Absolute Infinity

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Abstract

This article is concerned with reflection principles in the context of Cantor's conception of the set theoretic universe. We argue that within a Cantorian conception of the set theoretic universe reflection principles can be formulated that confer intrinsic plausibility to strong axioms of infinity.

1 Introduction

The perspective adopted throughout most of this article is that of set theoretic platonism, which holds that the mathematical universe consists of collections, possibly built on a collection of Urelements that are not themselves sets.

Within the set theoretic platonist outlook, we shall nevertheless take a 'pure' perspective: we shall assume that there are no mathematical objects that are not collections. In other words, we assume that there are no *Urelemente*. This entails that we take even mathematical objects that are often assumed to be somehow irreducible, such as real numbers and natural numbers, to be collections. This is merely a simplifying assumption. It reflects the fact that we are not concerned with the problems of indeterminacy of reference to which Benacerraf has drawn attention [Benacerraf 1965]. The arguments of the present article do not depend on the non-existence of Urelemente.

Even though he admitted Urelemente, Zermelo also held that there exist no collections beside sets. He held that the mathematical universe forms a potentially infinite sequence of sets of a special kind, which he called 'normal domains'. Quantification over sets is necessarily restricted: we cannot quantify over all sets. Zermelo's viewpoint allows the motivation of set theoretical principles that go beyond the standard principles of set theory and which lead to large cardinal axioms. Specifically, Zermelo's viewpoint leads to a set theoretic principle that posits the existence of many strongly inaccessible cardinals.

Cantor held that the set theoretic universe exists as a *completed* absolute infinity. The Burali-Forti paradox and Russell's paradox were initially interpreted as showing that Cantor's 'naive' set theory, as it is still sometimes called, is inconsistent. It was thought that Cantor had failed to recognise that the mathematical universe cannot itself constitute a set. Cantor himself protested that he never took the set theoretic universe as a whole to be a set. Nowadays, Cantor is rarely accused of having defended an outright inconsistent theory of sets. Nevertheless, according to the received view, Cantor's views about the set theoretic universe as whole are outdated, and ultimately philosophically untenable.[1]

Certainly there are, as we shall see, tensions in Cantor's view of the nature of the existence of the set theoretic universe. But we claim that a Cantorian viewpoint, when appropriately understood, is nevertheless more powerful and fruitful than Zermelo's view of the set theoretic universe. This is manifested in the motivation of reflection principles in set theory. It is known that on Zermelo's conception of the set theoretic universe, only weak reflection principles can be motivated, which give rise to small large cardinal principles [Zermelo 1930]. It is also known that a Cantorian conception of the set theoretic universe, as formalised by Von Neumann, can motivate somewhat stronger reflection principles [Bernays 1961], [Tait 2005]. We shall argue that a Cantorian viewpoint in fact motivates much stronger reflection principles, from which much stronger large cardinal axioms can be derived.

2 Zermelo

Zermelo was the first to hold that, Urelemente aside, the mathematical universe consists only of sets. Through the work of Zermelo, Fraenkel and von Neumann, it became established in the 1920s that sets are governed by the laws of ZFC.[2] This has become the most prevalent form of set theoretic platonism: there are only sets, and they obey the principles of ZFC.

Hereby the question is raised how the sets are related to the mathematical universe. Zermelo's viewpoint can be canvassed as follows [Zermelo 1930,

[1]For one expression of this view, see [Jané 1995].

[2]For an account of the role of Fraenkel and von Neumann in this development, in particular with respect to the axiom of Replacement, see [Kanamori 2004, section 5].

p. 1231–1233].[3] When we are engaged in set theory, our quantifiers always range over a domain of discourse D, which Zermelo calls a 'normal domain'. The entities over which our set theoretic quantifiers range are sets: they are governed by the principles of standard set theory (ZFC). Our domain of discourse D itself is also a collection. Since there are no collections other than sets (and Urelemente, for Zermelo, but we disregard them here), our domain of discourse must also be governed by the principles of ZFC. But, on pain of contradiction, D can then not be included as an element in our domain of discourse. Nonetheless, we can expand our domain of discourse so that it includes D as an element. The expanded domain of discourse D' can even be taken to be such that it also satisfies the principles of ZFC. But the expanded domain D' will again be a set. So the previous considerations apply to D' also: it cannot contain itself as an element, even though we can expand it further so as to remedy this defect. In sum, even though the domain of discourse can always be expanded, it never comprises all sets. The upshot is that for Zermelo, the mathematical universe is a potential infinite sequence of (actually infinite) domains of discourse that satisfy the principles of ZFC:[4]

> What appears in one model as an 'ultrafinite non- or superset' is in the succeeding model already a perfectly good, valid 'set' with a cardinal number and ordinal type, and it is itself a foundation-stone for the construction of a new domain. To the unlimited series of Cantor's ordinal numbers there corresponds a likewise unlimited double series [of V_α's and the membership relation restricted to these V_α's] of essentially different set-theoretic models in each of which the whole classical theory is expressed. The two opposite tendencies of the thinking spirit, the idea of creative *advancement* and that of collective completion [*Abschluss*] [...] are symbolically represented and reconciled in the transfinite number series based on well-ordering. This series in its unrestricted progress reaches no true completion; but it does possess relative stopping points, namely those 'limit numbers' which separate the higher from the lower models. (Zermelo 1930, p. 1233)

There are basic structural insights about the set theoretic universe that escaped Cantor. For instance, Zermelo in his later years after adopting the

[3] For a detailed description of Zermelo's technical results and philosophical view as articulated in [Zermelo 1930], see [Kanamori 2004, section 6].

[4] A similar view is expressed in [Parsons 1974, p. 219].

Axiom of Foundation viewed the set theoretic universe as structured into a layered hierarchy of initial segments V_α (with α ranging over the ordinals) that are sets.[5] And Zermelo even saw that there might be segments V_α that in a strong sense make all the axioms of ZFC true. Cantor did not see that far. Nonetheless, we shall argue that Cantor's conception of the set theoretic universe as a completed infinity is more powerful and more plausible than that of Zermelo.

Incidentally, Zermelo's picture does raise some difficult questions. For one thing, it is not clear in which dimension the mathematical universe is supposed to vary. The notion of *creative advancement* suggests some form of progression or growth but this is not a metaphor that can be stretched too far.[6] For another, there is the question how Zermelo can get his view across to us.[7] (*What* is it that we cannot quantify over?) This does not imply that Zermelo's thesis about the essential restrictedness of quantification is false; however, it does seem difficult to see how this thesis can be communicated. We shall not pursue these worries here,[8] but we note that this can be taken as a reason for preferring Cantor's view over Zermelo's.

3 Cantor on the set theoretic universe

Cantor's theory of the nature of the set theoretic universe as a whole is not easy to summarise. His views seem to have undergone a transformation around 1895. We first discuss his earlier views, and then turn to his later views.

3.1 The Absolutely Infinite

Cantor's basic convictions preclude Zermelo's potential infinity of (completed) normal domains ever to be the final word about the nature of the set theoretic universe. The set theoretic universe could not, in Cantor's view, form a potential infinity of actual infinities because of what Hallett

[5]Mirimanoff anticipated this hierarchy in [Mirimanov 1917], and the work of Von Neumann was also crucial: see [Kanamori 2004, p. 518].

[6]For an interesting attempt to unearth the literal content of this metaphor, see [Linnebo 2010].

[7]Zermelo calls his theory about normal domains 'meta-set theory' [Zermelo 1930, p. 1233]. But he does not elaborate on the question what the domain of discourse of meta-set theory is.

[8]For a discussion of various aspects of this problem, see the essays in [Rayo & Uzquiano 2006].

calls Cantor's *domain principle* [Hallett 1984, p. 7–8], which says that every potentially infinite variable quantity presupposes an underlying fixed and completed domain over which the potentially infinite entity varies:[9]

> There is no doubt that we cannot do without variable quantities in the sense of the potential infinite; and from this the necessity of the actual infinite can also be proven, as follows: In order for there to be a variable quantity in some mathematical inquiry, the 'domain' of its variability must strictly speaking be known beforehand through a definition. However, this domain cannot itself be something variable, since otherwise each fixed support for the inquiry would collapse. Thus, this 'domain' is a definite, actually infinite set of values. Thus, each potential infinite, if it is rigorously applicable mathematically, presupposes an actual infinite. (*Mitteilungen zur Lehre vom Transfiniten VII* (1887): [Cantor 1932, p. 410–411], our translation)[10]

In particular, this means that even every absolute infinity of transfinite sets that potentially exists presupposes an actual, completed absolutely infinite domain as its range of variation.

