Large Cardinals, Inner Models and Determinacy: an introductory overview

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Abstract. We sketch a brief development of large cardinals as they apply to determinacy results and to inner model theory.

1 Introduction

The following article grew out of a series of three tutorials at the conference on the development of large cardinal hypothesis and connections to proofs of the determinacy of two person perfect information Gale-Stewart games. The intention was to give a perspective, not to specialists in set theory, who in any case either would have no need for such, or would know where to go find it, but to those interested in the foundations of set theory already with some technical knowledge of Zermelo-Fraenkel axiomatics, and the construction of Gödel's universe L, but who may not have seen a systematic production of the hierarchy of increasing strong axioms of infinity arising from embeddings of the universe V and in particular to the connections of these axioms with the descriptive set theory of the real continuum. It was hoped that the lectures would provide a fast ascent without oxygen to some of the peaks of the last twenty years. In this we believed that a lot could be learnt, not in detail, nor even attempting to provide a working knowledge (for which a longer term devotion would be required, and again for which there are thorough texts) but that would bring about at least some familiarity with the technical tools, and some insight as to how they can be used. We should also like to urge the reader to study Jensen's masterly, and non-technical, overview [**9**] if they have not already done so.

1.1 *Preliminaries:*

Our notation and formalisms are quite standard and can be found in many text books, but in particular we mention [**8**] and [**10**]. We let $\mathcal{L}_{\dot\in}$ denote the first order language of set theory; by ZF we mean a formulation of first order Zermelo-Fraenkel.

By $\mathcal{L}_{\dot\in,\dot{A}}$ we mean the standard language of set theory with an (optional) predicate $\dot{A}$. In both these languages we freely make use of terms abstracts $t = \{z \mid \varphi(\vec{y}, z)\}$ as if they were part of the languages. ZF_A is then a formulation of ZF with instances of the predicate $\dot{A}$ allowed in the axioms. A set is *transitive*, $Trans(x)$, if every element of x is at the same time a subset of x.

The class of all ordinals is denoted On; Sing is the class of singular ordinals; Card is the class of all cardinals (we assume AC throughout and that cardinals are initial ordinals). $\mathrm{SingCard}$, Reg are the classes of singular cardinals and regular cardinals respectively. Inacc is the class of (strongly) inaccessible cardinals. For a limit ordinal τ, by the "*cofinality of* τ",

[1]The author would like to warmly thank the organisers of the meeting for the opportunity to give these lectures. He would also like to thank the Isaac Newton Institute where these notes were completed whilst the author was a long term Fellow on the program 'Syntax and Semantics.'

$\mathrm{cf}(\tau)$, is meant the least δ so that there is a function $f : \delta \longrightarrow \tau$ with $\mathrm{ran}(f)$ unbounded in τ ($\tau \in Reg$ then, if $cf(\tau) = \tau$). The rank function,"$\rho(x) = \alpha$" is Δ_1^{ZF} and the relation of y and α: "$y = V_\alpha$" is Π_1^{ZF}. We let ${}^x y$ denote the set of all functions from the set x to y. It is often customary to deviate from the practice when $x = \omega$ and we are considering, *e.g.*, ${}^\omega\omega$ or ${}^\omega X$ as spaces and then we write ω^ω or X^ω.

It is useful to fix a recursive enumeration $\langle \tau_i \mid i < \omega \rangle$ of $\mathrm{Seq} =_{\mathrm{df}} {}^{<\omega}\omega$ so that:
(i) $\tau_0 = (), \tau_1 = (0)$; $(ii)\ i \geq |\tau_i| \, (=_{\mathrm{df}} \mathrm{lh}(\tau_i))$; (iii) $\tau_i \subset \tau_j \longrightarrow i < j$. (We may similarly define recursive enumerating sequences $\langle \tau_i^k \mid i < \omega \rangle$ for ${}^k({}^{<\omega}\omega)$.)
The *Kleene-Brouwer ordering* is defined as follows: for $s, t \in {}^{<\omega}\mathrm{On}$ put

$$s <_{\mathrm{KB}} t \leftrightarrow s \supset t \vee (s(i) < t(i) \text{ where } i \in \omega \text{ is least with } s(i) \neq t(i).$$

Finally recall that a set $B \subseteq \mathbb{R}$ is Π_n^1 if it can be defined in Z_2 with

$$x \in B \leftrightarrow \exists f_1 \forall f_2 \cdots Q f_n \forall k R(k, \overrightarrow{f_i} \restriction k, \overline{x} \restriction k) \quad \text{with } R \in \Sigma_1^0$$

with the $f_i \in \mathbb{N}^{\mathbb{N}}$ and $k \in \mathbb{N}$. (Z_2 is a formalisation of second order number theory, or analysis, see [**26**].) As is usual in set theory, we make little distinction between $\mathbb{R}$ and ω^ω, or even $\mathcal{P}(\omega)$ occasion.

