Axiomatic Set Theory Exercises: Selected Solutions

Exercise 1.1 In ST (= MATH32000 Set Theory) TC was given an alternative (but equivalent) definition, and was shown to satisfy the definition of TC above. Rework this by showing, using the above definition, that: (i) If $\mathrm{Trans}(t) \wedge x \subseteq t \longrightarrow \mathrm{TC}(x) \subseteq t$. Hence $\mathrm{TC}(x)$ is the $\subseteq$-smallest transitive set t satisfying $x \subseteq t$. Moreover $\mathrm{Trans}(x) \longleftrightarrow \mathrm{TC}(x) = x$. (ii) $x \in y \longrightarrow \mathrm{TC}(x) \subseteq \mathrm{TC}(y)$. (iii) Define by recursion on ω:

$$\bigcup^0 x = x; \bigcup^{n+1} x = \bigcup (\bigcup^n x); \quad \mathrm{tc}(x) = \bigcup \{\bigcup^n (x) \mid n < \omega\}. \text{ Show that } \mathrm{tc}(x) = \mathrm{TC}(x)$$

Solution: For (i) By $\in$-induction assume for all $y \in x$ we have proven:

" $\mathrm{TC}(y)$ *is the $\subseteq$-smallest transitive set t satisfying $y \subseteq t$*".

We do the same for x. We first show $\mathrm{Trans}(\mathrm{TC}(x))$. Suppose $z \in y \in \mathrm{TC}(x) =_{\mathrm{df}} x \cup \bigcup \{\mathrm{TC}(w) \mid w \in x\}$. If $y \in x$ then $\mathrm{TC}(y) \subseteq \mathrm{TC}(x)$ and $z \in y \subseteq \mathrm{TC}(y)$. Hence $z \in \mathrm{TC}(x)$. If $y \notin x$ then for some $w \in x$ we must have $y \in \mathrm{TC}(w)$. By the Inductive Hypothesis, $\mathrm{Trans}(\mathrm{TC}(w))$. Hence $z \in y \subseteq \mathrm{TC}(w) \subseteq \mathrm{TC}(x)$. Hence $\mathrm{Trans}(\mathrm{TC}(x))$. Now let $t \supseteq x$ be any other transitive set containing x. Let $y \in x$. As $x \subseteq t \wedge \mathrm{Trans}(t)$, we have $y \subseteq t$. By the Ind. Hyp. $\mathrm{TC}(y)$ is the smallest transitive set containing y. Hence $\mathrm{TC}(y) \subseteq t$. But then, as y was arbitrary, $\mathrm{TC}(x) = x \cup \bigcup \{\mathrm{TC}(y) \mid y \in x\} \subseteq t$ as required.

That "$\mathrm{Trans}(x) \longleftrightarrow \mathrm{TC}(x) = x$" is immediate from this.

(ii) is immediate: $\mathrm{TC}(y)$ by (the notes) definition is $y \cup \bigcup \{\mathrm{TC}(x) \mid x \in y\}$. Hence $x \in y \longrightarrow \mathrm{TC}(x) \subseteq \mathrm{TC}(y)$.

For (iii):

First note as well that: (a) $y \in x \longrightarrow \mathrm{tc}(y) \subseteq \mathrm{tc}(x)$ as it is easy to see that $\bigcup^n y \subseteq \bigcup^{n+1} x$; and (b) $z \in \bigcup^{n+1} x$ for some n, implies that $z \in \bigcup^n y$ for some $y \in x$. (Both these statements could be proven rigorously by a (metatheoretic) induction on n, but they are reasonably obvious from the chain-like nature of the definition of $\bigcup^{n+1} x$: $z \in \bigcup^{n+1} x \leftrightarrow \exists y \exists z_0 \ldots \exists z_n (z = z_n \in z_{n-1} \cdots z_0 \in y \in x)$.)

Now we show: $\mathrm{tc}(x) = \mathrm{TC}(x)$: Proof by $\in$-induction. Suppose $\mathrm{tc}(y) = \mathrm{TC}(y)$ for all $y \in x$.

($\supseteq$) Then $\mathrm{TC}(x) = x \cup \bigcup \{\mathrm{tc}(y) \mid y \in x\} \subseteq \mathrm{tc}(x)$ using (a).

($\subseteq$) However if $z \in \mathrm{tc}(x)$, then either $z \in \bigcup^0 x = x$ in which case immediately $z \in \mathrm{TC}(x)$) or else $z \in \bigcup^{n+1} x$ for some n, and hence $z \in \bigcup^n y$ for some $y \in x$ by (b). But then $z \in \mathrm{tc}(y) = \mathrm{TC}(y)$ (by the Inductive Hypothesis). But $\mathrm{TC}(y) \subseteq \mathrm{TC}(x)$ by definition of $\mathrm{TC}(x)$. Hence $z \in \mathrm{TC}(x)$.

Exercise 1.2 (i) Show that the relation $xRy \longleftrightarrow x \in \mathrm{TC}(y)$ is wellfounded. (ii) Show for any x that $\rho(x) = \rho(\mathrm{TC}(x))$; (iii) Let X be any set. Show that $\mathrm{Trans}(X) \to \rho$ "$X \in \mathrm{On}$.

Solution: (i) Let t be a non-empty term, and pick $y \in t$. If y is not R-minimal in t, let y_0 be ρ-minimal in $\mathrm{TC}(y) \cap t$. Then y_0 is R-minimal in t. [Check: $zRy_0 \longrightarrow z \in \mathrm{TC}(y_0)$. But the latter implies that $\rho(z) < \rho(y_0)$. (Check!) So $z \notin t$.]

(ii) As $x \subseteq \mathrm{TC}(x)$, $\rho(x) \leq \rho(\mathrm{TC}(x))$ is trivial. But note that if $x \subseteq V_\alpha$ then $\bigcup^k x \subseteq V_\alpha$ for any $k \in \omega$ also. Hence $\mathrm{TC}(x) \subseteq V_\alpha$ too, and so $\rho(x) \geq \rho(\mathrm{TC}(x))$.

(iii) Suppose there is X a transitive set, so that ρ " $X \notin \mathrm{On}$; by $\in$-induction we may choose such an X with $\rho(X)$ least. Then there is some $\gamma < \rho(X)$ with no $z \in X$ with $\rho(z) = \gamma$, whilst at the same time there are $u \in X$ with $\gamma < \rho(u)$. By part (iii) $\rho(u) = \rho(\mathrm{TC}(u))$ and by hypothesis as $\rho(u) < \rho(X)$, ρ " $\mathrm{TC}(u) \in \mathrm{On}$. So there is $z \in \mathrm{TC}(u)$ with $\rho(z) = \gamma$. But $\mathrm{TC}(u) \subseteq X$, so $z \in X$. This is a contradiction.

Exercise 1.4 Convince yourself of the truth of Lemma 1.23. [Hint: At least set out the base cases of the induction: suppose φ is $v_0 \in v_1$ and let $t_0 = x$, $t_1 = \{z \mid \psi\}$. Then $(x \in t_1)^W \leftrightarrow (x \in \{z \mid \psi\})^W \leftrightarrow \psi(x/z)^W \leftrightarrow x \in \{z \mid z \in W \wedge \psi^W\} \leftrightarrow x^W \in (t_1)^W$. The other base cases are relatively straightforward, but a little lengthy to write out. The inductive step for non-atomic formulae is easy by comparison.]

Solution: Assume that $(x \in t_1)^W$ has been proven as shown. We do the other cases in the following order: (a) $t_1 = t_2$, (b) $x = t_1$, c) $t_1 \in t_2$, then (d) $t_1 \in x$. Suppose in the following $t_1 = \{z \mid \psi\}$, as above, and $t_2 = \{y \mid \varphi\}$.

(a)
$$\begin{aligned}
& (t_1 = t_2)^W \leftrightarrow (\{z \mid \psi\} = \{y \mid \varphi\})^W \\
& \leftrightarrow (\forall w (w \in \{z \mid \psi(z)\} \leftrightarrow w \in \{y \mid \varphi(y)\}))^W \\
& \leftrightarrow \forall w \in W ((w \in \{z \mid \psi(z)\} \leftrightarrow w \in \{y \mid \varphi(y)\}))^W \\
& \leftrightarrow \forall w (w \in \{z \mid \psi(z)\}^W \leftrightarrow w \in \{y \mid \varphi(y)\}^W)
\end{aligned}$$

(using that Trans(W) in this last equivalence, and $\{z|\psi(z)\}^W, \{y|\varphi(y)\}^W \subseteq W$)

$\leftrightarrow \{z|\psi(z)\}^W = \{y|\varphi(y)\}^W \leftrightarrow t_1^W = t_2^W$.

(b) $(x = \{z|\psi\})^W \leftrightarrow (\forall y(y \in x \leftrightarrow y \in \{z|\psi\}))^W$

$\leftrightarrow \forall y \in W(y \in x \leftrightarrow y \in \{z|\psi\})^W$

$\leftrightarrow \forall y(y \in x \leftrightarrow (y \in \{z|\psi\})^W$

$\leftrightarrow x^W = \{z|\psi\}^W$.

(c) $(\{z|\psi\} \in \{y|\varphi\})^W \leftrightarrow (\exists x(x = \{z|\psi\} \wedge x \in \{y|\varphi\}))^W$

$\leftrightarrow (\exists x \in W(x = \{z|\psi\} \wedge x \in \{y|\varphi\}))^W$

$\leftrightarrow (\exists x \in W(x = \{z|\psi\} \wedge \varphi(x/y))^W$

$\leftrightarrow \exists x(x \in W \wedge x = \{z|\psi\}^W \wedge \varphi(x/y)^W)$

$\leftrightarrow \{z|\psi\}^W \in \{x \in W | \varphi^W\} = \{y|\varphi\}^W$.

(d) $(\{z|\psi\} \in x)^W \leftrightarrow (\exists y(y = \{z|\psi\} \wedge y \in x))^W$

$\leftrightarrow \exists y \in W((y = \{z|\psi\})^W \wedge (y \in x)^W)$ (but $x \subseteq W$ and so:)

$\leftrightarrow ((\exists y(y = \{z|\psi\}))^W \wedge y \in x) \leftrightarrow \{z|\psi\}^W \in x^W$.

Now the lemma is established for atomic formula with terms substituted, the inductive steps for the more complex formulae are straightforward. We only consider the case φ is $\exists x\psi(x, y_1, \ldots, y_n)$ with terms $t_1, \ldots, t_n$. Then:

$\varphi(t_1, \ldots, t_n)^W \leftrightarrow \exists x \in W(\psi(x, t_1, \ldots, t_n))^W \leftrightarrow \exists x \in W\psi(x, t_1^W, \ldots, t_n^W)^W \leftrightarrow (\exists x\psi(x, t_1^W, \ldots, t_n^W))^W$

with the inductive hypothesis used at second equivalence.

Exercise 1.5 Show (ii) and (iv) of Lemma 1.25: Let W be a transitive term.

(ii) If $\omega \in W$ then (Ax.Infinity)^W.