Admittedly Cantor was in his writings not very explicit about what he did take the set theoretic universe as a whole to be. One problem is that it is not in every instance clear whether he has a theological or a mathematical conception of absolute infinity in mind. Indeed, he argues that it is the task not of mathematics but of 'speculative theology' to investigate what can be humanly known about the absolutely infinite [Cantor 1932, p. 378]. The following passage, for example, leans heavily to the theological side:

> I have never assumed a "Genus Supremum" of the acual infinite. Quite on the contrary I have proved, that there can be no such "Genus Supremum" of the actual infinite. What lies beyond all that is finite and transfinite is not a "Genus"; it is the unique, completely individual unity, in which everything is, which comprises everything, the 'Absolute', for human intelligence unfathomable, also that not subject to mathematics, unmeasurable, the "ens simplicissimum", the "Actus purissimus",

[9]The domain principle is extensively discussed in [Hallett 1984, chapter 1, section 2].

[10]In this quotation, Cantor speaks of the necessity of 'knowing' the domain of variation through a 'definition'. Surely Cantor is merely sloppy here, and we are should discount the epistemological overtones. Another slip can be detected in Cantor's use of the word 'set' in this quotation. Cantor means the argument to be applicable not just just to sets but also to absolute infinities.

> which is by many called “God”. (Letter to Grace Chisholm-Young (1908): [Cantor 1991, p. 454], our translation).

All this is related to the fact that in an Augustinian vein, Cantor takes all the sets to exist as ideas in the mind of God:[11]

> The transfinite is capable of manifold formations, specifications, and individuations. In particular, there are transfinite cardinal numbers and transfinite ordinal types which, just as much as the finite numbers and forms, possess a definite mathematical uniformity, discoverable by men. All these particular modes of the transfinite have existed from eternity as ideas in the Divine intellect. (Letter to Jeiler (1895): [Tapp 2005, p. 427], our translation)

Nonetheless, in many instances it is clear that Cantor has a mathematical conception of the absolutely infinite in mind:

> The transfinite, with its wealth of arrangements and forms, points with necessity to an absolute, to the ‘true infinite’, whose magnitude is not subject to any increase or reduction, and for this reason it must be quantitatively conceived as an absolute maximum. (*Mitteilungen zur Lehre vom Transfiniten V* (1887): [Cantor 1932, p. 405], our translation)

This is the notion of absolutely infinite that we shall concentrate on in this article. We shall from now on disregard what Cantor takes to be the theological aspects of the mathematical absolutely infinite; we shall instead concentrate on Cantor’s conception of the ‘quantitatively absolute maximum’, which is the set theoretic universe as a whole. From the passages discussed above, we conclude that he attributes to it the following properties. It is a fully determinate, fully actual (‘completed’), inaugmentable totality. It is composed of objects (sets) that are of a mental nature (‘ideas’). And unlike the sets in the mathematical universe, the universe as a whole cannot be uniquely characterised.

3.2 Inconsistent multiplicities

From around the time when Burali-Forti published his ‘paradox’ [Burali-Forti 1897], one finds a subtle change of terminology in Cantor’s writings. Whereas be-

[11] For Cantor’s most detailed account of the set theoretic universe in God’s mind, see [Tapp 2005, p. 414–417]. See also *Mitteilungen zur Lehre vom Transfiniten V*, footnote 3 [Cantor 1932, p. 401–403].

fore, Cantor used the expression 'the Absolutely Infinite' for characterising the set theoretic universe, he now categorises the set theoretic universe and other proper classes (such as the class of all ordinals) as *inconsistent multiplicities*:

> If we assume the concept of a determinate multiplicity (of a system of, of a realm ['Inbegriff'] of) things, then it has proved to be necessary to distinguish two kinds of multiplicity (I always mean determinate multiplicities).
>
> A multiplicity can be of such nature, that the assumption of the 'togetherness' ['Zusammenseins'] of its elements leads to a contradiction, so that it is impossible to conceive the multiplicity as a unity, as a 'finished thing'. I call such multiplicities absolutely infinite or inconsistent multiplicities. (Letter to Dedekind (1899), [Cantor 1932, p. 443], our translation)

Jané has argued that passages such as these indicate that Cantor no longer believed that the set theoretic universe forms a completed infinity [Jané 1995, section 6, section 7]. The strongest evidence for this thesis is perhaps the following quote from a letter from 1899 from Cantor to Hilbert:

> I am now used to call 'consistent' what before I referred to as 'completed', but I do not know if this terminology deserves to be maintained. (Letter to Hilbert (1899): [Cantor 1991, p. 399])

Jané speculated in [Jané 1995] that instead of conceiving of the set theoretic universe as a completed whole, Cantor tacitly moved to a conception of the set theoretic universe as an irreducibly potential entity, whereby he arrived at a pre-figuration of Zermelo's conception of the mathematical universe. This means that he must have by that time tacitly given up on the domain principle which says that every potential infinite has as its domain of variation an underlying completed infinite.

In his more recent [Jané 2010], Jané no longer claims that Cantor actually gave up the thesis of the existence of the mathematical universe as a completed infinity. But Jané rightly stresses that there remains a tension between Cantor's earlier commitments and Cantor's later terminology of inconsistent multiplicities:

> I submit that, owing to Cantor's allegiance to a changeless mathematical universe, Cantor's explanations [of the concept of inconsistent multiplicity] are indeed unconvincing. For how can

> the elements of a multiplicity fail to coexist if they all inhabit the same universe? [Jané 2010, p. 223]

And he thinks that the best way for Cantor to resolve this tension would be to embrace Zermelo's conception of the set theoretic universe as essentially open-ended.

Not everyone agrees with Jané's interpretation. It is true that Cantor's choice of words in the letter to Hilbert indicates that he no longer believed that the set theoretic universe can be mathematically understood as a whole. But the passages do not show that Cantor no longer believed that the set theoretic universe does not form an inaugmentable totality that forms the domain of our mathematical discourse past, present, and future. In Hauser's words:

> [B]y 'existing together' Cantor evidently means 'existing together as elements of a "finished" set'. Thus, what he is saying is merely that the totality of all transfinite numbers (or all alephs) does not constitute a set and therefore cannot be an element of some other set. But he is not denying that the transfinite numbers coexist in some other form, namely as *apeiron*, which is mathematically indeterminate, meaning that one cannot assign a cardinal or ordinal number to the totality of all numbers. [Hauser 201?, section 3]

The content of the notion 'apeiron' that one finds in Plato is notoriously unclear. So this does not really help much in the clarification of the nature of the set theoretic universe. In other words, there is an unresolved interpretative debt at this point on the side of the defender of the Cantorian viewpoint. It seems that Jané is right that Cantor (or his defender) is facing a choice. Either she upholds Cantor's earlier view of the set theoretical universe and tries to make good philosophical sense of it. Or she takes Cantor's characterisation of the mathematical universe as an inconsistent multiplicity as the final word, and tries to make sense of that. But both cannot be done at the same time.

What we propose to do is in the first instance to ignore Cantor's description of the set theoretic universe as an inconsistent multiplicity. In the following sections, we shall adopt Cantor's characterisation of the set theoretic universe as a completed whole, and discuss how it can be used to motivate 'top down' reflection principles. Then we shall formulate a stronger reflection principle. And we shall see that to make sense of this stronger reflection principle, elements both of Cantor's earlier views and elements of Cantor's later views on the nature of the set theoretic universe can be used.

4 Reflection

It is a central theme of the Judeo-Christian theological tradition that God is fundamentally ineffable.[12] Cantor was well aware of this tradition and he extended it it to mathematical absolutely Infinities. After Cantor's time, in modern set theory, this view has been given *positive* expressions, that somewhat surprisingly have mathematical strength. These statements are known as *reflection principles*.

4.1 The very idea

The starting point of set theoretic reflection is the early Cantorian view that the mathematical Absolutely Infinite is unknowable:

> The Absolute can only be acknowledged, but never known, not even approximately known. (*Grundlagen einer allgemeinen Mannigfaltigkeitslehre* (1883), endnote to section 4: [Cantor 1932, p. 205])

There are obvious connections with central themes in theology, especially with the medieval doctrine that only negative knowlege is possible of God (apophatic theology). As it stands, it is indeed a negative statement. However it can be given a positive interpretation as follows. Let us provisionally identify the mathematical Absolutely Infinite with the set theoretic universe as a whole (V). V is unknowable in the sense that we cannot single it out or pin it down by means of any of our assertions: no true assertion about V can be made that excludes unintended interpretations that make the assertion true. In particular—and this is stronger than the previous sentence—no assertion that we make about V can ensure that we are talking about the mathematical universe rather than an object *in* this universe. So if we do make a true assertion ϕ about V, then there exist sets s such that ϕ is also true when it is interpreted in s.