2 Inner Models, Elementary Embeddings, and Covering Lemmas

Recall that Gödel constructed an *inner model* of V (here identified with the hierarchy of *well-founded* sets $\mathrm{WF} = \bigcup_{\alpha \in \mathrm{On}} V_\alpha$) by taking a "definable power set" operation $x \longrightarrow \mathrm{Def}(x)$ where x is shorthand for the first order structure $\langle x, \in \rangle = \langle x, \in \restriction x \times x \rangle$ (on occasion this is augmented as relativised operation $x \longrightarrow \mathrm{Def}_A(x)$ using definability over $\langle x, A \restriction x, \in \restriction x \times x \rangle$ in the $L_{\dot\in, \dot A}$ language. These two Def functions are given by Δ_1^{ZF} or $\Delta_1^{\mathrm{ZF}_A}$ terms and hence are absolute.

Then ("$V = L$")^L was shown by Gödel ([**6**]), as well as ("AC")^L, ("GCH")^L thereby establishing: $\mathrm{Con(ZF)} \longrightarrow \mathrm{Con(ZFC + GCH)}$. Why should $V = L$? As this is a hierarchy of sets constructed by syntactico-semantical means, and not responding to any particular intuitions about set existence, most set-theorists do not believe $V = L$ (but see the discussion in [**9**]).

We shall see good reasons for asking whether there are other inner models, some associated with the notions of elementary embeddings. We first define our terms.

Definition 2.1 (Inner Model of ZF) $IM(M) \leftrightarrow \mathrm{Trans}(M) \wedge \mathrm{On} \subseteq M \wedge (\mathrm{ZF})^M$.

In the above we are assuming that M is given by some term of the language. The notion of being an inner model of ZF actually has a first order formalisation: it is well known (see Jech [**8**]) that

$$\mathrm{ZF} \vdash IM(M) \leftrightarrow \forall u \subseteq M \exists v \supseteq u (\mathrm{Trans}(v) \wedge \mathrm{Def}(\langle v, \in \rangle) \subseteq M).$$

We single out the following beautiful theorem as the motivation for this account:

Theorem 2.2 *(ZF) The following are equivalent:*
a) $\exists j : L \longrightarrow_e L$.
b) $\exists \gamma (\omega_2 \leq \gamma \in \mathrm{SingCard} \wedge (\gamma \notin \mathrm{SingCard})^L)$*;*
c) $\mathrm{Determinacy}(\Pi_1^1)$

This is actually a culminative theorem established over several years: a)$\leftrightarrow$b) Jensen [**3**]; a) $\rightarrow$ c) Martin [**15**]; c) $\rightarrow$ a) Martin-Harrington [**7**].

Remarks 1) a) is sometimes stated as: "$0^\#$ *exists*". Martin actually showed that the determinacy of games at the third level of the difference hierarchy of analytic sets, proved the existence of $0^{\#}$. This was then reduced by Harrington in the work cited. For an elegant proof of the Harrington result see [**22**].

2) Much large cardinal theory is about which ultrafilters can or do exist on (large) sets; in particular when those large sets are the power set of some cardinal of an inner model, then there is usually an equivalent formulation in terms of elementary embeddings, as defined below, of that inner model such as stated at a).

We explore the background to Jensen's a)$\leftrightarrow$b). Again we have to define some terms:

Definition 2.3 *Let M, N be inner models of* ZF*, $j : M \longrightarrow N$ is an* elementary embedding *if the function j takes elements $x \in M$ to elements $j(x) \in N$ in a 'truth preserving way': for any formula $\varphi(v_0, \ldots, v_{n-1})$ and any $\vec{x} = x_0, \ldots, x_{n-1} \in M$, then*

$$\varphi(\vec{x})^M \leftrightarrow \varphi(\overrightarrow{j(x)})^N.$$

In this case we write: $j : M \longrightarrow_e N$. (We shall always assume that $j \neq \mathrm{id}$, and shall write cp(j) for the critical point: the least ordinal α so that $j(\alpha) > \alpha$, if it exists.)

(ii) If the above holds, but with the formulae restricted to a certain class, eg. the Σ_k formulae, then we write $j : M \longrightarrow_{\Sigma_k} N$.

In the above scheme, we have assumed that the models M, N satisfy $IM(M), IM(N)$ above and are given by terms of our basic set theoretical language, and the same holds true for j. Our embeddings in this paper will all have critical points in the ordinals. It is an easy consequence of the ZF axioms (using the definition of the rank function, the V_α hierarchy, and Replacement) that if $j : M \longrightarrow_{\Sigma_1} N$, then by a (meta-theoretic) induction on k we may prove $j : M \longrightarrow_{\Sigma_k} N$ for any $k \in \omega$.