(iv) If for any $x \in W$, and term f with f^W is a function, then f^W“$x \in W$, then (AxReplacement)$_W$.

Solution: (ii) $(\exists x(\varnothing \in x \wedge \forall y(y \in x \longrightarrow y \cup \{y\} \in x)^W \leftrightarrow$

$\exists x \in W((\varnothing \in x \wedge \forall y(y \in x \longrightarrow y \cup \{y\} \in x))^W \leftrightarrow$

$\exists x \in W(((\varnothing \in x)^W \wedge \forall y \in W(y \in x \longrightarrow y \cup \{y\} \in x)^W)$.

But $\varnothing^W = \varnothing$, and $(y \cup \{y\} \in x)^W \leftrightarrow y \cup \{y\} \in x$ since $x \subseteq W$ as Trans(W). Taking x as ω which is in W we see the above last equivalence holds.

(iv) Suppose f given as hypothesized. We require (Fun(f) $\longrightarrow f$“$x \in V)^W$. We are given that Fun(f)^W and that f^W“$x \in W = V^W$. hence (with the help of Lemma 1.23 - meaning the Exercise ? 1.4 above on the atomic sentence of type $t_1 \in t_2$) we have $(f$“$x \in V)^W$.

Exercise 1.7 Let W be class term $\{\varnothing\}$. Which axioms of ZFC hold in $\langle W, \in\rangle$? Let U be the class term On. Which axioms of ZFC hold in $\langle U, \in\rangle$? (NB For the latter, $\langle$On$, \in\rangle$ just is $\langle$On$, <\rangle$ of course.)

Solution: $\langle W, \in\rangle$: Note that Trans(W) (hence AxExt. holds) and W has one member, namely $\varnothing$ (hence Ax.Empty holds). AxPair, Ax.Infinity, Ax. Power all fail. (For example, for Ax.Pairing: $(\{\varnothing, \varnothing\} = \{\varnothing\} \in V)^W$ is false.) The others hold more or less vacuously: Union, Foundation, Replacement, Separation. (For Foundation, note that the only non-empty term a is $V^W = W = \{\varnothing\}$. And $\exists x(x \in a \wedge x \cap a = \varnothing\}$. For Replacement, note that there are no terms f with f^W a function - since W has no ordered pairs.) Note that $\varnothing$ is a function: it is the empty function with empty domain! However technically, $\prod \varnothing = \{\varnothing\} \neq \varnothing$, (as the empty function $\varnothing$ itself satisfies the clauses for h to be in $\prod \varnothing$!) so AC (in our version) is vacuously true (but very silly).

$\langle U, \in\rangle$: Trans(On) so Ax.Ext. holds, as easily does Axs. Empty, Infinity, and Foundation. Similarly if $x \in$ On then $\bigcup x \in$ On. Hence Ax.Union holds. Ax.Pair fails, in particular, as the only unordered pair that exists is $\{\varnothing, \varnothing\} = 1$! Ax.Replacement thus holds vacuously in U (as the only term defining a function is $\varnothing$). For any $\alpha \in$ On, we have $(\mathcal{P}(\alpha) = \alpha + 1)^U$ (Check!) so AxPower holds. However Ax.Separation, fails. Does the Ax.Choice hold? This is Fun(f) $\wedge$ dom(f) $\in V \wedge \varnothing \notin$ ran(f) $\longrightarrow \prod f \neq \varnothing$. We've remarked that Fun(f) $\longrightarrow f = \varnothing$ is the only function term for U. As above $\prod \varnothing = \{\varnothing\} \neq \varnothing$ As $\{\varnothing\} \in$On it again looks like that Ax9 holds in U!

Exercise 1.6 Let $\varphi(v_0, \ldots, v_n)$ be any formula. Let $g_\varphi(\vec{y}) \approx$ the least β such that $\exists x\, \varphi(x, \vec{y}) \longrightarrow \exists x \in V_\beta\, \varphi(x, \vec{y})$ if such an x and β exist; let it be 0 otherwise. Show that $\forall \xi\, g_\varphi$“$V_\xi \in V$. Deduce that $f_\varphi(\xi) =_{\mathrm{df}} \sup(g_\varphi$“$V_\xi)$ is a welldefined function.

Sol: This is just a use of the AxRep.: g_φ is a function and its action on V_ξ, g_φ“V_ξ, yields a set for any ξ. Then the supremum of this set is an ordinal, and so f_φ is properly defined.

Exercise 1.8 Which axioms of ZFC hold in V_ω?

Sol: All axioms other than Ax. Infinity. (See Lemmata 1.24-1.26.)

Exercise 1.9 Check, or recheck, the following basic properties of the V_α using the Definitions 1.17, 1.18 of ρ and V_α:

(i) Trns(V_α) *and if* $x \in V_\alpha$ *then* $\forall y \in x (y \in V_\alpha \wedge \rho(y) < \rho(x))$;

(ii) $\alpha < \beta \longrightarrow V_\alpha \subseteq V_\beta$;

(iii) $V_{\alpha+1} = \mathcal{P}(V_\alpha)$;

(iv) *If* $x \in V$, *then* $\rho(x) =$ least α so that $x \subseteq V_\alpha =$ least α such that $x \in V_{\alpha+1}$.

(v) $\rho(\alpha) = \alpha$;

(vi) $\mathrm{On} \cap V_\alpha = \alpha$.

Solution: (i),(ii) are immediate from the definitions 1.17, 1.18.

For (iii): If $x \subseteq V_\alpha =_{\mathrm{df}} \{z | \rho(z) < \alpha\}$ then $\forall y \in x (\rho(y) < \alpha)$. Thus $\rho(x) =_{\mathrm{df}} \sup\{\rho(y) + 1 \mid y \in x\} \leq \alpha$. Hence $x \in V_{\alpha+1}$. Thus $V_{\alpha+1} \supseteq \mathcal{P}(V_\alpha)$; But $z \in V_{\alpha+1} \longleftrightarrow \rho(z) < \alpha + 1$, which implies that any $u \in z \in V_{\alpha+1}$ satisfies $\rho(u) + 1 \leq \rho(z) < \alpha + 1$. Hence $\rho(u) < \alpha$ and $u \in V_\alpha$. Thus $z \subseteq V_\alpha$. Hence $V_{\alpha+1} \subseteq \mathcal{P}(V_\alpha)$.

For (iv): By $\in$-induction. Assume as Ind. Hyp. that for any $y \in x$ the conclusions hold. Then by definition $\rho(x) = \sup\{\rho(y) + 1 | y \in x\} = \sup\{\alpha + 1 | y \in x \wedge y \in V_{\alpha+1}\}$ which is the least β with $x \subseteq V_\beta$. That this β is the least β for which $x \in V_{\beta+1}$ follows from (iii).

For (v): By induction on α: Trivial for $\alpha = 0$. Suppose true for $\beta < \alpha$: then $\rho(\alpha) =_{\mathrm{df}} \sup\{\rho(\beta) + 1 | \beta \in \alpha\} = \sup\{\beta + 1 | \beta \in \alpha\} = \alpha$.

(vi) is then immediate from (v) and Defs. 1.17, 1.18.

Exercise 1.10 Sol: That this is a total order is easy, even trivial, to see. We check that it is a wellorder. Let $X \neq \varnothing$ be a set of n-tuples of ordinals. Let $Z \subseteq X$ be the subset of X with least max. Now just pick the lexicographically least $\vec{\alpha}$ from Z . (That is, let $\vec{\alpha}_0$ be least first element of some $\vec{\alpha} \in Z$. Then let $\vec{\alpha}_1$ be least second element amongst those sequences starting out: $\vec{\alpha}_0$, $\vec{\alpha}_1$ etc for n-steps.)

Exercise 1.11*E Sol: first note that $<^*$ is the lexicographic ordering when restricted to *descending* finite sequences of ordinals p: $p_0 > p_1 > \cdots > p_k$, q: $q_0 > q_1 > \cdots > q_l$. So, given two sets p, q as above note that $p <^* q$ or $q <^* p$ depending on which is lexicographically first when written out in this descending fashion. So the ordering is a strict total ordering. Let $\varnothing \neq X$ be a set of finite sets of ordinals. Assume each $p \in X$ is enumerated in descending order. Let

$d_0 =$ least $\{\delta | \delta = p_0$ for some $p_0 \in p \in X\}$;

$d_1 =$ least $\{\delta | \delta = p_1$ for some $d_0 = p_0 > p_1 \in p \in X\}$. Then $d_1 < d_0$. Proceeding in this way, after a finite number of steps (as the ordinals are wellordered) we arrive at some $d_k < d_{k-1} \cdots < d_1 < d_0$ the minimal element of X.

(What is the order type of $\langle F(\omega + 1), <^* \rangle$, $\langle F(\omega^\omega + 1), <^* \rangle$ (where $F(\delta)$ is the set of all finite subsets of δ)?)

Exercise 2.1 Prove (iii) of Lemma 2.7 : *If* $f: \alpha \longrightarrow \beta$ *is cofinal and strictly increasing, then* $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$ and the corollary following: *If* $\mathrm{Lim}(\lambda)$ *then* $\mathrm{cf}(\omega_\lambda) = \mathrm{cf}(\lambda)$.

Sol: Suppose f, α, β are as hypothesised. By definition of cf let $g: \mathrm{cf}(\alpha) \longrightarrow \alpha$ and $h: \mathrm{cf}(\beta) \longrightarrow \beta$ be (1-1) strictly increasing maps unbounded in α,β respectively. (We can assume this by part (i) of Lem.2.7.) Then $f \circ g$ is cofinal into β and shows $\mathrm{cf}(\beta) \leq \mathrm{cf}(\alpha)$. To show $\mathrm{cf}(\alpha) \leq \mathrm{cf}(\beta)$ define $k: \mathrm{cf}(\beta) \longrightarrow \alpha$ by $k(\tau) =$ least ξ with $f(\xi) > h(\tau)$. As f, h are strictly increasing unbounded maps, k is cofinal into α.

Corollary: Let $\mathrm{Lim}(\lambda)$: since $F_\aleph \restriction \lambda: \lambda \longrightarrow \omega_\lambda$ is cofinal and strictly increasing this is immediate by what we have just shown.

Exercise 2.5 *Let* $f: \Omega \longrightarrow \Omega$ *be strictly increasing. Then* f *is normal iff* $\mathrm{ran}(f)$ *is c.u.b. in* Ω.

Sol: ($\Rightarrow$) Let $\xi < \Omega$. As f is increasing $f(\xi) \geq \xi$. Hence ran(f) is trivially unbounded in Ω. Let ζ be a limit ordinal with ran(f) unbounded in ζ. We require that $\zeta \in \text{ran}(f)$. Using the continuity of f we have that $f(\xi) = \zeta$, where $\xi = \sup\{\eta \mid f(\eta) < \zeta\}$.

($\Leftarrow$) Let ran(f) be cub in Ω. As we are told f is increasing we only have to check (ii) in the definition of normality. Let Lim(λ). Let $\tau = \sup f``\lambda$. But ran(f) is closed. Thus $\tau \in \text{ran}(f)$. Thus τ must be $f(\lambda)$, just as required by (ii) of normality.