Now in the late 1890s the Burali-Forti theorem (after some initial confusion on the part of Russell and even Burali-Forti himself) made it abundantly clear that V is not the only actual whole that is absolutely infinite. So

[12] This doctrine was for instance defended by the fifth century philosopher-theologian pseudo-Dionysius Areopagita [Dionysius 1920].

in light of this we must say that the mathematical absolutely infinite comprises, in addition to the mathematical universe as a whole all other proper classes.[13] But in fact, the above argument should hold true for any proper class. They can then be said to be unknowable in the sense that no assertion in the language of sets can be true of only a proper class. So if we do make a true assertion concerning a proper class, then there exists sets about which this assertion is already true. If we truly describe mathematical absolute infinities, then there are set proxies for the absolute infinities such that our description can also truly be taken to range over the proxies.

Cantor did not explicitly articulate this line of argument. Yet he was probably the first one to make use of reflection as a principle motivating the existence of sets. He argues that the finite ordinals form a set because they can be captured by a definite condition:

> Whereas, hitherto, the infinity of the first number-class (I) alone has served as such a symbol [of the Absolute], for me, precisely because I regarded that infinity as a tangible or comprehensible idea, it appeared as an utterly vanishing nothing in comparison with the absolutely infinite sequence of numbers. (Grundlagen einer allgemeinen Mannigfaltigkeitslehre (1883), endnote to section 4: [Cantor 1932, p. 205])

This can be seen as an application of a a reflection principle [14] Being closed under the successor operation is a set theoretic property of a mathematical absolute infinity (the ordinals). Reflection then allows us to infer that there must be a *set* that is closed under the successor operation, and hence that there must be a minimal such. This is the set ω. Anachronistically, one is tempted to say that Cantor is appealing to something like Montague-Levy reflection, which is a first-order reflection property that is provable in ZFC.[15]

4.2 Set reflection

On the face of it, Zermelo's viewpoint uses a form of set theoretic reflection: every admissible domain of discourse in set theory is a 'normal domain',

[13] Cantor's 1899 argument that the ordinals form an inconsistent totality is critically discussed in [Jané 1995, p. 395–396].

[14] Admittedly this passage is sufficiently vague as to be open to multiple interpretations. The view that this passage should be seen as an application of reflection is defended in [Hallett 1984, p. 117–118].

[15] The Montague-Levy reflection principle is discussed in [Drake 1974, Chapter 3, section 6].

and this can by a reflective movement be seen to be a set. We cannot quantify over or in any way make use of proper classes, for, in his view, no such things exist. The set theoretic universe as a whole is not something we can talk about, according to Zermelo, for it never exists as a completed realm. So, literally speaking, Zermelo cannot, according to his own view, truly say that "*the* set theoretic universe is so rich that it contains many normal domains".

The best Zermelo can do is simply to postulate that above every ordinal, there is an ordinal which is the 'boundary number' of a normal domain. In modern terms, this is expressed as an axiom that postulates unboundedly many strongly inaccessible cardinals:

Axiom 1 $\forall\alpha\exists\beta : \beta > \alpha \wedge$ " β *is a strongly inaccessible cardinal* ".

This *seems* to say exactly what is required. It says that a fundamental property of the set theoretic universe, namely making ZFC^2 true, is reflected in arbitrarily large set-sized domains. But closer inspection reveals that this cannot exactly be the case: there must be ordinal numbers that fall outside the quantifiers in this axiom. By Zermelo's own lights, the quantifiers in axiom 1 must range over a domain of discourse that forms a set in a wider domain of discourse. There will be ordinals in this wider domain of discourse that do not belong to the 'earlier' domain of discourse.

Nonetheless, axiom 1 and its relatives have some proof theoretic strength. They do postulate the existence of 'small large cardinals' to which ZFC is not committed [Drake 1974, chapter 4].

4.3 Class reflection

Stronger reflection principles can be formulated if we take Cantor's idea of absolutely infinite multiplicities seriously. However, to study these reflection principles in a precise setting, logical laws governing them have to be formulated. The language that is assumed is the language of second-order (or two-sorted, if you will) set theory, where the membership symbol is expressing the only fundamental non-logical relation, and where we have two types of variables: the first-order variables range over sets $(x, y, \ldots)$ and the second-order variables range over (proper and improper) classes $(X, Y, \ldots)$. we shall from now on take the sets and classes to be governed by the principles of Von Neumann-Bernays-Gödel (NBG) class theory (and worry about the justification for this later). Indeed, von Neumann's class theory, the pre-cursor to Bernays' formulation of NBG, can be seen as a formalisation of Cantor's viewpoint (but not as a conceptual clarification).[16]

If we take the point of view of Cantor's early theory of the mathematical universe, and take the point that there are more absolutely infinite collections than V alone, then we can express the reflection idea as follows:

Axiom 2 $\forall X : \Phi(X) \to \exists \alpha : \Phi^{V_\alpha}(X \cap V_\alpha)$,

where Φ^{V_α} is obtained by relativising all first- and second-order quantifiers to V_α and its power set, respectively, and where α does not occur free in Φ.

Zermelo's reflection principle (axiom 1) only expresses that certain true class theoretical statements are reflected downwards (the axioms of ZFC^2). Axiom 2 states that *every* true (second order parametrised) class theoretic statement is reflected down to some set sized domain. Axiom 2 is stronger than axiom 1: it implies large cardinal principles that postulate indescribable cardinals.[17] This axiom and its relatives were discussed in [Bernays 1961].

Of course it is then natural to formulate reflection principles of orders higher than two in an analogous manner. However already the full third-order class reflection principle is inconsistent, at least for formulae that involve general parameters [Reinhardt 1974],[Koellner 2009, section 3].[18] Third-order reflection restricted to a certain class of "positive" formulae is consistent and stronger than second-order class reflection [Tait 2005].[19] It entails the existence of ineffable cardinals, but does not prove the existence of the least ω Erdős cardinal $\kappa(\omega)$ [Koellner 2009, section 4]. Fourth-order reflection is inconsistent even when restricted to "positive" formulae [Koellner 2009, section 5]. Indeed, Koellner gives an argument to the effect that no "internal reflection prinicple" can ever entail large cardinal principle axioms that are as strong as the principle that postulates the existence of $\kappa(\omega)$.[20]

He issues the following challenge [Koellner 2009, section 7]:

[16] See [von Neumann 1925].

[17] For a discussion of indescribable cardinals, see [Drake 1974, chapter 9].

[18] Parameter free sentences of higher orders are unproblematic.

[19] See also [Marshall 1989] for an account of higher order reflection with restricted kinds of parameters.

[20] The essential point is that taking a skolem hull of the ω-sequence of indiscernibles guaranteed by $\kappa(\omega)$, we can obtain a countable transitive model $(M, \in)$, and then by shifting indiscernibles around, a non-trivial first-order elementary embedding

$$j_0 : (M, \in) \to (M, \in).$$

Then any "generalised reflection principle" weaker than, or obtainable in ZF from the supposed existence of a $j : (V, \in) \to (V, \in)$, would presumably be consistent relative to the

. . . Formulate a strong reflection principle which is intrinsically justified on the iterative conception of set and which breaks the $\kappa(\omega)$ barrier.

However he expresses scepticism that this challenge can be met.

In sum, the situation is this. From Zermelo's conception of the set theoretic universe as a potential infinity of sets, the region of small large cardinals in the neighbourhood of inaccessible cardinals can be motivated. Due to its recognition of proper classes alongside of sets, the Cantorian point of view can be said to lead to the above stronger reflection principles of class reflection. However even those principles do not take us beyond the small large cardinal principles consistent with V being Gödel's constructible universe L. In particular, the class reflection principles fall below the strength of postulating measurable cardinals. Indeed, the (tentative) conclusion of [Koellner 2009] is that class theoretic reflection principles are either weak or inconsistent.