If $\exists j : L \longrightarrow_e L$ then we may define a *derived measure* $U = U_j$ on $\kappa = \mathrm{cp}(j)$ as follows: we set

$$X \in U_j \leftrightarrow X \in \mathcal{P}(\kappa)^L \wedge \kappa \in j(X).$$

Then U_j is a *normal measure* ([**10**],p 52) on $\mathcal{P}(\kappa)^L$. Suppose we have $j : M \longrightarrow_e N$; $\mathrm{cp}(j) = \kappa$, define $U = U_j$. Then we construct an ultrapower defining:

$$|\,\mathrm{Ult}(M, U)| = \{[f]_\sim : f \in {}^\kappa M \cap M\}$$

where

$$f \sim g \leftrightarrow \{\alpha \mid f(\alpha) = g(\alpha)\} \in U$$

and on which we can define a pseudo-$\in$ relation E:

$$fEg \leftrightarrow \{\alpha \mid f(\alpha) \in g(\alpha)\} \in U.$$

Because of where U_j comes from, we are guaranteed E is wellfounded on $\mathrm{Ult}(M, U)$ and in fact we have:

$$\begin{tikzcd} j : (M, \in) \arrow[r] \arrow[d, dashed] & (N, \in) \\ \pi : ((Ult(M, U_j), E)) \cong (\overline{N}, \in) \arrow[ur, "k"'] & \end{tikzcd}$$

We can, and often do, have $(M, \in) = (V, \in)$. Also *starting* from *any* κ-complete U on $\mathcal{P}(\kappa)^V$, (for an uncountable regular κ, then we can define $\mathrm{Ult}(V, U)$ and we may prove outright that the E relation on $\mathrm{Ult}(V, U)$ will be wellfounded. In general for wellfounded cases we may define by recursion along E the Mostowski-Shepherdson transitivising collapse isomorphism $\pi : (\mathrm{Ult}(M, U_j), E) \longrightarrow (\overline{N}, \in)$ with $\overline{N}$ transitive. Taking these facts together then:

Theorem 2.4 *(ZFC) Let $\kappa > \omega$. The following are equivalent:*
(a)*There is a κ-complete non-principle ultrafilter on $\mathcal{P}(\kappa)$.*
(b) $\exists j : V \longrightarrow_e M$ with $\mathrm{cp}(j) = \kappa$

Theorem 2.5 (Scott) [**25**] *(ZF) $\exists \kappa$(κ a measurable cardinal)$\longrightarrow V \neq L$.*

Proof: If $V = L$, let κ be the least such measurable cardinal (MC), form the ultrapower and so the embedding above. Then from $j : V \longrightarrow_e N$, and elementarity we have: $(V = L)^V \longrightarrow (V = L)^N$; so as $\mathrm{Trans}(N)$, $N = L$. But "κ is the least MC "$\longrightarrow$ ("$j(\kappa)$ is the least MC")^N. But $N = V \wedge j(\kappa) > \kappa$! $\square$

The assumption of this theorem implies $\exists j : V \longrightarrow M$, but again by Gödel's results on the absoluteness of the L-construction, $L^M = L$, so $j \upharpoonright L : L \longrightarrow_e L$. Note that no first order formula $\varphi(v_0)$ can differentiate between κ and $j(\kappa)$: $\varphi(\kappa)^L \leftrightarrow \varphi(j(\kappa))^L$. Moreover both ordinals are inaccessible cardinals in the sense of L.

So we investigate the consequences of this embedding from L to L and shall discover that such *indiscernibility* of these critical points in fact characterises such embeddings. Kunen showed ([**12**]) that if $\exists \overline{j} : L \longrightarrow_\varepsilon L$, then a number of consequences follow:

(i) Then there is such a j: $L \longrightarrow L$ with $\mathrm{cp}(j) < \omega_1$. Moreover defining U_0 from such a j with critical point κ_0 least, we are guaranteed wellfoundedness of *iterated ultrapowers*: that is we may define $j_{01} : L \longrightarrow_e L$ by taking the ultrapower of L by U_0; define U_1 on $\mathcal{P}(\kappa_1)^L$ where $\kappa_1 =_{\mathrm{df}} j_{01}(\kappa_0)$, and then $\mathrm{Ult}(L, U_1)$ will also be wellfounded. We thus may take its transitive isomorph and then have the ultrapower map $j_{12} : L \longrightarrow L$ with critical point κ_1; we then define $\kappa_2 = j_{12}(\kappa_1)$ and U_1 on $\mathcal{P}(\kappa_2)^L$. The process may be iterated without breaking down, forming a directed system $\langle\langle N_\alpha\rangle, j_{\alpha\beta}, \kappa_a, U_\alpha\rangle_{\alpha\leq\beta\in\mathrm{On}}$ with (in this case) all $N_\alpha = L$ and elementary maps into direct limits at limit stages λ, and the κ_α forming a class C of L-inaccessibles, which is closed and unbounded below any uncountable cardinal.