Exercise 2.6 *For any $E \subseteq \text{On}$ define E^* to be the class of limit points of E: namely those limit ordinals β such that $E \cap \beta$ is unbounded in β. Show that E^* is a closed class, and if $E \in V$, with $\text{cf}(\sup(E)) > \omega$, then E^* is c.u.b. below $\sup(E)$.*

Sol: That E^* is closed is immediate: if $E^* \cap \beta$ is unbounded in β, then for any $\delta < \beta$ there is $\tau \in E^* \cap \beta$ with $\tau > \delta$. But as $\tau \in E^*$ then there is a $\tau' \in E \cap \tau$ with $\tau > \tau' > \delta$ too. Hence $E \cap \beta$ is also unbounded in β and thus $\beta \in E^*$. Suppose E is a set. Let $\eta_0 < \sup(E)$ be arbitrary. Any ω-sequence $\eta_0 < \eta_1 < \cdots < \eta_n < \cdots$ of elements from E has supremum in E^* which is below sup (E) as the latter has uncountable cofinality. Hence E^* is unbounded in $\sup(E)$. Closure was the first part.

Exercise 2.7 suppose Ω is a regular cardinal. (i) Let $C, D \subseteq \Omega$ be c.u.b.in Ω. Show that $C \cap D$ is c.u.b.in Ω. (ii) Now generalise this argument: let $\gamma < \Omega$. Let $\langle C_\xi \mid \xi < \gamma \rangle$ be a sequence of c.u.b.in Ω classes. Show that $\bigcap_{\xi<\gamma} C_\xi$ is c.u.b. in Ω.

Sol: (i) Let $\alpha < \Omega$ be arbitrary. Pick alternately: $\alpha \leq \alpha_0 < \beta_0 < \alpha_1 < \cdots < \alpha_n < \beta_n < \cdots$ for $n < \omega$ with $\alpha_n \in C$, $\beta_n \in D$. Let $\gamma = \sup\{\alpha_n\} = \sup\{\beta_n\}$. By closure of C and D $\gamma \in C \cap D$ provided that $\gamma < \Omega$. However this is true: if Ω is a regular cardinal it is because $\text{cf}(\Omega) = \Omega$ (and not ω!). Hence $C \cap D$ is unbounded in Ω. But it is also closed: if Lim(μ) and $(C \cap D) \cap \mu$ is unbounded in μ, then both $C \cap \mu$ and $D \cap \mu$ are unbounded in μ and thus $\mu \in C \cap D$. Hence $C \cap D$ is closed.

(ii): Let $\alpha < \Omega$ be arbitrary. Using repeatedly the fact that Ω is regular and $\gamma < \Omega$, we can successfully pick alternately: $\alpha \leq \alpha_0^0 < \alpha_1^0 < \alpha_2^0 < \cdots \alpha_\iota^0 < \cdots$ for $\iota < \gamma$ with $\alpha_\iota^0 \in C_\iota$. Let $\alpha^1 = \sup_\iota \alpha_\iota^0$. Then $\alpha^1 < \Omega$ again, as the latter is regular and $\gamma < \Omega$. Now repeat this again with $\alpha^1 \leq \alpha_0^1 < \alpha_1^1 < \alpha_2^1 < \cdots < \alpha_\iota^1 <$ for $\iota < \gamma$, and define $\alpha^2 = \sup\{\alpha_\iota^1 \mid \iota < \gamma\}$. Then $\alpha^2 < \Omega$. Etc for $n < \omega$. Let $\alpha^\omega = \sup_n \alpha^n < \Omega$ (once more by regularity of Ω). As $\alpha^\omega = \sup\{\alpha_\iota^n \mid n < \omega\}$ for any $\iota < \gamma$, then by closure of C_ι we have that $\alpha^\omega \in C_\iota$. Hence $\alpha^\omega \in \bigcap_{\xi<\gamma} C_\xi$. This shows the latter intersection is unbounded in Ω. Closure is immediate: if Lim(μ) then $((\bigcap_{\xi<\gamma} C_\xi) \cap \mu$ is unbounded in μ) implies $\forall \xi < \gamma$ ($C_\xi \cap \mu$ is unbounded in μ) which implies $\forall \xi < \gamma (\mu \in C_\xi)$.

Exercise 2.8 (Diagonal Intersections) *Let $\Omega \in \text{Reg}$. Let $\langle E_\xi \mid \xi < \Omega \rangle$ be a sequence of subsets of Ω. Define the diagonal intersection of the sequence to be the set $D =_{\text{df}} \Delta_{\xi<\Omega} \langle E_\xi \mid \xi < \Omega \rangle = \{\alpha < \Omega \mid \forall \beta < \alpha (\alpha \in E_\beta)\}$. Now suppose that the E_ξ are all c.u.b.in Ω. Show that (i) the diagonal intersection D is c.u.b.in Ω (ii) $D = \cap_{\alpha<\Omega}(E_\alpha \cup (\alpha + 1))$.*

Sol: (i) Closure is immediate again as in the last Exercise. ($x \subseteq D \longrightarrow \sup x \in E_\xi \cup \{\Omega\}$ for any $\xi < \sup x$.) Unboundedness: let $\alpha^0 < \Omega$ be arbitrary; by the unboundedness of $\bigcap_{\xi<\gamma} E_\xi$ for any $\gamma < \Omega$ (by the last Exercise) we may for any $\gamma < \Omega$, define $f(\gamma)$ to be the least element of $\bigcap_{\xi<\gamma} E_\xi$ above γ. Now define $\alpha^0 < \alpha^1 = f(\alpha^0) < \cdots < \alpha^{n+1} = f(\alpha^n)$, and $\alpha^\omega = \sup_n \alpha^n$; then check $\alpha^\omega \in D$. For (ii): Let $\beta \in D$. Then $\forall \gamma < \beta (\beta \in E_\gamma) \wedge \forall \gamma \geq \beta (\beta \in \gamma + 1)$. Conversely $\beta \in \cap_{\alpha<\Omega}(E_\alpha \cup (\alpha + 1))$ implies *a fortiori* that $\forall \gamma < \beta (\beta \in E_\gamma)$ and thus $\beta \in D$.

Exercise 2.9 *Show that $\forall \alpha (|V_{\omega+\alpha}| = \beth_\alpha)$.*

Sol: By induction on α: $|V_\omega| = \omega = \beth_0$; if $|V_{\omega+\alpha}| = \beth_\alpha$ is proven, then $|V_{\omega+\alpha+1}| = |\mathcal{P}(V_{\omega+\alpha})| = 2^{|V_{\omega+\alpha}|} = \beth_{\alpha+1}$; if $|V_{\omega+\alpha}| = \beth_\alpha$ for $\alpha < \lambda$, then $|V_{\omega+\lambda}| = |\bigcup_{\alpha<\lambda} V_{\omega+\alpha}| = \bigcup_{\alpha<\lambda} \beth_\alpha = \beth_\lambda$. (Remark: once $\alpha \geq \omega^2$ then we have $\omega + \alpha = \alpha$ in any case, so the annoying "$\omega +$" in the subscript can be dropped.)

Exercise 2.10 *(i) Show that the GCH (Generalised Continuum Hypothesis, that $\forall \alpha (2^{\aleph_\alpha} = \aleph_{\alpha+1})$) implies that $\forall \alpha (\aleph_\alpha = \beth_\alpha)$. (ii) Show that the first fixed point of the $\beth$ function has cofinality ω. (iii) Show that for any regular cardinal κ there is α, a fixed point of the $\beth$ function, with $\text{cf}(\alpha) = \kappa$. {Hint: this is simple, just consider an enumeration of the fixed points.]*

Sol: (i): This is almost by definition, GCH implies that $\forall\alpha\,(2^{\aleph_\alpha}=\aleph_\alpha^+)$, if, by induction we have proven $\beth_\alpha=\aleph_\alpha$, then $\beth_{\alpha+1}=2^{\beth_\alpha}=2^{\aleph_\alpha}=\aleph_{\alpha+1}$. Then $\beth_\lambda=\aleph_\lambda$ as both are defined continuously at limit λ.

(ii) Let $f(0)=\beth_0$; $f(n+1)=\beth_{f(n)}$; let $f(\omega)=\sup\{f(n)\,|\,n<\omega\}$. Then $\beth_{f(\omega)}=f(\omega)$ (For, if $\gamma<\beth_{f(\omega)}$ then $\gamma<\beth_{f(n)}=f(n+1)$ for some n, hence $\beth\gamma<f(\omega)$, hence $\beth_{f(\omega)}\leq f(\omega)$. As the $\beth$ -function is increasing, $\gamma\leq\beth_\gamma$ for all γ.) $f(\omega)$ obviously has cofinality ω. By construction we could not have any $\kappa<\beth_{f(\omega)}$ a smaller fixed point.

(iii) Just continue the construction of (ii) for κ many steps, then $\beth_{f(\kappa)}=f(\kappa)$, and $\mathrm{cf}(\beth_{f(\kappa)})=\kappa$.

Exercise 2.12 *Let $S\subseteq\Omega$ be stationary and $C\subseteq\Omega$ be c.u.b. Then $S\cap C$ is stationary.*

Sol: Let $D\subseteq\Omega$ be an arbitrary c.u.b. class. Then $D\cap C$ is c.u.b. Hence $(S\cap C)\cap D\neq\varnothing$ as required.

Exercise 2.13 *Let $S\subseteq\Omega$ be stationary. Show that $S\cap S^*$ is stationary.*

Sol: Immediate from the last Exercise, since S^* is c.u.b.

Exercise 2.14 *Can you generalise Example 1 to larger cardinals?*

Sol: For any $n\geq 2$ we could have $S_{\omega_k}=\{\alpha<\omega_n|\ \mathrm{cf}(\alpha)=\omega_k\}$ and these would be disjoint stationary sets for $k<n$. More generally for $\kappa>\lambda\geq\omega_0$, $\kappa,\lambda\in\mathrm{Reg}$, define $S_\lambda=\{\alpha<\kappa|\ \mathrm{cf}(\alpha)=\lambda\}$ and then for different such λ,λ' S_λ, $S_{\lambda'}$ are disjoint and stationary.

Exercise 2.15 *Find $S_n\subseteq\aleph_{\omega+1}$ stationary, for $n<\omega$, with $S_{n+1}\subseteq S_n$, but with $\bigcap_n S_n=\varnothing$.*

Sol: Let $S_n=\{\alpha<\aleph_{\omega+1}\,|\ \mathrm{cf}(\alpha)\geq\aleph_n\}$. Then each S_n is stationary, but for every $\alpha<\aleph_{\omega+1}$, for just one n is $\mathrm{cf}(\alpha)=\aleph_n$. Hence the intersection of the S_n must be empty.

Exercise 2.26 Let $\langle H,R\rangle\in\mathrm{WO}$. Apply the Collapsing Lemma. What is the outcome?