5 Global reflection

Gödel thought that *all* sound large cardinal principles can be reduced to reflection principles:

> All the principles for setting up the axioms of set theory should be reducible to Ackermann's principle: The Absolute is unknowable. The strength of this principle increases as we get stronger and stronger systems of set theory. The other principles are only heuristic principles. Hence, the central principle is the reflection principle, which presumably will be understood better as our experience increases. Meanwhile, it helps to separate out more specific principles which either give some additional information or are not yet seen clearly to be derivable from the reflection principle as we understand it now. [Wang 1996, 8.7.9].

This sentiment goes against the conclusions that Koellner reached, and is often regarded as implausible, because the familiar reflection principles are compatible with the principle that $V = L$. Nonetheless, we shall now argue that from a Cantorian point of view there may be more to Gödel's conjecture than is commonly thought.

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countable model case of j_0 and $(M, \in)$. Here the "generalised reflection principles" intended are those resembling those of the kind already discussed above.

Gödel himself was an adherent of Cantor's actualist viewpoint regarding the set theoretic universe rather than of Zermelo's potentialist viewpoint:

> To say that the universe of all sets is an unfinished totality does not mean objective undeterminateness, but merely a subjective inability to finish it. (Gödel, as reported in [Wang 1996, 8.3.4])

We have seen that the set theoretic universe as a whole and all classes of sets are recognised by Cantor to (actually) exist: let us call this structure $\langle V, \in, \mathcal{C} \rangle$. Then the reflection idea tells us that we cannot single this structure out by means of any of our assertions. Positively put, any assertions that hold in $\langle V, \in, \mathcal{C} \rangle$ must also hold in some set-size structure.[21]

There are various possible ways of trying to making this more precise. we shall not try to give a catalogue of the pro's and contra's of various options. Rather, we shall concentrate on one way that seems to us especially powerful, natural, and fruitful. Consider the following principle:

Axiom 3 *There is an ordinal κ and a nontrivial elementary embedding*

$$j : \langle V_\kappa, \in, V_{\kappa+1} \rangle \longrightarrow \langle V, \in, \mathcal{C} \rangle$$

with critical point κ (i.e., $j(\kappa) > \kappa$ whereas below κ, j is the identity transformation).

As was mentioned earlier, it is to be assumed that $\langle V, \in, \mathcal{C} \rangle$ makes at least the principles of NBG true. Let us call this principle the *Global Reflection Principle* (GRP). What the embedding function does is to act as the identity function on all elements of V_κ but to send the elements of $V_{\kappa+1}$ to elements of $\mathcal{C}$: $j(\kappa) = On, j(V_\kappa) = V, j(Card \cap \kappa) = Card, \ldots$ So axiom 3 says that the set theoretic universe (with all its proper classes) is reflected in a particular way to a set-size initial segment of the universe.

The level of elementarity that is insisted upon can be varied. For the most part of this article we will require only Σ^0_1-elementarity where formulae are allowed to have class variables $X, Y, X, \ldots$ but are restricted to have only existential quantifiers which range over sets alone: we denote the resulting global reflection principle as $GRP_{\Sigma^0_1}$. But we could also insist on Σ^0_∞-elementarity or even Σ^1_∞-elementarity (denoted as $GRP_{\Sigma^0_\infty}$, $GRP_{\Sigma^1_\infty}$,

[21] We will see later (section 6.4) that the expression "any assertions" in this statement may need to be qualified.

respectively). Often, however, we leave the level of elementarity required by the principle unspecified and simply speak of *GRP*.[22]

The principle *GRP* says that the universe with its parts is, to a certain degree, indistinguishable from at least one of its initial parts V_κ and *its* parts. It says that the whole set theoretic universe with all its proper classes is mirrored in a set-sized initial segment $\langle V_\kappa, \in, V_{\kappa+1}\rangle$, where the first-order quantifiers range over V_κ, and where the reflection of a proper class C is obtained by 'cutting it off' at level V_κ.

GRP expresses the idea of reflection in a more powerful way than axiom 2. Axiom 2 just says that each (second-order) statement is reflected from the set theoretic universe to some V_κ (where possibly different second-order statements are reflected in different V_κ's): therefore it does not entail that the universe as a totality particularly resembles any one *single* set-like initial segment. However *GRP* postulates that the whole universe $\langle V, \in, \mathcal{C}\rangle$ is indistinguishable from an initial 'cut' $\langle V_\kappa, \in, V_{\kappa+1}\rangle$ in a very specific way, namely in a way such that no large 'set' and no proper class can be distinguished from a proper subset of itself (its intersection with V_κ and with $V_{\kappa+1}$, respectively).

Thereby *GRP* is a more robustly ontological form of reflection than axiom 2. In this respect, there is a striking connection with theological ideas that have a long history, as the following passage shows (Odo Reginaldus, quoted in [Côté 2002, p. 78, our translation].):

> How can the finite attain [knowledge of] the Infinite? On this question some said that God will show Himself to us in a mediated way, and that he will show Himself to us not in His essence, but in created beings. This view is receding from the aula...[23]

Of this passage, van Atten remarks [van Atten 2009, footnote 84, p. 22]:

> From here it is only a small step to: "Suppose creature *A* has a perception of God. Then God is capable of making a creature *B* such that *A*'s perception cannot distinguish between God and *B*."

[22]It is natural to strengthen Axiom 3 by requiring that for every ordinal α, there is a reflection $\langle V_\kappa, \in, V_{\kappa+1}\rangle$ that includes this ordinal α. This results in a *strong* version of *GRP*. We shall in the sequel concentrate on explicating and motivating the slightly weaker version of *GRP* that is captured by axiom 3. However everything we say generalises straightforwardly to the stronger axiom.

[23]"Quomodo potest finitum attingere ad infinitum? Propter hoc dixerunt alii quod deus contemperatum se exhibebit nobis, et quod ostendet se nobis non in sua essentia, sed in creatura."

Indeed, we conjecture that the "view that is receding from the aula" to which Reginaldus is referring traces back to Philo of Alexandria, who writes in his *On Dreams*:[24]

> Thus in another place, when he had inquired whether He that is has a proper name, he came to know full well that He has no proper name, [the reference is to Exodus 6:3] and that whatever name anyone may use for Him he will use by licence of language; for it is not in the nature of Him that is to be spoken of, but simply to be. Testimony to this is afforded also by the divine response made to Moses' question whether He has a name, even "I am He that is (Exodus 3:14)." It is given in order that, since there are not in God things that man can comprehend, man may recognise His substance. To the souls indeed which are incorporeal and occupied in His worship it is likely that He should reveal himself as He is, conversing with them as friend with friends; but to souls which are still in the body, giving Himself the likeness of angels, not altering His own nature, for He is unchangeable, but conveying to those which receive the impression of His presence a semblance in a different form, such that they take the image to be not a copy, but that original form itself.

Although we have seen that Cantor was deeply familiar with the idea of God as ineffable, there is no textual evidence to suggest that he was familiar with theological literature in which the uncharacterisability of God is transformed into a *positive* principle, as was done in the passages above. Yet we have seen that Cantor at least once only more or less explicitly made use of a mathematical reflection principle. But then it was done only a fairly minimal way, namely, to argue for the existence of ω as a set. *GRP* is clearly a *much* stronger reflection principle than the one that Cantor implicitly appealed to in the quoted passage (Montague-Levy). But it can be seen as the class-theoretic translation of the theological thesis that is defended by Philo of Alexandria. Just that in the theological context, there are 'angels' such that every humanly describable property of God also applies to them, in the class theoretic context there are some sets such that every property of the universe also holds when relativised to them.

But *is GRP* a reflection principle? In contrast with traditional reflection principles such as axiom 2, the reflection effected by *GRP* is mediated by

[24] As quoted in [Segal 1977, p. 163].

the embedding function j. For this reason, Koellner suggests that therefore GRP is more aptly called a *resemblance principle* (or perhaps a *projection principle*) than a reflection principle.[25]

In our view, this is largely a terminological dispute. GRP captures the idea that the theory of the universe is reflected in an initial segment of the V-hierarchy. It also posits a connection (the mapping j) between the initial segment and the universe. Perhaps it is possible to 'split' the content of GRP into an embeddingless reflection principle on the one hand, and a strong class choice principle on the other hand (which yields the embedding function). We will not pursue this possibility in this paper, but merely note that Zermelo viewed choice axioms as *logical* principles [Zermelo 1904, p. 141].

6 Sets, parts, and pluralities

Now that the philosophical motivation behind, and the content of, GRP has been explained, we turn to the ontological assumptions of the framework in which it is formulated.