(ii) The iteration points of such ultrapowers enjoy full-blooded indiscernibility properties in L if $\varphi(v_0, \ldots, v_n)$ is any formula of $\mathcal{L}$ and $\vec{\gamma}$,$\vec{\delta}$ any two ascending sequences from $[C]^{n+1}$ then $(\varphi(\vec{\gamma}) \leftrightarrow \varphi(\vec{\delta}))^L$.

Definition 2.6 (The $0^{\#}$-mouse) *Let $j_{\alpha\beta}$ etc. be as above. Let $M_0 = \langle L_{\kappa_0^{+L}}, \in, U_0\rangle$. This structure is called the "$0^{\#}$-mouse" which itself has iterated ultrapowers using maps that are the restrictions of the*

$$j_{\alpha\beta} \upharpoonright M_\alpha : M_\alpha \longrightarrow M_\beta \quad \textit{where } M_\alpha = \langle L_{\kappa_\alpha^{+L}}, \in, U_\alpha\rangle \textit{ etc.}$$

Remark 2.7 *The viewpoint is shifted to that of the* mouse (M_0) generating the model *(in this case L). All of this is a paradigm for* generalised constructible *inner models K - the core models.*

By these means we argue for

Theorem 2.8 (ZF) *If* $\exists j : L \longrightarrow L$ *then* $\forall \gamma((\gamma \in \mathrm{SingCard}) \longrightarrow (\gamma\,\mathrm{Inacc})^L)$.

Proof: The above implies that $C \cap \gamma$ is unbounded below γ. But C is closed, so $\gamma \in C$. Each $\gamma \in C$ is inaccessible in L. $\square$

Theorem 2.9 (Jensen)(ZF) *Suppose* $\gamma \in \mathrm{SingCard}$, $\gamma \geq \omega_2$ *but* $(\gamma \in \mathrm{Reg})^L$. *Then* $\exists j : L \longrightarrow L$, *with* $j \neq \mathrm{id}$*; that is "*$0^{\#}$ *exists."*

Proof: Suppose $\neg\exists j : L \longrightarrow L$, but γ is chosen least with $\gamma \in \mathrm{SingCard}$ but $(\gamma \in \mathrm{Reg})^L$. Without loss of interest, we shall assume that (i) $\mathrm{cf}(\gamma) > \omega$ (ii) $\delta < \gamma \longrightarrow \delta^\omega < \gamma$. Let $\tau = \mathrm{cf}(\gamma)$. By assumption then $\tau < \gamma$ and so we may choose $X_0 \subseteq \gamma$ with $|X_0| = \tau$ but X_0 unbounded in γ. By (ii) we'll assume also that for some $X \supset X_0$ we have (a) $\gamma \in X \prec L_{\gamma^{+L}}$ (b) ${}^\omega X \subseteq X$ (c) $|X| = \tau^\omega < \gamma$.

Let $\pi : \langle X, \in\rangle \longrightarrow \langle M, \in\rangle = \langle L_{\overline{\delta}}, \in\rangle$ be the collapsing isomorphism with $\pi(\gamma) = \delta$ say. Then we have:

(1) $\mathrm{cf}(\delta) = \tau$ also, with $|\overline{\delta}| = |M| = |X| < \gamma$.

Suppose we had $\mathcal{P}(\delta)^M = \mathcal{P}(\delta)^L$. Then we could define a measure derived from π^{-1} in the usual manner: let $\alpha = \mathrm{crit}(\pi^{-1})$ and define U by

$$Z \in U \iff Z \in \mathcal{P}(\delta)^M \wedge \alpha \in \pi^{-1}(X).$$

Then U would be a *countably complete* ultrafilter on $\mathcal{P}(\delta)^L$ (that is why we chose ${}^\omega X \subseteq X$ as this implies ${}^\omega M \subseteq M$), and this in turn implies that $\mathrm{Ult}(L, U)$ is wellfounded. But that implies $\exists j : L \longrightarrow L$.

$$\begin{tikzcd}
(X,\in) \arrow[r, phantom, "\cong"] \arrow[d, "\mathrm{id}"'] & (L_{\overline{\delta}},\in) \arrow[r, phantom, "\subseteq"] \arrow[dl, "\pi^{-1}"'] & (L,\in) \arrow[dl, "j \supset \pi^{-1}"'] \arrow[d, dashed] \\
(L_{\gamma^+},\in) \arrow[r, phantom, "\subseteq"] & (L,\in) \arrow[r, phantom, "\cong"] & (Ult(L,U),E)
\end{tikzcd}$$

Hence we must have: $\mathcal{P}(\delta)^M \subsetneq \mathcal{P}(\delta)^L$. So:

(2) $\exists\beta \geq \overline{\delta}(\mathrm{Def}(L_\beta) \cap \mathcal{P}(\delta)) \nsubseteq L_{\overline{\delta}}$.