Sol: $\pi{:}\,\langle H,R\rangle\cong\langle\alpha,\in\rangle$ where α is the order type of $\langle H,R\rangle$.

Exercise 2.27 Show that V_ω can be 'coded' as a subset of ω: that is there is $E\subseteq\omega$ so that $\langle\omega,E\rangle\cong\langle V_\omega,\in\rangle$. [Hint: Define $nEm\longleftrightarrow_{\mathrm{df}}$ the "2^n" column in the binary expansion of m contains a 1; (thus $\{n|nE11\}=\{0,1,3\}$) ; check there is u satisfying $\langle\omega,E\rangle\cong\langle u,\in\rangle$ with Trans(u). Show $u=V_\omega$.]

Sol: We use the given E as our coding method. We check that E is (i) wellfounded (this is because $\langle\mathbb{N},<\rangle$ is wellfounded) (ii) E is extensional (this is because two different n,n' have different binary expansions!). Hence by the Mostowski-Shepherdson theorem there is a $\langle u,\in\rangle$ with $\pi{:}\,\langle\omega,E\rangle\cong\langle u,\in\rangle$. ($u\subseteq V_\omega$): suppose $v\in u$ and all members of v are in V_ω. As v has only finitely many members $v\subseteq V_n$ for some n. Hence $v\in V_{n+1}\subseteq V_\omega$. The result follows by $\in$-induction.

($u\supseteq V_\omega$): conversely by induction on n suppose $V_n\subseteq u$. If $X\subseteq V_n$ is an arbitrary subset, X is of course finite, and its members are coded by π^{-1}"$X\subseteq\omega$. If π^{-1}"X is $\{n_1,n_2,\ldots,n_k\}$ it is a simple matter to find an n so that $\{m|\,mEn\}=\{n_1,n_2,\ldots,n_k\}$. Then $\pi(n)=X\in u$. Hence $\mathcal{P}(V_n)=V_{n+1}\subseteq u$.

Exercise 2.28 Show that if A, B are transitive sets, and f: $\langle A,\in\rangle\longrightarrow\langle B,\in\rangle$ is an isomorphism, that $f=\mathrm{id}\restriction A$.

Sol: By $\in$-induction: suppose for any $x\in A$, that we have shown for all $y\in x$ that $f(y)=y$. (Note that as Trans(A), $x\subseteq A$.) As f is an isomorphism for any $y\in A$: $y\in x\longleftrightarrow f(y)\in f(x)$; Thus $f(x)=\{f(y)|\,y\in x\}$. But then $f(x)=\{f(y)|\,y\in x\}=\{y|\,y\in x\}=x$. As this is true for any $x\in A$ we have the desired conclusion by "$\in$-induction", that is we have used the following principle:

$$\forall x[\forall y\in x(\Phi(y)\longrightarrow\Phi(x))]\longrightarrow\forall x\Phi(x).$$

Exercise 2.30 Find an example of an $\langle x,\in\rangle$ which is not extensional. If we nevertheless apply the Mostowski-Shepherdson Collapse function π to it, what happens?

Sol: *e.g.* take $x=\{0,\{1\}\}$. Then $\pi(0)=0=\pi(\{1\})$. That is $\pi{:}\,\langle x,\in\rangle\to\langle\{0\},\in\rangle$, and the map is not (1-1).

Exercise 2.31 Prove (i),(iii)-(iv) here.

(i) $\mathrm{On}\cap H_\kappa=\kappa$; Trans($H_\kappa$);

(iii) $y\in H_\kappa\wedge x\subseteq y\longrightarrow x\in H_\kappa$;

(iv) $x,y\in H_\kappa \longrightarrow \bigcup x, \{x,y\}\in H_\kappa$;
(v) Give an example to show that $x\in H_\kappa \leftrightarrow x\subseteq H_\kappa \wedge |x|<\kappa$ fails if κ is a singular cardinal.

Sol: (i): All ordinals are already transitive; hence $\alpha\in \mathrm{On}\cap H_\kappa \longleftrightarrow |\alpha|<\kappa \longleftrightarrow \alpha<\kappa$. Let $y\in x\in H_\kappa$, then $\mathrm{TC}(y)\subseteq \mathrm{TC}(x) \longrightarrow |\mathrm{TC}(y)|\le|\mathrm{TC}(x)|<\kappa$. Hence $y\in H_\kappa$ and so $\mathrm{Trans}(H_\kappa)$.

(iii): $|\mathrm{TC}(y)|<\kappa \wedge x\subseteq y \longrightarrow \mathrm{TC}(x)\subseteq \mathrm{TC}(y)\wedge |\mathrm{TC}(x)|<\kappa$.

(iv) : $|\mathrm{TC}(\bigcup x)|\le|\mathrm{TC}(x)|$ (since $\mathrm{TC}(\bigcup x)\subseteq \mathrm{TC}(x)$); hence $x\in H_\kappa \longrightarrow \bigcup x\in H_\kappa$ is immediate. Similarly $|\mathrm{TC}(\{x,y\})|\le\max\{|\mathrm{TC}(x)|,|\mathrm{TC}(y)|\}$, from which $x,y\in H_\kappa \longrightarrow \{x,y\}\in H_\kappa$ is also immediate.

(v) fails if $\kappa=\aleph_\omega$ for example: $A=\{\aleph_n|n<\omega\}\subseteq H_{\aleph_\omega}\wedge|A|=\aleph_0<\aleph_\omega$ but $A\notin H_{\aleph_\omega}$.

Exercise 2.32 Show that $V_\omega=\mathrm{HF}$. [Hint: For ($\subseteq$) use induction on n to show $V_n\in\mathrm{HF}$. For ($\supseteq$) use $\in$-induction].

Sol: Follow the Hint: ($\subseteq$) suppose as inductive hypothesis that $V_n\in\mathrm{HF}$. Then V_n is finite, as is $\mathcal{P}(V_n)=V_{n+1}$. But $V_{n+1}\subseteq\mathrm{HF}$, as any $X\in V_{n+1}$ is a subset of $V_n\subseteq\mathrm{HF}$ and is finite, and so is in HF. (see Lemma 2.31(v)). For ($\supseteq$) suppose $x\in\mathrm{HF}$ and as an $\in$-inductive hypothesis that $\forall y\in x(y\in V_\omega)$. As x is finite, $m=\max\{\rho(y)\,|\,y\in x\}<\omega$. So $x\subseteq V_m$. Hence $x\in V_{m+1}\subseteq V_\omega$.

Exercise 2.33 Check that HF is closed under all the assumptions of Lemmata 1.24, 1.25, (except 1.25 (ii)) and even the power set operation. Hence $(\mathrm{ZF}-\mathrm{Ax.Inf})^{\mathrm{HF}}$.

Sol: This is very much routine checking given the definition of HF. As Trans(HF) we have Ax.Extensionality. For the basic operations of pairing and union (and Power set) we can appeal to the Ex. 2.31(iv), and the fact that we know that V_ω is closed under such operations; likewise Ax.Empty is trivial. Since HF is closed under the power set operation Ax.Subsets becomes trivial. Ax.Replacement holds because the range of a function on a finite set is finite. (More detail: if $u\in\mathrm{HF}$ and has size k and F is a function $F\colon\mathrm{HF}\longrightarrow\mathrm{HF}$ then $F``u\subseteq\mathrm{HF}$ and has size at most $k$ also. Hence $F``u\in\mathrm{HF}$.) The negation of the Ax.Infinity is true, since no set $x\in\mathrm{HF}$ is closed under the successor function $S(u)=_{\mathrm{df}}u\cup\{u\}$. Ax.Found. holds in any transitive structure $\langle M,\in\rangle$.

Exercise 2.35. *If $x\in\mathrm{HC}$ then we have $|\mathrm{TC}(x)|\le\omega$. Define a wellfounded extensional relation E on ω so that $\langle\omega,E\rangle\cong\langle\mathrm{TC}(x),\in\rangle$. [Hint: We have a bijection $f\colon N\longleftrightarrow\mathrm{TC}(x)$ for some $N\le\omega$; define $nEm\leftrightarrow f(n)\in f(m)$.] If we use a recursive pairing bijection $p\colon\omega\longleftrightarrow\omega\times\omega$ (for example $p^{-1}(\langle k,l\rangle)=2^k.(2l+1)-1$) we may further code E as a subset $\bar{E}\subseteq\omega$. We thus have effectively coded up* TC(x) *as a subset of ω.] (By using further such coding devices we may take any countable structure with domain in* HC *and code it up as a subset of ω.)*

Sol: The more interesting case here is with TC(x) infinite. The Hint here really says it all: as opposed to the coding we did earlier of V_ω onto $\langle\omega,E\rangle$ in Ex.2.27, we don't really need to fix on a particular fancy coding device: *any* bijection $f\colon N\longleftrightarrow\mathrm{TC}(x)$ will do, as will any pairing function $p\colon\omega\longleftrightarrow\omega\times\omega$. The point is just to code up the $\in$-information up by natural numbers. There are many $\bar{E}$ with $\langle\omega,\bar{E}\rangle\cong\langle\mathrm{TC}(x),\in\rangle$ (in fact $2^{\aleph_0}$ many).

Exercise 2.36 Which axioms of ZF hold in V_α if $\mathrm{Lim}(\alpha)$? Find a wellordering $\langle A,R\rangle\in V_{\omega+\omega}$ but for which there is no ordinal β with $\langle A,R\rangle\cong\langle\beta,<\rangle$; hence find an instance of the Ax.Replacement that fails in $V_{\omega+\omega}$. [The latter is a model of Z, the system Zermelo which is more or less ZF with Replacement removed. For almost all regions of mathematical discourse, $V_{\omega+\omega}$ is a sufficiently large "universe" - mathematicians rarely need sets outside of this set.]

Sol: If $\mathrm{Lim}(\alpha)$ then all axioms of ZF relativise to V_α except the Ax.Replacement. (One easily sees that such a V_α is closed under pairing, union, subsets etc, by calculating ranks of sets as we did in ST). (1) Let $\langle A,R\rangle$ be $\langle\mathbb{N},<'\rangle$ where $<'$ is the ordering that puts all the Evens before all the Odds, but otherwise retains the usual order. Then $\mathrm{ot}(\langle A,R\rangle)=\omega+\omega\notin V_{\omega+\omega}$. Then $G(2k+1)=\mathrm{ot}(\langle\mathbb{N},<'\restriction 2k+1\rangle)$ is a definable function that maps Odds unboundedly into $\omega+\omega$. So ran(G) is not a set in $V_{\omega+\omega}$ although Odds is. Thus AxRep. fails.

(2) Alternative viewpoint: Consider $V_{\omega+\omega}$ and the class term function $F(\delta)=A_\omega(\delta)=\omega+\delta$. However $F``\omega=\{\omega,\omega+1,\ldots\omega+k,\ldots\}\notin V_{\omega+\omega}$. ($F``\omega\in V_{\omega+\omega}\longrightarrow\sup F``\omega=\bigcup F``\omega=\omega+\omega\in V_{\omega+\omega}$. This is absurd as $\mathrm{On}\cap V_{\omega+\omega}=\omega+\omega$.) Thus again Ax.Replacement fails.

