omega_3$.

   ii. Give an example of a limit cardinal κ with $\mathrm{cf}(\kappa) = \omega_1$, justifying your choice.

   iii. Let $\omega < \kappa < \lambda$ be cardinals, and let $f : \kappa \longrightarrow \lambda$ be a *cofinal map*. Show that $\beta = \bigcup_{\gamma<\kappa} X_\gamma$ for some sets X_γ for $\gamma < \kappa$, with the property that $|X_\gamma| < \lambda$.

   (b) (**5 marks**)
   Prove carefully that if $f : \alpha \longrightarrow \beta$ is cofinal and strictly increasing, then $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$.

   (c) (**6 marks**)
   What can you say about V_κ if κ is both regular and a fixed point of the $\beth$-function, *i.e.* with $\kappa = \beth_\kappa$, ?

   (d) (**5 marks**)
   Suppose λ is a regular limit cardinal. Discuss the possibilities for there being a fixed point of the $\beth$-function less than λ.

Q1

(a)

(i) Let $C_0 \subseteq \omega_3$ be any unbounded set (different from ω_3). Let C^* be the set of its closure points. Then C^* is closed and unbounded in ω_3.

(ii) $\aleph_{\omega_1}$ is by definition the ω_1'st infinite cardinal. So $f\colon \omega_1 \longrightarrow \aleph_{\omega_1}$ is cofinal where $f(\alpha) = \aleph_\alpha$.

(iii) Let f be the function hypothesized, and $X_\gamma = f(\gamma)$. Because f is cofinal, every $\eta < \lambda$ is in some such X_γ. Because λ is a cardinal $|X_\gamma| < \lambda$.

(b) Suppose f, α, β are as hypothesised. By definition of cf let g: $\mathrm{cf}(\alpha) \longrightarrow \alpha$ and h: $\mathrm{cf}(\beta) \longrightarrow \beta$ be (1-1) *strictly increasing* maps unbounded in α,β respectively. (We can assume this by a Lemma from the lectures.)

Then $f \circ g$ is cofinal into β and shows $\mathrm{cf}(\beta) \leq \mathrm{cf}(\alpha)$. To show $\mathrm{cf}(\alpha) \leq \mathrm{cf}(\beta)$ define k: $\mathrm{cf}(\beta) \longrightarrow \alpha$ by $k(\tau) =$ least ξ with $f(\xi) > h(\tau)$. s f, h are strictly increasing unbounded maps, k is cofinal into α.

(c)

Principally: (i) V_κ equals H_κ the class of sets hereditarily of cardinality $<\kappa$: $H_\kappa = \{x \mid |\mathrm{TC}(x)| < \kappa\}$.

(ii) s κ is such a fixed point we have that $\gamma < \kappa \longrightarrow 2^\gamma < \kappa$. This implies that $(\mathrm{ZFC})_{V_\kappa}$ since for any regular κ we have that $(\mathrm{ZFC}^-)_{H_\kappa}$.

(iii) We cannot prove in ZFC the existence of such κ for that would prove the existence of a transitive model of ZFC contradicting G del's Second Incompleteness theorem.

(d) This may or may not be true: it depends on the nature of λ: if λ is merely weakly inaccessible it is possible that $2^{\aleph_0} > \lambda$ in which case there is no such fixed point. If however λ is strongly inaccessible then V_λ is a model of ZFC. Since in ZFC we can prove that the $\beth$-function has fixed points, there must be such below λ in this latter case.