Choose β least so that (2) holds. By so-called fine structural methods Jensen showed how there is a superstructure L_η for some $\eta > \gamma^{+L}$ and a sufficiently elementary map $\tilde{\pi} \supset \pi^{-1}$, $\tilde{\pi} : L_\beta \longrightarrow L_\eta$, and because there is a 'new' subset of δ definable over L_β there must also be a 'new' subset of $\gamma = \pi^{-1}(\delta)$ that is not in L_η. But this is absurd as by L's construction $(\mathcal{P}(\gamma) \subset L_{\gamma^+})^L$. $\square$

Remark 2.10 *The assumptions (i) and (ii) can be dropped, but not without some difficulty, in particular when $cf(\gamma) = \omega$; however the format of the argument remains roughly the same.*

Theorem 2.11 ((Jensen - the full L-Covering Lemma) (ZF+$\neg 0^\#$ exists))
For any $X \subseteq \mathrm{On}$, if $|X| > \omega$ then there exists $Y \in L$ with (a) $|Y| = |X|$ and (b) $Y \supseteq X$.

The above result Theorem 2.9 is then a corollary of this. As are:

Corollary 2.12 (ZF+$\neg 0^\#$ exists) *(a) Let $(\tau \in \mathrm{Reg})^L$ with $\tau \geq \omega_2$, then $\mathrm{cf}(\tau) = |\tau|$.*
(b) Let $\tau \in \mathrm{SingCard}$. Then $\tau^+ = \tau^{+L}$.

Remark 2.13 Part (b) of the last Corollary above is sometimes called WCL the Weak Covering Lemma. The reason being that for other inner models M we may have $\mathrm{WCL}(M)$ provable (obtained by replacing L by M in the Corollary's statements) whilst the full $\mathrm{CL}(M)$ is not.

Generalizations

If $0^{\#} = M_0$ exists as above, perhaps there is no non-trivial $j : L[0^{\#}] \longrightarrow_e L[0^{\#}]$ and then we have a CL($L[0^{\#}]$)? This is indeed the case; however if this new assumption fails then we have "$(0^{\#})^{\#}$". We then get a theorem along the lines of a new Covering Lemma in the form of CL($L[0^{\#}]$) which holds iff $\neg j : L[0^{\#}] \longrightarrow_e L[0^{\#}]$. $(0^{\#})^{\#}$ is again a countable object and we can repeat this process. After we have done this uncountably often our $\#$-like mouse objects are no longer countable and we have to resort to uncountable mice M.

Instead of toiling inductively through seas of such objects, Dodd & Jensen first proved the following:

Theorem 2.14 (Dodd-Jensen [**4**]) (ZF) *There is an inner model K^{DJ}, so that if there is no inner model with a measurable cardinal, then (a) there is no non-trivial embedding $j; K^{\mathrm{DJ}} \longrightarrow K^{\mathrm{DJ}}$ and (b)* $\mathrm{CL}(K^{\mathrm{DJ}})$.

This was the first core model to go beyond L (if one discounts the models $L[0^{\#}]$ etc).

Theorem 2.15 (Steel [**28**]) (ZFC) *If there is no inner model for a Woodin cardinal, then there is a model K^{Steel}, which is again rigid, and over which* $\mathrm{WCL}(K^{\mathrm{Steel}})$ *holds.*

Perhaps there is some ultimate model which, although not L, is an inner model $L[E]$ say, which is the "core" of V, with E coding up all possible 'measures' or means of generating embeddings of models, and hopefully one has say $\mathrm{WCL}(L[E])$? Maybe, but to date the following 'inner model program' has worked on the following inductive template: assuming you have built core models for any strictly weaker large cardinal assumption than "There is an inner model with a Large Cardinal Γ"; so: assume there is no inner model for such a cardinal as Γ and build K^{Γ}, which, you should show under this assumption, is rigid and over which $\mathrm{WCL}(K^{\Gamma})$ holds. Then if the latter fails you have an inner model with a Γ-cardinal. Continue. This programmatic approach to filling out V with wider and thicker inner models depending on the strength of the large cardinals existing in (inner models of) V, has become known as the "*Inner Model program*". Currently, however, the programme is stuck at the level roughly of Γ a Woodin limit of Woodin cardinals.

Further remarks:

Theorem 2.16 (Magidor, [**14**]) (ZF) *Assume $\neg 0^{\#}$. Then if $X \subseteq \mathrm{On}$ is uncountable and closed under the primitive recursive set functions, then $X = \bigcup_{n<\omega} Y_n$ with each $Y_n \in L$.*

This is thus a decomposition rather than a covering theorem, but its proof is essentially (but perhaps surprisingly) a variant on Jensen's argument. The primitive recursive set functions mentioned are a mild collection of absolute functions on sets generalizing those on numbers, *cf* [**2**].