Exercise 2.38 Let $\kappa\in\mathrm{Card}$. Show that $|H_\kappa|=2^{<\kappa}$.

Sol: The Hint was to use the last Exercise: that concluded that $|H_{\kappa^+}| = |\mathcal{P}(\kappa)|$. This equals $2^{\kappa} = 2^{<\kappa^+}$. So this is proven for successor cardinals κ^+. Let now κ be a limit cardinal, and as an inductive hypothesis that $|H_\mu| = 2^{<\mu}$ for $\mu \in \mathrm{Card} \wedge \mu < \kappa$. Then $H_\kappa = \bigcup \{H_\mu | \mu \in \mathrm{Card} \cap \kappa\}$. As the cardinality of H_μ is increasing with μ, we have: $|H_\kappa| = \sup\{|H_\mu| \,|\, \mu \in \mathrm{Card} \cap \kappa\} = \sup\{2^{<\mu} \,|\, \mu \in \mathrm{Card} \cap \kappa\} = 2^{<\kappa}$.

Exercise 2.39 (Levy) Let $h(\kappa)$ be the class of sets x with (i) $|x| < \kappa$ (ii) $\forall y \in \mathrm{TC}(x)$ ($|y| < \kappa$). Show that if $\kappa \in \mathrm{Reg}$, then $H_\kappa = h(\kappa)$; find an example where this fails if κ is singular.

Sol: Suppose κ is $\aleph_\omega$. Then the set $x = \{\aleph_n | n < \omega\}$ is a counterexample: it is in $h(\kappa)$ but not H_κ. Suppose now $\kappa \in \mathrm{Reg}$. Let $x \in H_\kappa$. Then $|\mathrm{TC}(x)| < \kappa$. This implies both that $|x|$ and $|\mathrm{TC}(y)| < \kappa$ for any $y \in x$, and so $x \in h(\kappa)$. Conversely let $x \in h(\kappa)$. Then $\mathrm{TC}(x)$ is the union of less than κ many sets of cardinality $< \kappa$. Hence, as $\kappa \in \mathrm{Reg}$ it has cardinality $< \kappa$ and thus is in H_κ.

Exercise 2.41 Let W be a transitive class term. Then (i) any Σ_1-formula φ is upwards absolute for W; (ii) any Π_1-formula φ is downwards absolute for W.

Sol: (i) Suppose φ is the formula $\exists x \psi(x, y)$ with the single free variable y displayed (for simplicity) and with $\psi(x, y)$ Δ_0. Let W be a transitive class. Then for any $x, y \in W$ we have $\psi(x, y) \leftrightarrow \psi(x, y)^W$ since Δ_0 formulae are absolute by the last Exercise). By the definition of upward absoluteness we must show: $\forall y \in W(\varphi^W \longrightarrow \varphi)$. Let $y \in W$ be arbitrary and assume $(\exists x \psi(x, y))^W$. Let $x_0 \in W$ with $(\psi(x_0, y))^W$. By the comment above $\psi(x_0, y)$ holds (in V). Hence so does $\exists x \psi(x, y)$ *i.e.* φ.

(ii) This follows from (i) where now φ is Π_1 that is (for simplicity) of the form $\forall x \psi(x, y)$ with ψ in Δ_0. But then if $y \in W$ we have that $\forall x \psi(x, y) \rightarrow \forall x \in W \psi(x, y) \leftrightarrow \forall x \in W(\psi(x, y))^W \leftrightarrow \varphi^W$.

Exercise 2.42 Verify the very last part of Lemma 2.40.

Sol: [This just asks that you understand that the Lemma follows from the *Claim*.] The β just constructed above is a closure point of the functions F_i. By appealing to Lemma 2.40, we then see that any formula on the list of the $\vec{\varphi}$ is absolute between Z_β and Z. There are arbitrarily large such β by the construction, which we performed within the ZF axiom system. So the Lemma 2.40 follows.

Exercise 2.43 Show that for every formula φ of $\mathcal{L}$:

ZF$\vdash$ "There is a c.u.b. class $C \subseteq \mathrm{On}$ so that $\forall \alpha \in C \forall \vec{x} \in V_\alpha(\varphi(\vec{x}) \leftrightarrow (\varphi(\vec{x}))^{V_\alpha})$"

[Hint: The reasoning of Lemma 2.40 pretty much gives the relevant cub class as the closure points of the F_i.] Remark: One might think that one could enumerate all the axioms of ZF $\varphi_0, \varphi_1, \ldots$, find the appropriate classes C_{φ_n} and take $D = \bigcap_n C_{\varphi_n}$. This appears then only to be an intersection of only countable many c.u.b. classes and so must be c.u.b. in On? But for any element $\alpha \in D$ we'd have $(\mathrm{ZF})^{V_\alpha}$, and we appear to have proven the existence of models of ZF - contradicting Goedel. What is wrong with this reasoning?

Sol: This is really only an observation on the proof of the Montague-Levy Reflection Theorem. Let $\vec{\varphi} = \varphi_0, \ldots, \varphi_n$ be a subformula closed list with φ equal to φ_n (without loss of generality). Just note that the function F_n defined in that proof is not just monotone but *normal*, hence $\mathrm{ran}(F)$ is just the desired c.u.b. class C_φ. [It is more interesting to ask what is wrong with the given reasoning. The point is just as we cannot prove the Reflection theorem for *all* formulae of the language at once, so we cannot prove more than the existence of *finitely many* such C_φ at any one time in ZF. Hence the given reasoning cannot be given a formal version within ZF. (A platonist who believes absolutely in the universe of all sets V might also believe in the existence of all such classes and (s)he could then take such an intersection without any qualms!))

Exercise 2.44 Find a sentence σ so that if σ is absolute for V_α then α is a limit ordinal. Repeat the exercise and find τ so that if τ is absolute for V_β then $\beta = \omega_\beta$ (the β'th infinite cardinal). [Hint: consider the statement: "*For every β ω_β exists*".]

Sol: $\sigma \equiv \forall \gamma \exists \beta(\beta \in \mathrm{On} \wedge \beta > \gamma)$. Then if σ is absolute for V_α there can be no largest ordinal in V_α. In other words $\mathrm{Lim}(\alpha)$ since $V_\alpha \cap \mathrm{On} = \alpha$ holds for any $\alpha \in \mathrm{On}$.

$\tau \equiv$ "$\forall \gamma \exists f(\mathrm{ran}(f) \subseteq \mathrm{Card} \wedge \mathrm{dom}(f) = \gamma \wedge f$ is a monotone increasing function)"

Then if τ is absolute for V_α it must be that for every $\gamma < \alpha$ ("there are γ many cardinals below α")$^{V_\alpha}$, in other words ("ω_γ exists for any γ")$^{V_\alpha}$. By absoluteness if (κ is a cardinal)$^{V_\alpha}$ then κ really is a cardinal: ${}^\kappa\kappa \subseteq V_\alpha$, so V_α knows all functions $f: \gamma \longrightarrow \kappa$ for any f, any $\gamma \leq \kappa$. hence $(\omega_\gamma)^{V_\alpha} = \omega_\gamma$; hence α is a limit point of the cardinal enumeration function $\gamma \longrightarrow \omega_\gamma$, and we have just reasoned that α must be ω_α itself.

Exercise 2.45 Let x be any set, and f_i: ${}^{n_i}V \longrightarrow V$ for $i<\omega$ be any collection of finitary functions (meaning that $n_i < \omega$); show that there is a $y \supseteq x$ which is closed under each of the f_i (thus f_i" ${}^{n_i}y \subseteq y$ for each i) and $|y| \leq \max\{\omega, |x|\}$. [Hint: no need for a formal argument here: build up a y in ω many stages $y_k \subseteq y_{k+1}$ at each step applying all the f_i.]

Sol: Let $y_0 = x$; let $y_{k+1} = y_k \cup \{f_i\text{“ }{}^{n_i}y_k \mid i<\omega\}$. Let $y = \bigcup_{i<\omega} y_i$. Suppose $|x| \geq \omega$. Then each $|y_{k+1}| = |y_k|$ as y_{k+1} is the result of adding countably many functions applied to finite tuples from y_k. But ${}^{<\omega}y_k$ (the set of finite sequences from y_k) has cardinality that of y_k. So altogether $|y_{k+1}| = |y_k| + \aleph_0 \cdot |{}^{<\omega}y_k| = |y_k| + |y_k| = |y_k|$. Then y is the union of countably many sets of cardinality that of x. So $|y| = |x| \cdot \aleph_0 = |x|$. If x is finite then possibly $y_1 = \aleph_0$ (as there are countably many f_i), but thereafter all the y_k and y will have cardinality $\aleph_0$.

Exercise 2.46 *Verify that κ is weakly inaccessible iff κ is regular and $\kappa = \aleph_\kappa$*

Sol: ($\Rightarrow$) By definition κ is regular. If $\kappa = \aleph_\alpha$ for an $\alpha < \kappa$ then $\mathrm{cf}(\kappa) = \mathrm{cf}(\alpha) < \kappa$ which contradicts κ's regularity. Hence, as $\beta \leq \aleph_\beta$ for any β, κ must be $\aleph_\kappa$.

($\Leftarrow$) $\mathrm{Lim}(\kappa) \longrightarrow \aleph_\kappa$ is a limit cardinal. As it is regular, it is weakly inaccessible.

Exercise 2.47 *Does $\kappa > \omega \wedge V_\kappa = H_\kappa$ imply that κ is strongly inaccessible?*

Sol: No. Consider κ the least fixed point of the beth function: $\kappa = \beth_\kappa$. This has cofinality ω and so is singular. Check that it satisfies $V_\kappa = H_\kappa$. [$H_\kappa \subseteq V_\kappa$ is Lemma 2.31 (ii). Conversely $|V_\alpha| = 2^{\beth_\alpha}$ (for $\omega^2 \leq \alpha$; see Ex.2.7). So if $\alpha < \kappa$, V_α is a transitive set of cardinality $<\kappa$. So $V_\alpha \in H_\kappa$.]

Exercise 2.48 *Let λ be the least weakly inaccessible cardinal which is itself a limit of weakly inaccessible cardinals (meaning the weakly inaccessibles below λ are unbounded in λ). Show that λ is not weakly Mahlo. Repeat the exercise replacing "weakly" by "strongly" throughout.*

Sol: Suppose λ were weakly Mahlo. Then by Lemma 2.50, $E = \{\alpha < \kappa \mid \alpha$ is weakly inaccessible$\}$ is stationary below λ. As usual let E^* be the set of limit points of E. Let $f_E: \lambda \longrightarrow E^*$ be the enumerating function of E^*. The set of $\alpha < \lambda$ which are fixed points of f_E, so for which $f_E(\alpha) = \alpha$, is a cub set below λ (see Lemma 2.13), $\bar{E}$ say, and consists of cardinals α with order type α-many weakly inaccessibles below it. But then $E \cap \bar{E}$ is stationary, and any $\alpha \in E \cap \bar{E}$ is then the α'th weakly inaccessible cardinal. But λ was supposed to be the least such. The argument is exactly the same for "strongly" in place of "weakly".