6.1 GRP as a second-order principle

So far we have only expressed GRP in a semi-formal way — in a manner of speaking often adopted by set theorists. If we want to formally express GRP, then at first blush it seems that we need a language of third order: the function j that is postulated to exist pairs sets of V_κ with themselves and sets of $V_{\kappa+1}$ with proper classes (elements of $\mathcal{C}$). But of course the mapping j that is postulated by GRP can in fact be *coded* as a second-order object: as a proper class K consisting of ordered pairs such that its first element a is in the domain of j (namely: $V_{\kappa+1}$) and the second element $j(a)$ is an element of $V_\kappa \bigcup \mathcal{C}$.

We also need a satisfaction predicate to express the elementarity of the embedding. GRP deploys two notions of truth: truth in the structure $\langle V_\kappa, V_{\kappa+1}, \in \rangle$, and truth in $\langle V, \mathcal{C}, \in \rangle$. Truth in $\langle V_\kappa, V_{\kappa+1}, \in \rangle$ can of course be defined in the language $\mathcal{L}^2_\in$, whereas truth in $\langle V, \mathcal{C}, \in \rangle$ cannot. But for our proof-theoretic purposes, adding a Tarskian compositional satisfaction predicate T to $\mathcal{L}^2_\in$ and postulating that the compositional truth axioms hold for $\mathcal{L}^2_\in$ suffices to express what it means for a statement of $\mathcal{L}^2_\in$ to be true and to prove basic properties of truth. In sum, the fact that GRP postulates the

[25] Peter Koellner, personal communication with the first author.

elementarity of the embedding j even if we take a strong version of GRP that is Σ^1_∞ preserving does not necessitate us to go up to third order.

6.2 Parts of V

As mentioned earlier, Cantor's distinction between sets and Absolute Infinities is a prefiguration of the distinction between sets and proper classes, which was articulated explicitly by von Neumann. The difference with Cantor's theory is that von Neumann did take classes as well as sets to be governed by mathematical laws. It is just that classes are objects *sui generis*: they obey different laws. Proper classes are objects that have elements, but they are not themselves elements. So, in particular, there is no analogue of the power set axiom for proper classes.

However it remained an open question how talk of proper classes ought to be interpreted. In particular, if proper classes are taken to be super-sets in some sense, then it is somewhat mysterious why they can have elements but not be elements. In Maddy's words [Maddy 1983, p. 122]:

> The problem is that when proper classes are combinatorially determined just as sets are, it becomes very difficult to say why this layer of proper classes a top V is not just another stage of sets we forgot to include. It looks like just another rank; saying it is not seems arbitrary. The only difference we can point to is that the proper classes are banned from set membership, but so is the κth rank banned from membership in sets of rank less than κ.

And then why is there no singleton, for instance, that contains the class of the ordinals as its sole element? An alternative would be to say that proper classes can be collected into new wholes, but that these could (for obvious reasons) not themselves be proper classes. They would be again a *sui generis* kind of objects: super-classes. But in this way we embark on a hierarchical road that few find worth traveling. On this picture, classes, super-classes, et cetera, look too much like sets. We seem to be replicating the cumulative hierarchy of sets whilst incurring the cost of introducing a host of different kinds of set-like objects.

We propose to adopt a *mereological* interpretation of proper classes. We could say that the mathematical universe is a mereological whole and classes, proper as well as improper, are parts of the mathematical universe. We identify those parts of V that are also parts of set, i.e., that are set sized, with sets. The threat of a hierarchy of super- and hyper-wholes is not looming here. The fusion of the parts of a whole does not create a super-whole, but just the whole itself. So there is no mereological analogue of the creative force of the power set axiom.[26]

The mereological interpretation of *classes* that we are proposing here is similar to David Lewis' interpretation of *sets* [Lewis 1991], [Lewis 1993]. Lewis takes sets to be generated by the singleton function and unrestricted mereological fusion. So sets have subsets as their mereological parts. Similarly, in our proposed interpretation of classes, every class is a fusion of a bunch of singletons, and classes have sub-classes (and not their elements) as their proper parts. Also sets will have sub-sets as their parts; but in contrast to proper classes, they *are* also elements (of sets and of classes).

The difference between our proposed interpretation of the range of the second-order quantifiers and Lewis' theory of classes is that we take set theory as given. Lewis regards the relation between an entity and its singleton as thoroughly "mysterious" [Lewis 1991, section 2.1]. Reluctantly, he takes it to be a structural relation [Lewis 1991, section 2.6]. Derivatively, there is, in Lewis' approach, something mysterious about all sets. We are in a different position. As stated in the introduction, in this article we take set theory as given, and do not commit ourselves to any specific interpretation (reductive or non-reductive) of the membership relation. Given the singleton-relation that is part of set theory, we can explain the elementhood relation for classes in a straightforward way. Explaining the element-relation for sets is outside the scope of this article.

The mereological interpretation of classes satisfies the two desiderata that according to Maddy an interpretation of class theory has to satisfy simultaneously [Maddy 1983, p. 123]:[27]

1. Classes should be real, well-defined entities;

2. Classes should be significantly different from sets.

The first desideratum is satisfied because classes are just as real and well-defined as sets. The second desideratum is satisfied because the laws of parthood are significantly different from the laws governing sets.

[26] Even the Augustinian idea that sets are ideas in God's mind is compatible with this view. Within such a framework, the mereological conception of classes would result in conceiving of classes (proper and improper) as *parts* of God's mind.

[27] It seems to us that Maddy's own view of classes does not completely satisfy the first desideratum. The reason is that she takes the class membership relation to be governed by partial logic. According to her theory, there is in many cases no fact of the matter whether a given class is an element of another given class.

6.3 Pluralities of sets?

An alternative explication of the range of the second-order quantifiers appeals to plural quantification. Boolos has argued that second-order quantification can be interpreted in such a way that it does not commit one to *classes* of elements of the underlying domain [Boolos 1984]. This is done by reading a formula of the form $\exists X : \phi(X)$ as: "there are some entities such that ϕ holds of *them*". In [Boolos 1985], this plural interpretation is applied to second-order ZFC. This gives us a plural interpretation of class theory. In other words, we can recognise the truth of systems of class theory without recognising anything beyond sets in our ontology: *class theory without proper classes*.[28]

There is then also no prima facie reason to think that a plural reading of the second-order quantifiers commits us to a potentialist conception of the set theoretic universe: if the first-order quantifiers are interpreted in an actualist fashion, then the plural interpretation of the second-order quantifiers rides piggy-back on that. Also, it is usually not assumed that the existence of pluralities is tied, even loosely, to the existence of defining conditions (over V) [Boolos 1985], [Hazen 1993]. Thus it is commonly held that the impredicative second-order comprehension scheme of Morse-Kelley (MK) holds for pluralities of sets. The interpretation of second-order quantification as plural quantification can be taken to be implicit in Cantor's later terminology of 'inconsistent multiplicities'.

Again, the hierarchy of super- and hyper-classes that was threatening von Neumann's project is not looming here. Whether super-plurals occur in ordinary English is controversial. Uzquiano has argued that higher-order pluralities do not make real sense [Uzquiano 2003, p. 74]:

> Is there, for one, a distinction to be drawn between a plurality of pluralities of sets and a plurality of sets? The answer to this question would certainly seem to be negative; a plurality of pluralities of sets is nothing over and above a plurality of sets, some sets, that is.

Against this, [Linnebo & Nicolas 2008] argue that super-plurals at least do occur in ordinary non-mathematical English. But the outcome of this debate is not really important for our purposes, and this is so for two reasons. First of all, in the coded second-order formalisation of GRP (where

[28] Or so the argument goes. For arguments to the effect that appeal to plural reference to avoid commitment to proper classes ultimately is a cheat: see [Resnik 1988], [Hazen 1993], [Linnebo 2003]. It falls outside the scope of this article to adjudicate in this matter.

j is coded as a proper class) we have no need to appeal to super-plurals. On this coded second-order rendering of *GRP*, there is no need to collect the pluralities into one whole ($\mathcal{C}$), and there is no need to quantify over pluralities of pluralities in order to express the existence of j. Second, the meaningfulness of super-plural quantification in English is still a long way removed from an analogue of the transfinite hierarchy of super- and hyper-classes that was threatening von Neumann's project.

However, it is difficult to see how the intrinsic philosophical motivation of *GRP* in terms of the notion of *resemblance* can be upheld if the second-order quantifiers are interpreted in a plural way. Put bluntly, how can there be a question of resemblance of V_κ and its subsets if there is no entity for it to resemble to? So even though on a plural reading of the second-order quantifiers *GRP* may have a fairly clear meaning, it is difficult to see why we should accept it. For this reason, we opt for the mereological interpretation of the second-order quantifiers instead of the plural interpretation.