The significance of the WCL over a model M say is that as $\gamma^{+} = \gamma^{+M}$ for singular cardinals of V, then for many ordinal combinatorial properties whose statements or constructions which are performed inside M up to γ^{+M} we have that that combinatorial property is valid in V as well and is available for us for that construction to be performed in V. The Covering Lemma (under the rigidity assumption as always) has thus delivered for us a piece of absolute knowledge about V: one such combinatorial principle is called:

$\Box_\gamma$: *There is* $\langle C_\alpha \mid \alpha \in \mathrm{Sing} \cap \gamma^+\rangle$ *with*
(i) $C_\alpha \subseteq \alpha$ *is closed; it is unbounded in* α, *if* $\mathrm{cf}(\alpha) > \omega$;
(ii) $\mathrm{otp}(C_\alpha) < \alpha$;
(iii) $\beta \in C_\alpha \longrightarrow C_\beta = \beta \cap C_\alpha$.

(Such a sequence can be thought of as a uniformly presented sequence of witnesses to the singularity of ordinals below γ^+, which cohere or glue together nicely as expressed in (iii). This is useful for inductive constrictions up to γ^+ in particular for singular γ. Since the properties (i)-(iii) are absolute between M and V, if such can be constructed in M and $\gamma^+ = \gamma^{+M}$ we have such a sequence good in V.)

Remark 2.17 Jensen first established $\Box_\gamma$ for all $\gamma \in \mathrm{Card}$, in L. This was then established in the core models over the years as they were developed (in K^{DJ} by the author, in K^{Steel} by Schimmerling-Zeman [**24**], and with other results by other set theorists for intermediate models which we have not defined, see also [**23**] for an overview.)

Remark 2.18 It is possible to establish the "correct successor cardinal" computation for certain classes of regular limit cardinals γ than singulars: for γ weakly compact in V, (Jensen originally for L); γ Ramsey (Mitchell [**19**] originally for K^{DJ}, later [**20**] K^{Steel}); γ Jonsson (Welch [**29**] for K^{Steel}) again showing that for cardinals in these classes, their successors in V also enjoy certain combinatorial properties. However for inaccessibles in general this will fail.

3 Determinacy: The Martin-Harrington Theorem

We don't give an account of the history of determinacy here which goes back to the work of Banach-Mazur in the 1930's (for this see Kanamori) but suffice it to say that when a pointclass of sets of reals Γ (such as $\Gamma = \Sigma^0_\alpha, \Delta^1_1, \Pi^1_n, \ldots, \Sigma^1_n, \ldots$) (thus the *projective sets* are those sets of reals in some Σ^1_n for some $n < \omega$) are all determined, then all sets in that pointclass enjoy substantial *regularity properties*, such as being Lebesgue measurable, if uncountable then having a perfect subset (and hence size that of the continuum), and the Baire property (that is all have meagre symmetric difference from some open set). Besides Kanamori a complete source for this material is Moschovakis [**21**]. Another significant property is *Uniformization:* suppose Γ, H are two pointclasses of sets of reals. We say that $\mathrm{Unif}(\Gamma, \mathrm{H})$ holds if whenever $Q \in \Gamma \cap \mathcal{P}({}^k\mathbb{R} \times \mathbb{R})$ ($k < \omega$) then there is $P \subseteq Q$, $P \in H$ and
$\forall x[\exists y Q(y,x) \longrightarrow \exists! y P(y,x)]$
(P thus acts as a function *uniformising* the relation Q by choosing a unique element in the relation for each x for which such is possible.) Classically one had (The Novikoff-Kondo-Addison Theorem) $\mathrm{Unif}(\Pi^1_1, \Pi^1_1)$. We let PU (Projective Uniformisation) abbreviate the statement that every projective relation can be projectively uniformised.

Games.

Definition 3.1 *For* $A \subseteq \omega^\omega$ *(or more generally* X^ω*) the infinite perfect information game* G_A *is defined between two players I, II playing elements* $n_i, m_i \in X$*:*

I	n_0		n_1		n_2		$\cdots$		n_k		$\cdots$
II		m_0		m_1			$\cdots$			m_k	$\cdots$

together constructing $x = (n_0, m_0, \ldots, n_k, m_k, \ldots) \in X^\omega$.

We say that I *wins* iff $x \in A$; otherwise *II* wins. Notions of *strategy* and *winning strategy* are defined in the obvious fashion. Notice that for $A \subseteq \omega^\omega$ a strategy σ for, *e.g.* I is a map from $\bigcup_k^{2k} \omega \longrightarrow \omega$. Since there is a recursive bijection ${}^{<\omega}\omega \leftrightarrow \omega$, we can think of σ as essentially a subset of ω or again, as a real number.

We speak of a topology of open and closed sets of a space X^ω by letting a typical *basic open set* to be a neighbourhood N_s where $s \in \mathrm{Seq}_X$ is a finite sequence of elements of X:

$$N_s =_{\mathrm{df}} \{x \in X^\omega \mid \exists k < \omega(x \upharpoonright k = s\}.$$

An *open set* is then a countable union of basic open neighbourhoods; a *closed set* is a complement of an open *etc.* When $X = \omega$ and an open set, U, is given as a union of basic open neighbourhoods defined by a *recursively* given set of sequences numbers s coding elements of ${}^{<\omega}\omega$, then we say that U is *semi-recursive*, or Σ^0_1. More generally we speak of open sets and Σ^0_1 sets in *parameter codes* which are themselves subsets of ω or again real numbers.