Exercise 3.1 (i) Show that "*z is a total order of y*" can be expressed in a Δ_0 fashion.

(ii) Complete (ix), (x), (xvi) and find a correct t to finish (xvii) of Lemma 3.7.

Sol:(i) $\forall u, v, w \in \mathrm{dom}(x) \cup \mathrm{ran}(x)[(u = v \vee \langle u, v\rangle \in x \vee \langle v, u\rangle \in x) \wedge (\langle u, v\rangle \in x \wedge \langle v, w\rangle \in x \rightarrow \langle u, w\rangle \in x) \wedge \langle u, u\rangle \notin x]$ (The last clause if the ordering is strict.)

(ii): (ix): $z \upharpoonright x = \{u \in z \mid (u)_0 \in x\}$; (x) $z^{-1} = \{\langle (u)_1, (u)_0\rangle \mid u \in z\}$

(xvi): "x is unbounded in β"; $\forall u \in \beta \exists v \in x (u \in v)$.

"$z: \alpha \longrightarrow \beta$ is a cofinal function": $\mathrm{Fun}(z) \wedge \mathrm{ran}(z)$ is unbounded in β.

"$x \subseteq \beta$ is a closed and unbounded set": $\forall u \in \beta(\sup(x \cap u) = u \rightarrow u \in \beta) \wedge x$ is unbounded in β.

(xvii) Let $t(u, v) = u \cup \bigcup v$. Then define $f(x) = t(x, \{f(w) \mid w \in x\})$. Hence f is definite. Check that $\mathrm{TC}(x) = f(x)$.

Exercise 3.2. Suppose $\mathrm{Trans}(W) \wedge (\mathrm{ZF}^-)^W$. Show $(V_\alpha)^W = V_\alpha \cap W$. {Hint: use that the rank function is absolute.]

Sol: Two ways: (i) Use that the rank function $\rho(x)$ is a.d. and that we know that $V_\alpha = \{y \mid \rho(y) < \alpha\}$. Then $(\rho(y) < \alpha)^W \longleftrightarrow \rho(y) < \alpha$. Thence $(V_\alpha)^W = \{y \in W \mid (\rho(y) < \alpha)^W\} = \{y \in W \mid (\rho(y) < \alpha)\} = V_\alpha \cap W$.

(ii) Or directly by induction on $\alpha \in W$ (without using the rank function). For $\alpha = 0$ this is trivial. If true for $\beta < \alpha$ then (if $\alpha = \beta + 1$) we have that $(V_\alpha)^W = (\mathcal{P}(V_\beta))^W = \{X \mid X \in W \wedge (X \subseteq V_\beta)^W\}$ by Ind. Hyp.:

$= \{X \mid X \in W \wedge (X \subseteq V_\beta \cap W)\}$ (as "$u \subseteq v$" is absolute for transitive models)

$= \{X \mid X \in V_\alpha \cap W\}$.

If $\mathrm{Lim}(\alpha)$ then $(V_\alpha)^W = (\bigcup_{\beta<\alpha} V_\beta)^W = \bigcup_{\beta<\alpha} V_\beta \cap W = V_\alpha \cap W$ (using that $\bigcup$ is absolute, and the Ind.Hyp. for the second equality.

Exercise 3.3 Let λ be a limit; show that the following are absolute for V_λ: (i) $\mathcal{P}(x)$ (ii) "α is a cardinal" (and hence $(\mathrm{Card})^{V_\lambda} = \mathrm{Card} \cap \lambda$); (iii) $\mathrm{cf}(\alpha)$ (iv) "α is (strongly) inaccessible" (v) V_α (vi) $\aleph_\alpha$ (vii) $\beth_\alpha$.

Sol: These are really rather immediate from the definition of the V_α-hierarchy and the assumption that $\mathrm{Lim}(\lambda)$: (i) if $x \in V_\lambda$ then for some $\alpha < \lambda$ $x \in V_\alpha \longrightarrow \mathcal{P}(x) \in V_{\alpha+1} \subseteq V_\lambda$; (ii) if $\alpha \in V_\lambda$ then (as $\mathrm{On} \cap V_\lambda = \lambda$ we have that $\alpha < \lambda$. But ${}^\alpha\alpha \subseteq V_{\alpha+5}$. So V_λ is correct about whether α is a cardinal or not; likewise V_λ knows if $\mathrm{cf}(\alpha) = \gamma \leq \alpha$ or not, as all the relevant functions are in V_λ - which is (iii). It is then clear that "α is inaccessible" is also absolute for V_λ. For *strongly inaccessible* note this comes from (i) and (iii) (as $\alpha \in \mathrm{Reg} \leftrightarrow \mathrm{cf}(\alpha) = \alpha$):

$$(\alpha \text{ is strongly inaccessible})^{V_\lambda} \leftrightarrow (\forall x \in V_\alpha(\mathcal{P}(x) \in V_\alpha) \wedge \alpha \text{ is regular})^{V_\lambda}.$$

Then for any $\alpha < \lambda$ we shall have $V_\alpha = (V_\alpha)^{V_\lambda}$; if ($y$ *is the* α*'th cardinal*)$^{V_\lambda}$ then y *is the* α*'th cardinal*. (More fully expanded, one should use that "y *is the* α*'th cardinal*" iff $\exists f(\mathrm{Func}(f) \wedge \mathrm{dom}(f) = \alpha + 1 \wedge \forall \xi < \zeta < \alpha + 1((f(\xi) < f(\zeta)) \wedge \mathrm{ran}(f) = \mathrm{Card} \cap y + 1 \wedge f(\alpha) = y)$. Then, if ($y$ *is the* α*'th cardinal*)$^{V_\lambda}$ there will be such a function $f \in V_{\alpha+3}$, and as by (ii) $(\mathrm{Card})^{V_\lambda} = \mathrm{Card} \cap \lambda$, all parts of the definition of f are absolute for V_λ, and so y really is the α'th cardinal, that is y is $\aleph_\alpha$). Similarly if $y = \beth_\alpha < \lambda \leftrightarrow (y = \beth_\alpha)^{V_\lambda}$. (v) That $V_\alpha = (V_\alpha)^{V_\lambda}$ may formally be proven by induction on $\alpha < \lambda$, but really we have already proven this in Ex 3.2.

Exercise 3.4 Finish (ii) $\mathrm{Trans}(x) \longrightarrow x \subseteq \mathrm{Def}(x)$;and (iii) $\forall z \subseteq x(|z| < \omega \rightarrow z \in \mathrm{Def}(x)$; of Lemma 3.19.

Sol: (ii) Let $y \in x$. Then $y = \{z \in x \mid \langle x, \in \rangle \models \ulcorner v_0 \in v_1 \urcorner [z, y]\}$.

(iii) Suppose $z = \{w_1, \ldots, w_n\}$. Then

$$z = \{w \in x \mid \langle x, \in \rangle \models \ulcorner v_0 = v_1 \vee \cdots \vee v_0 = v_n \urcorner [w, w_1, \ldots, w_n]\}$$

Exercise 3.5 Let $\langle x, \in \rangle$ be a transitive $\in$-model. Show that $\mathrm{Trans}(\mathrm{Def}(x))$. If $y, z \in x$ then is $\langle y, z \rangle \in \mathrm{Def}(x)$? Is $\{x\}$? [Hint (for the last question): If $\rho(x) = \alpha$, compute $\rho(\mathrm{Def}(x))$ and compare this with the given sets.]

Sol: Let $z \in \mathrm{Def}(x)$. Then for some φ some $\vec{y} \in Q_x$, $z = \{u \in x \mid \langle x, \in \rangle \models \ulcorner \varphi \urcorner [u, \vec{y}]\}$ so obviously $z \subseteq x \subseteq \mathrm{Def}(x)$ by the last Exercise. Thus $\mathrm{Trans}(\mathrm{Def}(x))$.

However in general $\langle y, z \rangle \notin \mathrm{Def}(x)$. Nor is $\{x\}$. For the latter this is for the simple reason that $\{x\} \in \mathrm{Def}(x)$ would imply that $x \in x$! Note that $z \in \mathrm{Def}(x) \longrightarrow z \subseteq x \longrightarrow \rho(z) \leq \rho(x)$. But $x \in \mathrm{Def}(x)$. Hence $\rho(\mathrm{Def}(x)) = \rho(x) + 1$. But if, *e.g.*, $\rho(x) = \gamma + 1$ and $y \in x$ with $\rho(y) = \gamma$, then $\rho(\langle y, y \rangle) = \gamma + 2$. Hence $\langle y, y \rangle$ cannot be in $\mathrm{Def}(x)$ as this has rank only $\gamma + 2$ itself.

Exercise 3.6 Let us say that w is *outright definable in the set* $\langle x, \in \rangle$ if for some formula φ with only free variable v_0 then w is the unique element in x so that $\langle x, \in \rangle \models \varphi[w]$. We may thus define a variant on the Def function by:

$$\mathrm{Def}_0(x) = \{z \mid \{z\} = \{w | \langle x, \in \rangle \models \varphi[w]\}, \mathrm{Fmla}(\varphi) = 1, \mathrm{FVbl}(\varphi) = \{v_0\}\ w \in x\}$$

of the sets outright definable in $\langle x, \in \rangle$, definable without use of parameters. Show that $|\mathrm{Def}_0(x)| \leq \omega$ for any x.

Sol: Just note that irrespective of the size of x, there are only countably many possible definitions using formulae φ from $\mathcal{L}$.

Exercise 3.7 (i) Show that: (a) $On \subseteq \mathrm{OD}^*$; (b) $\forall \beta\, V_\beta \in \mathrm{OD}^*$; (c) $\forall x\,(x \in \mathrm{OD}^* \longrightarrow \{x\} \in \mathrm{OD}^*)$.

(ii) ($*$) Show that there is a (countable) set X so that for unboundedly many ordinals β $X \in \mathrm{Def}_0(V_\beta)$. Hint:consider the theory of each V_β: the set of all codes of sentences σ so that $V_\beta \models \ulcorner \sigma \urcorner$. This is a subset of V_ω.]

Sol:(i) (a) $\{\alpha\} = \{w \mid V_{\alpha+1} \models$ "*w is the largest ordinal*"$\}$. Similarly (b) $\{V_\beta\} = \{w \mid V_{\beta+1} \models$ "*w is the set of all sets of rank less than the largest ordinal*"$\}$. (c) Suppose $x \in \mathrm{Def}_0(V_\beta)$ for some β. Then $\{x\} = \{w \mid V_\beta \models \varphi[w]\}$ for some suitable $\varphi(v_0)$. But then

$\{\{x\}\} = \{w \mid V_{\beta+1} \models$ "*w is the set of t so that $V_y \models$ '$\varphi[t]$' where y is the largest ordinal* "$\}$.