6.4 Mathematical and mereological reflection

In the Appendix, we show that even weak versions of *GRP* yield strong large cardinal consequences. They guarantee the existence of an unbounded class of Woodin cardinals, and thus in some sense 'complete' the theory of countable sets. In fact, the strength of strong versions of *GRP* lie between the statement that postulates a 1-extendible cardinal and the statement that postulates a subcompact cardinal

Large cardinal axioms can be formulated as postulating elementary embeddings from a model M of set theory into another model N of set theory ("$\exists j : M \longrightarrow N$"). The principle that postulates the existence of 1-extendible cardinals marks a watershed in the theory of large cardinals. For all weaker large cardinal axioms (with critical point κ), the embeddings that they postulate are *continuous* at κ^+, in the sense that

$$\sup\{j(\alpha) \mid \alpha \in On \wedge \alpha < \kappa^+\} = j(\kappa^+).$$

But from the axiom of 1-extendible cardinals onward, the '=' in this equality must be replaced by '$<$'. This discontinuity property is exploited time and time again in the theory of large large cardinals.

As noted above,[29] variants of *GRP* can be ordered according to the level of elementarity that they require. It is shown in the Appendix to this article that changing the level of elementarity required changes the strength of the resulting global reflection principle:

[29] See section **??**.

1. $NBG + GRP_{\Sigma^0_\infty} \nvdash$ "there is a 1-extendible cardinal."

2. $NBG + GRP_{\Sigma^1_1} \vdash$ "there is a 1-extendible cardinal."

So increasing the elementarity-requirement from Σ^0_∞ to Σ^1_1 carries us over an important dividing line in the theory of large cardinals.

We have classified Σ^0_∞ statements as *mathematical* statements because they only quantify over sets. We have classified Σ^1_∞ statements as *mereological* (or in Cantorian vein one might say *theological*) statements because they quantify over proper classes, which we do not regard as extra- or supra-mathematical objects. In other words, we might call $GRP_{\Sigma^0_\infty}$ mathematical global reflection, whereas $GRP_{\Sigma^1_1}$ must already be regarded as mereological global reflection. The divide between mathematical and mereological reflection then coincides with the discontinuity threshold in the theory of large cardinals discussed above.

We have mentioned before that already $GRP_{\Sigma^0_1}$ gives us the large cardinal consequences that are required for a proof of Projective Determinacy.[30] In other words, a limited version of mathematical global reflection is sufficient for these purposes. On the other hand, one can insist on mereological global reflection ($GRP_{\Sigma^1_1}$, for instance). In fact, even $GRP_{\Sigma^1_\infty}$ does not express the reflection idea in its strongest form. Recall that the guiding idea was that the set theoretic universe is *absolutely indistinguishable* from some set-like initial segment of V. GRP requires that the embedding j is elementary with respect to the second-order language of set theory *without the satisfaction predicate*. If we have a satisfaction predicate in our arsenal, we might require even stronger elementarity, viz. with respect to the second-order language *including the primitive satisfaction predicate*. Since this same satisfaction predicate is used to express the elementarity of j, it will have to be a non-Tarskian, type-free satisfaction predicate. But it is known from the theory of the semantic paradoxes that type-free truth and satisfaction quickly leads to contradictions. So we do not want to travel down this road in this article.

In its most extreme form, negative theology states that no property can be truly predicated of God. In a more positive vein, one might say that everything that we can truly say about God, also holds for some being that is less exalted than God. (Note that this is not equivalent to the first sentence of this paragraph.) Both these theories raise a difficult question, that was perhaps first articulated forcefully by Dionysius the Areopagite

[30] For a detailed analysis of this, see [Welch 201?].

[Dionysius 1920]. If everything we truly say about God is also true about some angel(s), or if nothing we say about God is true at all, then how can we name God in the first place? What is it that makes our uses of the word 'God' refer to God rather than to angels in the first place? The mirror image of this challenge for *GRP* is just as troublesome. If *everything* we truly say about V and $\mathcal{C}$ is also true about some set κ and its subsets, then what makes it the case that when we are using 'V' and '$\mathcal{C}$' in this article, these terms refer to the set-theoretic universe and its classes, respectively, rather than to some set and its subsets? If we insist on articulating *GRP* as requiring full Σ^1_∞ elementarity, then we only have the primitive notion of Satisfaction to single out V and $\mathcal{C}$. However, if we articulate *GRP* as insisting only on a form of mathematical elementarity (Σ^0_1 elementarity or Σ^0_∞ elementarity), then this worry is not pressing. Then we can say that mereological (or "theological") statements allow us to distinguish V and $\mathcal{C}$ from every set together with its subsets.

Moreover, there is a less philosophical reason for being hesitant to endorse mereological reflection. $GRP_{\Sigma^1_\infty}$ entails the axioms *MK* of Morse-Kelley class theory: *MK* holds at $(V_\kappa, V_{\kappa+1}, \in)$, and is then sent up by virtue of the Σ^1_∞ elementarity of j. So one might as well have started with the very "class-impredicative" theory *MK* as one's background theory. But this argument does not go through if instead j is only Σ^0_1 elementary: there is then not enough elementarity to preserve the impredicative second order comprehension scheme upwards.[31]

We see these as reasons for being cautious. Thus we do not endorse mereological global reflection here.

7 Conclusion

According to many, Cantor's early view of the mathematical universe as a whole is hopelessly entangled with his theological views [Tapp 200?]. In contrast, his later view of the set theoretic universe and proper classes more generally as 'inconsistent multiplicities' is less so, and can be seen as a first step in the direction of a modern view of the set theoretic reality. It can then either be seen as a prefiguration of a plural interpretation of proper classes (as in [Boolos 1985]) or as a potentialist conception of the mathematical universe *à* la Zermelo [Jané 1995].

[31]On the other hand, since *MK* holds at $(V_\kappa, V_{\kappa+1}, \in)$, accepting $GRP_{\Sigma^0_1}$ still commits one to believing that impredicative second-order logic is coherent. So $GRP_{\Sigma^0_1}$ is certainly not free of second-order involvement.

In this article we have argued that good secular (non-theological) sense can be made of Cantor's earlier view of the set theoretic universe. It is an ontological view on which both proper classes (Absolute Infinities) and pluralities (multiplicities) of classes are recognised. Sets are all the mathematical objects there are. All the sets together form, as Cantor The Younger said, a completed whole: the mathematical universe V. However V itself is not a *mathematical* object. Proper classes are parts of the universe. Every part of V is a completed whole. Every set is an element of V. The parthood relation corresponds to the subclass relation, which is a transitive relation. So parthood is not the same as membership, even for sets: not all sets are transitive. The language of sets and parts of V is the language of second-order set theory $\mathcal{L}^2_\in$. The first-order quantifiers range over all sets. The second-order quantifiers range over all the parts of V. So we are ontologically committed to the existence of sets, the universe of all sets (V), and all of its parts: we make no further ontological commitments. The sets certainly satisfy ZFC. The parts of V satisfy at least predicative second-order comprehension. And the class replacement axiom also holds. So we are licensed to postulate NBG class theory in the language $\mathcal{L}^2_\in$. If one takes a Gödelian stance towards impredicative definitions, then even impredicative second-order comprehension is acceptable. If that is so, then the axioms of Morse-Kelly class theory are motivated. However such a strong class theory is not needed for the results in this paper. Therefore we do not here take a stance on the matter.

Not only is this interpretation of Cantor's earlier view perfectly coherent. It is also mathematically fruitful. It allows us to indirectly motivate strong principles of infinity (large cardinal axioms). Large cardinal principles play an important role in contemporary set theory. However whereas the axioms of ZFC seem to be fairly generally accepted to hold of the set theoretic universe, there is no general agreement that most of the large cardinal principles hold.

Gödel argued that mathematical axioms can be motivated in two ways: intrinsically, and extrinsically [Gödel 1964]. Extrinsic support for an axiom derives from its consequences. Thus extrinsic motivations are success arguments; they are instances of *Inference to the Best Explanation*. Many believe that intrinsic justification for mathematical principles is more reliable than extrinsic justification. Indeed, many do not think that external motivation for a mathematical axiom can provide strong confirmation of its truth [Tait 2001, p. 96]. So it is an important question to what extent large cardinal principles can be motivated intrinsically.