The basic theorem here is:

Theorem 3.2 (Gale-Stewart) *Let $A \subseteq X^\omega$ be an open set. Then G_A is* determined*, that is one of the players has a winning strategy.*

Proof: Note that if *I* wins it is because he has manœuvred the play so that there is a finite stage $(n_0, m_0, \ldots, n_k)$ so that $N_{(n_0, m_0, \ldots, n_k)} \subseteq A$. Essentially he has won by this stage as it matters not what n_l he plays for $l > k$. *II* however, playing into the closed set which is $X^\omega \backslash A$ must be vigilant to the end if she is to win. Suppose then *I* has no winning strategy. Then for every n_0 *II* has a reply m_0 so that *I* has no winning strategy in the game $G_{A/(n_0,m_0)}$ where $A/(n_0, m_0) =_{\mathrm{df}} \{x \in A \mid x(0) = n_0, x(1) = m_0\}$. For, if there was an n_0 so that *I* did always have a winning strategy in this latter game for whichever m_0 *II* played, $\sigma(m_0)$ say, then this would amount to a winning strategy for him in G_A: first play n_0, wait for m_0 and then use $\sigma(m_0)$. Thus given n_0 *II* should play m_0 so that *I* has no such strategy. But if she continues in this way, *this* is a winning strategy for *II*: always respond so that I has no winning strategy from that point on. The resulting play x cannot be in A. $\square$

The motto here is that we can always in ZFC, prove $\mathrm{Det(Open)}$ for any space X^ω and so by taking complements $\mathrm{Det(Closed)}$ too. (AC is needed only to wellorder X if need be.) Many proofs of determinacy of complicated sets in X^ω, involve reducing the game to an open game in some larger space Y^ω. The latter are determined by Gale-Stewart. The difficulty arises in showing that the player with the winning strategy for the open set on the space Y^ω also has one for the related, but much more, complicated set in X^ω.

Definition 3.3 *(i) A* tree T on $\omega \times X$ *(for $X \neq \varnothing$) is a set of sequences in $\bigcup_k {}^k\omega \times {}^kX$ where if (σ, u) and $k \leq |\sigma| = |u|$ then $(\sigma \upharpoonright k, u \upharpoonright k) \in T$. Similarly defines trees on ${}^n\omega \times X$.*

(ii) For such a tree T we set

$$T_\sigma =_{\mathrm{df}} \{u \mid (\sigma, u) \in T\}\ ;\ T^{\subseteq}_\sigma =_{\mathrm{df}} \bigcup_{k \leq |\sigma|} T_{\sigma\upharpoonright k}\ ;\ \textit{and}\ T_x = \bigcup_k T_{x\upharpoonright k}.$$

Definition 3.4 *For T a tree:*

(i) $[T] =_{\mathrm{df}} \{(x, f) \mid \forall k(x \upharpoonright k, f \upharpoonright k) \in T\}$ *- is the set of* branches through T.

(ii) $p\,[T] =_{\mathrm{df}} \{x \in {}^\omega\omega(\text{or } {}^k({}^\omega\omega)) \mid \exists f(x, f) \in [T]\}$ *- the* projection *of* T.

(iii) A set $A \subseteq {}^k({}^\omega\omega)$ is κ-Suslin *(for $\kappa \geq \omega$, $\kappa \in \mathrm{Card}$) if $A = p[T]$ for some tree on $\omega \times \kappa$.*

Clearly a tree is wellfounded (under $\supseteq$) if $[T] = \varnothing$. For a $C \subseteq \omega \times X$ a closed set, there is a tree T on $\omega \times X$ with $C = [T]$. In particular for $X = \omega$. If C has a recursively open complement, then we may take T as recursive set of sequences from $\bigcup_k ({}^k\omega \times {}^k\omega)$.

If $A \subseteq \omega^\omega$ is Σ^1_1 then $A = p[T]$ for a tree on $\omega \times \omega$ - thus such sets are projections of closed sets, and conforms to the idea that $x \in A \leftrightarrow \exists y \in \omega^\omega (x, y) \in [T]$. This classical result (due to Suslin) is sometimes stated that analytic (= $\mathbf{\Sigma^1_1}$) sets are "ω-Suslin". Many of the classical properties of analytic sets as studied by analysts can be attributed to this (and similar) representations.

Given this last fact, we can represent $\mathbf{\Pi^1_1}$ sets, being complements of $\mathbf{\Sigma^1_1}$'s, as those sets where the tree T for the complement is wellfounded. We may thus define *rank functions* for such trees, by finding functions that map the nodes into the ordinals in a tree-order preserving way. We do this next.