(ii) For $\beta \in \mathrm{On}$ let $\mathrm{Th}(\beta)$ be the 'theory' of $\langle V_\beta, \in \rangle$, that is

$$\mathrm{Th}(\beta) = \{\ulcorner \sigma \urcorner \mid \ulcorner \sigma \urcorner \textit{ is a code of a sentence } \wedge V_\beta \models \ulcorner \sigma \urcorner\}.$$

Then note that $\mathrm{Th}(\beta) \in \mathrm{Def}_0(V_{\beta+1})$. But each $\mathrm{Th}(\beta)$ is a, necessarily, countable subset of HF, and then it is easy to see that for unboundedly many β the $\mathrm{Th}(\beta)$ must be the same T (as the countable subsets of HF form a set not a proper class!) So this $T \in \mathrm{Def}_0(V_{\beta+1})$ for unboundedly many β.

Exercise 3.8 Suppose κ is strongly inaccessible. Verify that $\langle V_\kappa, \in \rangle \models \ulcorner \mathrm{ZFC} \urcorner$.

Sol: Just note that we have shown (Lemma 2.48 (iii)) for every axiom σ of ZFC that we have $(\sigma)^{V_\kappa}$, so by Correctness, Thm 3.21, $\langle V_\kappa, \in \rangle \models \ulcorner \sigma \urcorner$.

Exercise 3.9 (∗) (E) Let $\mathcal{A}, \mathcal{B}$ be structures. We write $\mathcal{A} \prec \mathcal{B}$ if for every formula u, every $h \in Q_{\mathcal{A}}$ if $\mathcal{B} \models u[h]$ then $\mathcal{A} \models u[h]$. Suppose that $\kappa < \lambda$ are such that $\langle V_\kappa, \in\rangle \prec \langle V_\lambda, \in\rangle$. Show that κ is a strong limit cardinal and that both $\langle V_\kappa, \in\rangle, \langle V_\lambda, \in\rangle$ are models of ZFC.

Sol: Note that both κ, λ are limit ordinals: if $V_\kappa \models$ "α is the largest ordinal", then $V_\lambda \models$ "α is the largest ordinal", but $\alpha + 1$ cannot be both κ and λ. So for any limit τ every axiom of ZFC holds in V_τ except possibly for the Ax Replacement. Let f^{V_κ} be a term which defines a function over V_κ. Let $u = \mathrm{dom}(f^{V_\kappa})$ and suppose that $\mathrm{ran}(f^{V_\kappa})$ is not a set of V_κ. (In other words $\mathrm{ran}(f^{V_\kappa}) \not\subseteq V_\tau$ for any $\tau < \kappa$, because otherwise $\mathrm{ran}(f^{V_\kappa}) \in V_{\tau+1}$.) By the definition of $\prec$ we have that if we define f^{V_λ} we have $f^{V_\kappa}(v) = f^{V_\lambda}(v)$ for every $v \in u$. But $V_\lambda \models$ " there exists κ' with $\mathrm{ran}(f^{V_\lambda}) \subseteq V_{\kappa'}$". However this is not true of f^{V_κ} in V_κ - a contradiction. So our supposition is false, and $\mathrm{ran}(f^{V_\kappa})$ is a set of V_κ; so AxRep holds in both V_κ (and by definition of $\prec$, in V_λ too). So $\langle V_\kappa, \in\rangle, \langle V_\lambda, \in\rangle$ are models of ZFC. And thus in both models the statement "$\forall\gamma(\gamma \in \mathrm{Card} \to \exists\eta(\eta = 2^\gamma)$" holds (as this follows from ZFC). So both κ (and λ) are strong limit cardinals. NB we have not shown, and it does not follow from our assumptions, that either of κ, λ is regular.

Exercise 3.12 Suppose $\langle X, \in\rangle \models T$ for some set of sentences T. Show that there is a countable transitive x with $\langle x, \in\rangle \models T$. [Hint: The Downward-Loewenheim Skolem Theorem says for any cardinal λ with $\omega \leq \lambda \leq |X|$ there is a Y with $\langle Y, \in\rangle \prec \langle X, \in\rangle$ and $|Y| = \lambda$. Then use the Mostowski-Shepherdson Collapsing Lemma.] In particular if there is an $\in$-structure which is a model of ZFC then there is a countable transitive one.

Sol: The Hint really says it all here: by the quoted theorem there is $\langle Y, \in\rangle \prec \langle X, \in\rangle$ with $|Y| = \omega$. Hence $\langle Y, \in\rangle \models T$ and Y has the correct cardinality. It may simply lack transitivity. But this is what the Collapsing Lemma provides: an $\langle x, \in\rangle \cong \langle Y, \in\rangle$ with $\mathrm{Trans}(x)$. As the latter two structures are isomorphic, $\langle x, \in\rangle \models T$ also. The final sentence applies when T is ZFC itself.

Exercise 4.1 (i) Verify that for all $\alpha \in \mathrm{On}$, $\rho_L(\alpha) = \rho(\alpha) = \alpha$; (ii) Prove that for $n \leq \omega$ $L_n = V_n$.

Sol: (i) The second equality is already proven. By (iv) of Lemma 4.4 $\alpha = \mathrm{On} \cap L_\alpha$. Hence $\alpha \in \mathrm{Def}(\langle L_\alpha, \in\rangle)$. ($\alpha = \{x \in L_\alpha | \langle L_\alpha, \in\rangle \models \ulcorner v_0 \in \mathrm{On}\urcorner[x]\}$.) Hence $\alpha \in L_{\alpha+1}$ and $\rho_L(\alpha) \leq \alpha$. But α cannot be a member of L_α, hence we have equality here.

(ii) By induction on n: for $n = 0$ this is trivial. Suppose true for $n = k$. Since $\mathrm{Def}(\langle L_k, \in\rangle \subseteq \mathcal{P}(L_k)$, then by the induction hypothesis we have $\mathrm{Def}(\langle L_k, \in\rangle \subseteq \mathcal{P}(V_k) = V_{k+1}$. Let $x = \{x_1, \ldots, x_m\} \in \mathcal{P}(V_k)$. Then $x \subseteq L_k$, and $x \in \mathrm{Def}(\langle L_k, \in\rangle = L_{k+1}$, by (iii) of Exercise 3.1.

Exercise 4.3 Show that "*x is a cardinal*" and "*x is regular*" are downward absolute from V to L. Deduce that if κ is a (regular) limit cardinal then (κ *is a (regular) limit cardinal*)^L

Sol: Being a cardinal, and being regular are downward absolute (so this is nothing special about L) as follows.

x is a cardinal $\leftrightarrow x \in \mathrm{On} \wedge \forall f(\mathrm{Func}(f) \wedge \mathrm{dom}(f) < x \longrightarrow \mathrm{ran}(f) \neq x)$.

Being an ordinal is definite hence absolutely definite, and the second conjunct is a universal quantifier in front of a definite, Σ_0, matrix. Hence if this holds for all functions f it certainly holds for all functions in any subclass of V (such as L). Similarly

x is a regular $\leftrightarrow x \in \mathrm{Card} \wedge \forall f(\mathrm{Func}(f) \wedge \mathrm{dom}(f) < x \wedge \mathrm{ran}(f) \subseteq x \longrightarrow \sup \mathrm{ran}(f) \neq x)$

is also (equivalent to) a universally quantified statement, and both are downward absolute to L. Finally then, if x is a limit of cardinals λ, then those λ are cardinals in L, hence (x is a limit of cardinals)^L. If x is regular in addition, then x will be a regular limit cardinal (and hence inaccessible) in L.

Exercise 4.5 Show that $\mathrm{ot}(L_\kappa, <_\kappa) = \kappa$ for κ an infinite cardinal; Show that $\mathrm{ot}(L, <_L) = \mathrm{On}$.

Sol: If $\alpha < \kappa$ we saw (Lemma 4.5 (iii)) that $|L_\alpha| = |\alpha| < \kappa$. Hence $\mathrm{ot}(L_\alpha, <_\alpha)$ must be strictly less than the cardinal κ. As $<_\kappa = \bigcup_{\alpha<\kappa} <_\alpha$ and $\alpha < \beta < \kappa \longrightarrow <_\beta$ is an end extension of $<_\alpha$, we must have that $\mathrm{ot}(L_\kappa, <_\kappa) \leq \kappa$ (for if $x \in L_\kappa$ is the κ'th set in $L_\kappa = \bigcup_{\alpha<\kappa} L_\alpha$ then for some $\beta < \kappa$ we should have that x is the κ'th set in L_β which is a contradiction). As $\alpha < \kappa \longrightarrow \alpha \in L_{\alpha+1}$ we must have that $\kappa \subseteq \mathrm{Field}(<_\kappa)$ and so $\mathrm{ot}(L_\kappa, <_\kappa) \geq \kappa$. Entirely the same argument works to show $\mathrm{ot}(L, <_L) = \mathrm{On}$ thinking now of On in place of κ.

Exercise 4.6 Suppose there is a transitive set model of ZFC. Show that there is a *minimal model* of ZFC, that is for some countable ordinal β_0, $(\text{ZFC})^{L_{\beta_0}}$ and that L_{β_0} is a subclass of any other transitive set model of ZF.

Sol: Suppose there is a transitive set model of ZFC. Then there is such of minimal rank, say M_0, with $\rho(M_0)=\beta_0$. Then $\langle M_0,\in\rangle\models\ulcorner\text{ZFC}\urcorner$ implies $\langle M_0,\in\rangle\models\ulcorner\sigma_1\urcorner$ where σ_1 is the conjunction of finitely many axioms of ZF needed to prove absoluteness and results about the L construction (as in Lemma 4.12). By the Correctness Theorem we have $(\sigma_1)^{M_0}$ and thus by Lemma 4.12, $L^{M_0}=L_{\beta_0}$; moreover $\langle M_0,\in\rangle\models$ "$L_{\beta_0}\models\ulcorner\text{ZFC}\urcorner$ ". L_{β_0} must then be the minimal model of ZFC as required.

Exercise 4.7 (Shepherdson) Show that there is no class term W so that $\text{ZFC}\vdash\text{IM}(W)$ and $\text{ZFC}\vdash(\neg\text{CH})^W$. [This Exercise shows that Goedel's argument was essentially a "one-off": there is no way one can define in ZFC alone an inner model and hope that it is a model of all of ZF plus, *e.g.*,$\neg$CH.]

Sol: Suppose there were such a term W. Then let U be the term W relativised to L: $U=(W)^L$. Thus $U\subseteq L$. However then By Lemma 4.12 $(L)^U=L=U$. But $(\neg\text{CH})^U$ since by assumption $\text{ZFC}\vdash(\neg\text{CH})^W$ and $(\text{ZFC})^L$. Contradiction.

Exercise 4.8 Show that if there is a weakly inaccessible cardinal κ then $(\text{ZFC})^{L_\kappa}$. Hence $\text{ZFC}\nvdash\exists\kappa(\kappa$ a weakly inaccessible cardinal.) [Hint: Use the fact that $(\text{GCH})^L$.]