Mathematical reflection principles are intrinsically motivated. These arguments follow a pattern of reasoning that has its roots in the Judeo-Christian theological tradition. This argument starts from the negative premise of the transcendence of God: there is no defining condition in any human language that is satisfied by Him and by Him alone. From this it follows that if we can truly ascribe a property to God, this property must hold of some entity that is different from God as well. This conditional positive statement can justly be called a *first theological reflection principle*. This argument can be strengthened if we assume the stronger negative premise that not even an infinite body of humanly describable conditions characterise God uniquely. This means that there must be an entity that is different from God and that satisfies all properties that can be truly ascribed to God. This then is a *second theological reflection principle*. We have seen how it was clearly articulated by Philo of Alexandria.

The first theological reflection principle is the exact analogue of Bernays' second-order reflection principle. The second reflection principle is the analogue of the Global Reflection Principle. The respective motivations for these set theoretic reflection principles are also analogous. In the foundational literature they are called *richness arguments*. In this article, we have focused on GRP. The thought is that the mathematical universe with its parts is so rich that there is a rank κ such that $\langle V_\kappa, V_{\kappa+1}, \in \rangle$ is elementarily equivalent with V with all its classes. Then given a strong global choice principle, the class function $j : \langle V_\kappa, V_{\kappa+1}, \in \rangle \longrightarrow \langle V, \mathcal{C}, \in \rangle$ that is postulated by GRP, can be shown to exist. Of course, for someone who is sceptical about the richness of the set theoretic universe, the whole idea of reflection principles will not be compelling.

Cantor did not see this far. On the theological side, there is no evidence that he was aware of statements of the second theological reflection principle. On the class theoretic side, he did not have the resources to even articulate the Global Reflection Principle: the cumulative rank structure of the set theoretic universe had yet to be discovered. We have seen that Cantor himself did (somewhat implicitly) appeal to a reflection principle on one occasion. But what he appealed to was a first-order reflection principle (Montague-Levy), and it is known that first-order reflection principles are provable in ZF. In general, Cantor mostly referred to the epistemic transcendence of the set theoretic universe as a whole instead of focussing on its positive consequences (reflection principles).

The global reflection principle in its stronger forms is essentially a second-order reflection postulate. So to interpret it, we have to assign a clear meaning to the second-order quantifiers. On Zermelo's potentialist picture, this seems a tall order. Perhaps what Zermelo calls 'meta-set theory' allows quantification over absolute infinities, but this seems counter to the potentialist spirit of Zermelo's view, and 'meta-set theory' has never been articulated in any detail anyway. The pluralist interpretation of second-order quantification fares better. It may well give us a fairly clear interpretation of the second-order quantifiers. But on this interpretation, and therefore also on the interpretation of the second order quantifiers as ranging not only over sets but also over 'inconsistent multiplicities', the motivation for *GRP* becomes opaque. It is on this interpretation hard to make sense of the motivation for *GRP* in terms of a notion of resemblance. As far as we can see, it is only in terms of the interpretation of the second-order quantifiers as ranging over parts of the universe that the intrinsic motivation of *GRP* can be articulated. For this reason we conclude that the early Cantorian view of the set theoretical universe is mathematically the most fruitful one. Theology is not conservative over mathematics.

7.1 Appendix: the strength of global reflection

We now expand on the mathematical properties of *GRP*.[32] As mentioned before, we assume *NBG* as our background class theory.

To start with, already weak forms of *GRP* have important large cardinal consequences.

Theorem 1 *If $GRP_{\Sigma^0_1}$ holds, then there exist unboundedly many measurable cardinals.*

The existence of measurable cardinals entails the existence of the ω-Erdős cardinal $\kappa(\omega)$. Therefore, if $GRP_{\Sigma^0_1}$ is a (sound) reflection principle, then Koellner's challenge can be met.

In the above the critical point κ is itself easily shown by standard arguments to be a measurable cardinal and hence strongly inaccessible; by easy elementarity arguments one sees that there are unboundedly many such. These leads also to the following proposition. If we assume the Axiom of Choice (AC) for sets, then the latter entails that there is a subset of $V_\kappa \times \kappa$ that enumerates V_κ in order type κ. By applying j we obtain that there is a subclass of $V \times On$ that enumerates V in order type that of the ordinals On. This yields a global choice principle:

Proposition 1 *If the Axiom of Choice is assumed for sets in $\langle V, \in \rangle$ and the $GRP_{\Sigma^0_1}$ holds, then Global Choice holds in $\langle V, \in, \mathcal{C} \rangle$.*

[32] For a more elaborate discussion, including proofs of theorems that are merely stated here, see [Welch 201?].

The existence of an enumeration of all the sets of V in order type that of the ordinals On allows the global form of choice to be obtained on all of V.

Theorem 1 can in fact be strengthened:

Theorem 2 *If $GRP_{\Sigma^0_1}$ holds, then κ is additionally a Woodin cardinal; hence there exist unboundedly many such measurable Woodin cardinals.*

It therefore follows from work of Martin, Steel [Martin & Steel 1989] and Woodin [Woodin 1999], that:

Corollary 1 *$GRP_{\Sigma^0_1}$ implies the axiom of Projective Determinacy (PD). $GRP_{\Sigma^0_1}$ implies in addition that the full Axiom of Determinacy (AD) holds in $L(\mathbb{R})$, the minimal inner model containing all of the real continuum. Moreover these facts, indeed the theory of $L(\mathbb{R})$, are absolute with regard to any Cohen-style set forcing.*

This is significant in many ways, because PD is something of a 'complete theory' of countable sets, much as Peano Arithmetic is something of a 'complete theory' of the finite natural numbers, in the sense that we have no examples of sentences σ about the Hereditarily Countable sets (HC)[33] that are not decided by $ZFC^- + PD +$"$V = HC$" other than Gödel-style diagonal sentences.[34] (ZFC^- is the theory of ZFC with the power set axiom removed.) The absoluteness results of Woodin that show, *inter alia* the fixity of the theory of $L(\mathbb{R})$ under set forcing, require the assumption of a proper class of Woodin cardinals in V in order to work. This is also the hypothesis needed to establish many of his results on Ω-logic, leading to his formulation of the Ω-Conjecture.

Obviously, then, versions of GRP are situated somewhere in the hierarchy of strong axioms of infinity. Mathematical versions of GRP are situated just below the axiom that states the existence of *1-extendible cardinals*; mereological versions of GRP are situated just below the axiom that states the existence of *subcompact cardinals*.

The notion of an α-extendible cardinal was introduced in [Reinhardt 1974]:[35]

Definition 1 *(i) A cardinal κ is α-extendible if there is $\lambda > \kappa$ and an elementary embedding j with $j_\alpha : \langle V_{\kappa+\alpha}, \in \rangle \longrightarrow \langle V_{\lambda+\alpha}, \in \rangle$.*
(ii) A cardinal κ is extendible *if it is α-extendible for all α.*

[33] More generally, $H(\kappa)$ denotes the set of sets that are hereditarily of cardinality $< \kappa$. See [Kunen 1980, chapter 4, section 6].

[34] See [Woodin 2001].

[35] More precisely Silver's reformulation of Reinhardt's earlier definition here (which Reinhardt also used in [Reinhardt 1974]). For a fuller discussion of extendible cardinals, see [Kanamori 1994, chapter 5, section 23].

The elementarity here is with respect to the full first-order language of set theory (so Σ^0_ω-preserving in our earlier notation). Notice in the above that the 'large cardinal' is κ on the domain side of the embedding; and this is kept fixed in the definition of extendibility.

In the light of this definition, we can set $\alpha = 1$ and state the following axiom:

Axiom 4 (The existence of 1-extendible cardinals.) There are ordinals κ, λ (with $\lambda > \kappa$) and a nontrivial elementary embedding

$$j : \langle V_\kappa, \in, V_{\kappa+1} \rangle \longrightarrow_{\Sigma^0_\omega} \langle V_\lambda, \in, V_{\lambda+1} \rangle$$

with critical point κ.

Theorem 3 *The existence of a 1-extendible cardinal entails the consistency of $GRP_{\Sigma^0_\infty}$.*

However we shall see below (Theorem 5) that $GRP_{\Sigma^1_\infty}$ entails the existence of many 1-extendible cardinals. So, to prove the consistency of $GRP_{\Sigma^1_\infty}$, we need stronger large cardinal principles.

Mereological global reflection can be conceived of as a weakening of a property derived from that of a subcompact cardinal:

Definition 2 *A cardinal κ is said to be* subcompact *if, for any $A \subseteq H(\kappa^+)$ there is a $\mu < \kappa$ and $\overline{A} \subseteq H(\mu^+)$ and an elementary embedding j with*

$$j : \langle H(\mu^+), \in, \overline{A} \rangle \longrightarrow_{\Sigma^0_\omega} \langle H(\kappa^+), \in, A \rangle.$$

We want to connect principles concerning subcompact cardinals (formulated in terms of principles concerning $H(\kappa^+)$) with principles concerning extendible cardinals or with GRP (formulated in terms of $V_{\kappa+1}$). Fortunately, in the situation under consideration, we may identify $H(\kappa^+)$ with $V_{\kappa+1}$: for any strong limit cardinal α (as μ and κ are in the above), $|V_\alpha| = \alpha$ and thus $V_\alpha \in H(\alpha^+)$. Thus $V_{\alpha+1} \subseteq H(\alpha^+)$. Moreover any $\langle x, \in \rangle \in H(\alpha^+)$ (where, without much loss of generality, we take x to be a transitive set) is isomorphic to $\langle \alpha, E \rangle$ for some $E \in \alpha \times \alpha$ (just define E to be the set of $\langle \xi, \zeta \rangle$ with $g(\xi) \in (\zeta)$ where $g : \alpha \twoheadrightarrow x$ is any onto function). Using a pairing function, we have that E, and thus ultimately $\langle x, \in \rangle$, is coded by some $E_0 \subseteq \alpha$, which is in $V_{\alpha+1}$.

It is easy to see then (by varying A over singletons of ordinals below κ), that in fact there are unboundedly many μ below κ with some j_μ : $\langle H(\mu^+), \in \rangle \longrightarrow_{\Sigma^0_\omega} \langle H(\kappa^+), \in \rangle$, or equivalently (using the reasoning above) with a $j_\mu : \langle V_{\mu+1}, \in \rangle \longrightarrow_{\Sigma^0_\omega} \langle V_{\kappa+1}, \in \rangle$.

Theorem 4 *The existence of a subcompact cardinal entails that $GRP_{\Sigma^1_\infty}$ is consistent.*

Proof. Let κ be a subcompact cardinal. Let $Sat_{\langle V_\kappa, V_{\kappa+1}, \in\rangle}$ be the satisfaction predicate for Σ^1_∞-formulae in the structure $\langle V_\kappa, V_{\kappa+1}, \in\rangle$, and let $Sat_{\langle V_\mu, V_{\mu+1}, \in\rangle}$ be the satisfaction predicate for Σ^1_∞-formulae in the structure $\langle V_\mu, V_{\mu+1}, \in\rangle$.

Then $\langle V_\kappa, V_{\kappa+1}, \in\rangle \models NBG$. Notice that in the definition of subcompactness the range of the embedding j is small, that is j itself is an element of $V_{\kappa+1}$. Using the properties of the satisfaction relation we thus have $\langle V_\kappa, V_{\kappa+1}, \in\rangle \models \exists J \exists \mu$:
1. J is (class-)functional and $J \neq id$ and μ is the critical point of J;
2. $Dom(J) = V_{\mu+1}$;
3. $\forall \overline{x} \in V_\mu \forall \overline{X} \in V_{\mu+1} \forall \varphi \in \Sigma^1_\infty$:

$$Sat_{\langle V_\mu, V_{\mu+1}, \in\rangle}(\overline{x}, \overline{X}, \varphi) \leftrightarrow Sat_{\langle V_\kappa, V_{\kappa+1}, \in\rangle}(J(\overline{x}), J(\overline{X}), \varphi),$$

i.e., $\langle V_\kappa, V_{\kappa+1}, \in\rangle \models GRP_{\Sigma^1_\infty}$.

So if there is a subcompact cardinal, then there is a set structure that makes $GRP_{\Sigma^1_\infty}$ true, whereby $GRP_{\Sigma^1_\infty}$ is consistent. ■

This means that the consistency strength of strong versions of GRP does not extend far above that of the axiom of 1-extendible cardinals. Indeed, the consistency strength of $GRP_{\Sigma^1_\infty}$ lies "just above" that of the existence of a 1-extendible cardinal and well below that of the existence of a 2-extendible cardinal, since below the first 2-extendible cardinal there are many subcompact cardinals.

As stated the notion of subcompact is a strongly third order one over $H(\kappa)$, but we may take a 'parameter-free' version by dropping the reference to $A, \overline{A}$'s. Thus we essentially obtain $GRP_{\Sigma^1_\infty}$.

Of course we do not intend the existence of subcompact cardinals to support versions of GRP. Epistemologically, it is rather the other way round: GRP is intended to lend intrinsic support to certain large cardinal axioms. The foregoing theorem then shows that GRP is not strong enough to motivate the existence of subcompact and a fortiori the existence of supercompact cardinals.

The motivation for the notion of subcompact is thus entirely different from those in Reinhardt's discussions in [Reinhardt 1974] (and elsewhere) which emphasised rather the possibility of projecting $(V, \in)$ into some imaginary realm. Indeed, the motivation of the axiom of subcompact cardinals is closer to the motivation of versions of GRP than Reinhardt's motivation of 1-extendible cardinals is. We can view the motivation of GRP as 'just turning around' the motivation for Reinhardt's embedding.

If a strong version of GRP holds in the universe, then there are many levels of the V-hierarchy that witness weaker versions of GRP. Such arguments can typically take the following form:

Theorem 5 *Assume $GRP_{\Sigma^1_\infty}$. Then there is a proper class of commuting 1-extendibles. Namely there is a system of commuting maps $\langle j_{\alpha\beta}\rangle_{\alpha\leq\beta<On}$, with ordinals $\mu_\alpha < On$, with*

$$j_{\beta,\gamma} : \langle V_{\mu_\beta}, V_{\mu_{\beta+1}}, \in\rangle \longrightarrow_{\Sigma^0_\infty} \langle V_{\mu_\gamma}, V_{\mu_{\gamma+1}}, \in\rangle$$

where the V_{μ_β}'s witness the truth of $GRP_{\Sigma^0_\infty}$, and where also for $\beta < On$ there are maps j_β also commuting with the $j_{\alpha,\beta}$ so that:

$$j_\beta : \langle V_{\mu_\beta}, V_{\mu_{\beta+1}}, \in\rangle \longrightarrow_{\Sigma^0_\infty} \langle V, \mathcal{C}, \in\rangle.$$

Proof. We are assuming that $GRP_{\Sigma^1_\infty}$ holds in the universe. Σ^0_∞-satisfaction can be expressed in the second-order language and thus so can the notion of a Σ^0_∞-elementary map. Thus there is a second-order statement expressing $GRP_{\Sigma^0_\infty}$ that holds in $\langle V, \mathcal{C}, \in\rangle$. So this statement must also hold in some $\langle V_\delta, V_{\delta+1}, \in\rangle$ connected to the universe *via* a Σ^1_∞-elementary embedding k:

$$k : \langle V_\delta, V_{\delta+1}, \in\rangle \longrightarrow_{\Sigma^1_\infty} \langle V, \mathcal{C}, \in\rangle$$

Now suppose for a contradiction that there is a maximal system $\mathcal{S} = \langle\langle V_{\mu_\alpha}\rangle, \langle j_{\alpha,\beta}\rangle\rangle_{\alpha\leq\beta<\lambda}$, with a sequence of maps $\langle j_\beta\rangle_{\beta<\lambda}$ into $\langle V, \in \mathcal{C}\rangle$ as posited in the conclusion of the theorem. By the reasoning just given $\lambda > 0$. Then the second-order statement expressing this fact about $\mathcal{S}$ being true in $\langle V, \mathcal{C}, \in\rangle$, is reflected down to $\langle V_\delta, V_{\delta+1}, \in\rangle$ and we thus have $\mathcal{S}$ in V_δ with now maps $\bar{j}_\beta$:

$$\bar{j}_\beta : \langle V_{\mu_\beta}, V_{\mu_{\beta+1}}, \in\rangle \longrightarrow_{\Sigma^0_\infty} \langle V_\delta, V_{\delta+1}, \in\rangle.$$

But this means that there is in V an extension of the system $\mathcal{S}$ to a longer chain of maps, of the relevant kind (by adding as λ'th model $V_{\mu_\lambda} = V_\delta$ and defining $j_\lambda = k$, $j_{\alpha,\lambda} = \bar{j}_\alpha$ for $\alpha < \lambda$). Contradiction. ■

In the above $GRP_{\Sigma^1_1}$ would suffice.

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