Sol: If κ is a regular cardinal then H_κ is a model of ZFC^- by Lemma 2.29 (although Power Set may fail). As $(H_\kappa)^L=L_\kappa$ we have all instances of ZFC axioms with the possible exception of Power Set holding in L_κ. But if $\alpha<\kappa$ is any cardinal, then $(|\mathcal{P}(\alpha)|=\alpha^+<\kappa)^L$ as $(\text{GCH})^L$. (Note that $(\alpha^+)^L\leq\alpha^+<\kappa$ as κ is a limit cardinal.) Thus Power Set does hold in L_κ after all. Thus if κ is weakly inaccessible cardinal, we have deduced that κ is a strongly inaccessible cardinal of L. But then (Lemma 2.48) $(\text{ZFC})^{L_\kappa}$, and we have a model of the ZFC axioms. The result follows by the Second Incompleteness theorem.

Exercise 4.10 Assume $V=L$. When does $V_\alpha=L_\alpha$?

Exercise 4.12 Show that (i) if κ is a weakly inaccessible cardinal, then $(\kappa$ is strongly inaccessible$)^L$; (ii) if κ is a weakly Mahlo cardinal, then $(\kappa$ is strongly Mahlo$)^L$.[Hint: See Exercises 4.3 & 4.8. For (ii) show that the property of being cub in κ is preserved upwards from L to V.]

Sol: (i) Similar to Ex 4.8, if $\alpha<\kappa$ is a cardinal then $(\alpha$ *is a cardinal*$)^L$ (We check this one more time: for if α was not a cardinal in L, then in L for some $\gamma<\alpha$ there is a function $f\in L$ with $f\colon\gamma\to\alpha$ onto; but this is absolute: so f really does show in V that α is not a cardinal!) Hence $(\kappa$ is a limit cardinal$)^L$. But κ is not a singular cardinal in L for the same reason that there is no constructible function demonstrating its singularity. So κ is a regular limti cardinal of L. Hence it is (strongly) inaccessible, because - as in Execise 2.8 - no $\alpha<\kappa$ has its constructible power set of cardinality $\geq\kappa$.

(ii) This is slightly different. Suppose $C\in L$ and $(C$ is closed and unbounded in $\kappa)^L$. Note how being "closed and unbounded in κ" is in fact an absolute notion. So C really is such. As κ is weakly Mahlo in V there is a regular cardinal $\alpha\in C$. As being regular is downward absolute to L (Lemma 4.3 again) $(\alpha$ is regular$)^L$. As C was arbitrary, L sees that κ is weakly Mahlo. As weakly Mahlo implies that κ is weakly inaccessible, we have that κ is strongly inaccessible in L by part (i) and therefore *strongly* Mahlo there.

Exercise 4.13 (i) Let $\langle x,\in\rangle\prec L_{\omega_1}$ where $\omega_1=(\omega_1)^L$. Show that already Trans(x) and so $x=L_\gamma$ for some $\gamma\leq\omega_1$. [Hint: For $\delta<\omega_1$ note that $(|\delta|=|L_\delta|=\omega)^{L_{\omega_1}}$. Hence for $\delta\in x$, in L_{ω_1}, and thus in x, there is an onto map $f\colon\omega\longrightarrow L_\delta$. Thus, as $\omega\subseteq x\wedge f\in x$ we deduce that $\text{ran}(f)=L_\delta\subseteq x$. Deduce that Trans$(x)$.]

Sol: (i) Notice that it suffices to show that Trans(x): for then $(\text{ZF}^-)^x$ since $(\text{ZF}^-)^{L_{\omega_1}}$ (as $(\text{ZF}^-)^{H_{\omega_1}}$ by Lemma 2.29, but in L, $H_{\omega_1}=L_{\omega_1}$). By Lemma 4.12 this means $L^x=L_\gamma$ where $\gamma=\text{On}\cap x$. However x is itself a model of "$V=L$" (*i.e.* $\forall z\exists\beta(z\in L_\beta)$ because the same is true of L_{ω_1}. Thus x can only be L_γ! We now show transitivity: first note that L_{ω_1} is a model of the statement that everything is countable. Following the Hint, we see that for every δ in x, there is, also in x, an onto function $f_\delta\colon\omega\to L_\delta$. But $\text{ran}(f_\delta)\subseteq x$ since for every $n\in\omega$, n has a first order definition and so $n\in x$; as $f_\delta\in x$, we conclude that $f_\delta(n)\in x$. We deduce that $L_\delta\subseteq x$. As x is a model of $V=L$, for every $z\in x$ there is some $\beta\in x$ with $z\in L_\beta$. But we have just seen that $L_\beta\subseteq x$. Hence $z\subseteq L_\beta\subseteq x$ (the first inclusion coming from Trans(L_β)).

Exercise 4.18 Show (i) $z \in \mathrm{HOD} \longleftrightarrow z \in \mathrm{OD} \wedge \forall y \in z (y \in \mathrm{HOD})$. (ii) $\mathcal{P}(\omega) \cap \mathrm{OD} = \mathcal{P}(\omega) \cap \mathrm{HOD}$.

Sol: (i) $z \in \mathrm{HOD} \longleftrightarrow_{\mathrm{df}} z \in \mathrm{OD} \wedge \mathrm{TC}(z) \subseteq \mathrm{OD} \longleftrightarrow z \in \mathrm{OD} \wedge (z \cup \bigcup \{\mathrm{TC}(y) | y \in z\}) \subseteq \mathrm{OD}$
$\longleftrightarrow z \in \mathrm{OD} \wedge \forall y \in z (\{y\} \cup \mathrm{TC}(y)) \subseteq \mathrm{OD} \longleftrightarrow z \in \mathrm{OD} \wedge \forall y \in z (y \in \mathrm{HOD})$.

(ii): ($\supseteq$): trivial. ($\subseteq$): Every integer is in HOD. hence if $x \in \mathcal{P}(\omega) \cap \mathrm{OD}$, then $x \in \mathrm{HOD}$ by part(i).

Exercise 4.19 Show that for any β, $V_\beta \cap \mathrm{HOD} \in \mathrm{HOD}$.

Sol: If $z = V_\beta \cap \mathrm{HOD}$, then $z \subseteq \mathrm{HOD}$ so by the last exercise it suffices to show $z \in \mathrm{OD}$. But z is uniquely defined from the ordinal α as "the set of $y \in V_\beta$ so that $\forall u \in \mathrm{TC}(\{y\}) \exists \alpha (u \in \mathrm{Def}_0(V_\alpha))$".

Exercise 4.20 Show that the following are equivalent: (i) $V = \mathrm{OD}$, (ii) $V = \mathrm{HOD}$, (iii) Trans(OD), (iv) $(\mathrm{AxExt})^{\mathrm{OD}}$. [Hint: Use that for any α $V_\alpha \in \mathrm{OD} \wedge V_\alpha \cap \mathrm{OD} \in \mathrm{OD}$.]

Sol: (i) $\Rightarrow$ (ii) is straightforward from the definition of HOD. (ii) $\Rightarrow$ (iii): $\mathrm{HOD} \subseteq \mathrm{OD} \subseteq V \wedge \mathrm{Trans}(V)$. (iii) $\Rightarrow$ (iv): $\mathrm{Trans}(W) \longrightarrow (\mathrm{AxExt})^W$ for any class term W. (iv) $\Rightarrow$ (i): As $V_\alpha \cap \mathrm{OD} \in \mathrm{OD}$ (almost identical to Ex. 4.18) and also $V_\alpha \in \mathrm{OD}$, then by $(\mathrm{AxExt})^{\mathrm{OD}}$ if these are different, then for some $x \in \mathrm{OD}$, $x \in V_\alpha \cap \mathrm{OD} \,\triangle\, V_\alpha$. But these facts are contradictory.

Exercise 4.21 Show that $\mathrm{HOD} \cap \mathcal{P}(\omega)$ is the largest subset of $\mathcal{P}(\omega)$ with a definable wellorder.

Sol: If $B = \mathrm{HOD} \cap \mathcal{P}(\omega)$, then $B \subseteq \mathrm{OD}$ and so by restricting the definable wellorder of OD to B, it has a definable wellorder. By Lemma 4.23 if $A \subseteq \mathcal{P}(\omega)$ has a definable wellorder then $A \subseteq \mathrm{OD}$. But by Ex 4.17 then $A \subseteq \mathrm{HOD} \cap \mathcal{P}(\omega)$.

Exercise 4.22 Suppose that W is a term defining an inner model of ZF and there is a definable global wellorder of W (that, as in L, there is a formula defining a wellorder $<_W$ of the whole of W in order type On). Show that $W \subseteq \mathrm{HOD}$. (Consequently HOD is the largest inner model W with a definable bijection F: $On \longleftrightarrow W$.)

Sol: Again HOD has a definable wellorder by restricting the definable wellorder of OD to it. By Lemma 4.23 $W \subseteq \mathrm{OD}$. But $\mathrm{IM}(W) \Rightarrow \mathrm{Trans}(W)$, so $W \subseteq \mathrm{HOD}$.

Exercise 4.23 Define "Π_2-OD" (and Π_2-HOD) just as we did for OD and HOD but now restrict the formulae allowed in definitions to be Π_2 only. Show that Π_2-OD = OD and Π_2-HOD = HOD. Now do the same for Σ_2-OD and Σ_2-HOD.

Sol: First note that "$\rho(z) < \alpha$" can be expressed both in Σ_1 and Π_1-form: this is $\exists f [\Phi(f, z) \wedge f(z) < \alpha]$ and $\forall f\ [\Phi(f, z) \longrightarrow f(z) < \alpha]$ where

$\Phi(f, z) \equiv \mathrm{Fun}(f) \wedge \mathrm{TC}(\{z\}) \subseteq \mathrm{dom}(f) \wedge \forall t \in z (f(t) = \sup \{f(s) + 1 | s \in t\})$.

Then note that "$y = V_\alpha$" is Π_1. This is $\forall z (z \in y \longleftrightarrow \rho(z) < \alpha)$ (using each form in turn for $\rho(z) < \alpha$).

Let $x \in \mathrm{OD}$. Then there is some $\varphi(v_0)$ and some β so that

$\{x\} = \{w | V_\beta \vDash \varphi(w)\} = \{w | \forall y (y = V_\beta \rightarrow \langle y, \in \rangle \vDash \varphi[w])\}$.

Now use that "$\langle u, \in \rangle \vDash \varphi[w]$" is Δ_1 expressible and we have a Π_2 definition of x. But we can also write:

$\{x\} = \{w | \exists y \exists \beta (y = V_\beta \wedge \langle y, \in \rangle \vDash \varphi[w])\}$ in a Σ_2 form. Thus Σ_2-OD will be OD.

The equalities for Π_2-HOD = HOD and similarly Σ_2-HOD = HOD now just follow from the definition of HOD.

Exercise 4.24* Show that there is a *single* formula $\varphi_0(v_0)$ with just the free variable shown, so that OD is the class of all those x so that $x \in \mathrm{Def}_0(V_\beta)$ for some β, that is for some β, $\{x\} = \{z | \varphi_0(z)^{V_\beta}\}$.