Let then $A \in \Pi^1_1$ be a (lightface) co-analytic set. Then by the above discussion there is a recursive tree T (meaning there is a recursive set of sequence numbers) on $\omega \times \omega$ with:

$$\forall x(x \in A \leftrightarrow T_x \text{ is wellfounded}).$$

T_x being countable, we have that $\mathrm{rk}(T_x) < \omega_1$. Consequently:

$$x \in A \leftrightarrow \exists g(g : T_x \longrightarrow \omega_1 \text{ in an order preserving way}).$$

That is we may define a tree $\hat{T}$ on $\omega \times \omega_1$ as follows:

$$\hat{T} = \{(\tau, u) \mid \forall i, j < |\tau| : \tau_i \supset \tau_j \wedge (\tau \restriction |\tau_i|, \tau_i) \in T \longrightarrow u(i) < u(j)\}.$$

Then one can see that

$$x \in A \leftrightarrow \exists g \in {}^\omega\omega_1((x, g) \in [\hat{T}]) \leftrightarrow x \in p[\hat{T}].$$

The above reasoning thus shows that any Π^1_1 set A is ω_1-Suslin. Further:

(i) $\hat{T} \in L$, and by the absoluteness of well founded relations on ω between ZF$^-$-models containing all countable ordinals $[\hat{T}] \neq \varnothing \leftrightarrow ([\hat{T}] \neq \varnothing)^L$.

(ii) If the underlying set is a $\Pi^1_1(a)$ set for some real parameter a then $\hat{T} \in L[a]$.

Now the argument can be stepped up to Σ^1_2 sets: suppose $x \in B \leftrightarrow \exists y(x, y) \in A$ (we write '$B = pA$') with $A \in \Pi^1_1$ and hence there is a tree $\hat{T}_A$ on $(\omega \times \omega) \times \omega_1$ with

$$(x, y) \in A \leftrightarrow \exists g \in {}^\omega\omega_1\big(((x, y), g) \in [\widehat{T}_A]\big).$$

However there is a Δ^{ZF}_0 definable bijection $\omega \times (\omega \times \omega_1) \leftrightarrow (\omega \times \omega) \times \omega_1$ thus re-defining the tree $\widehat{T}_A$ as $\overline{T}$ but on different sequences, so that for any x, y we have

$$(x, y) \in p[\widehat{T}_A] \leftrightarrow \exists g \in {}^\omega\omega_1\big(((x, y), g) \in [\widehat{T}_A]\big) \leftrightarrow \exists g \in {}^\omega\omega_1\big((x, (y, g)) \in [\overline{T}]\big).$$

However now we again use a Δ^{ZF}_0 bijection $\omega \times \omega_1 \leftrightarrow \omega_1$to recast $\overline{T}$ as a tree $\overline{\overline{T}}$on ω_1 alone, then:

$$x \in B \leftrightarrow x \in pA \leftrightarrow x \in p[\overline{\overline{T}}].$$

We have:

Theorem 3.5 (Shoenfield) *Any Σ^1_2 set is ω_1-Suslin, as a projection of a tree $T \in L$.*

Corollary 3.6 *Let B be any Σ^1_2 relation. The $\exists x B(x) \leftrightarrow (\exists x B(x))^L$. In particular Σ^1_2 sentences are absolute between L and the universe V. Moreover if $A \subseteq \mathbb{N}$ is Σ^1_2 then $A \in L$.*

Theorem 3.7 (Levy) *There is an ordinal $\sigma_1 < \omega_1^L$ so that $(L_{\sigma_1}, \in) \prec_{\Sigma_1} (V, \in)$.*

Proof: Let A be a Σ_1 sentence in $\mathcal{L}$ true in V. By Löwenheim-Skolem, the following holds:
"*there is a countable wellfounded transitive model* $(M, \in)$ *with* $(A)^M$".
The latter can be expressed as a Σ^1_2 assertion about a real number x (coding such an $(M, \in)$). By the last corollary there is such a x, $M_x \in L_{\omega_1^L}$. By the upward persistence of Σ_1 sentences $(A)^{M_x} \Longrightarrow (A)^L$. Hence $L_{\omega_1^L} \prec_{\Sigma_1} V$. $\square$

We sometimes wish to be more specific about the tree for Π^1_1 and for this we use the Kleene-Brouwer ordering: $<_{\mathrm{KB}}$ is a linear ordering, and it restricts to linear orderings on ${}^{<\omega}\kappa$ for cardinals $\kappa \geq \omega$. One may check:

If T is a tree on κ then T is wellfounded iff T is wellordered by $<_{\mathrm{KB}}$. In particular we have for a Π^1_1 set A that there is a tree T on $\omega \times \omega$ so that

$$x \in A \leftrightarrow T_x \text{ is wellfounded } \leftrightarrow T_x \text{ is wellordered by } <_{\mathrm{KB}} .$$

For a tree T on $\omega \times \kappa$ we then have a linear ordering $<_x$ corresponding to the KB ordering of T_x with the following definition and properties:

$$i <_x j \longleftrightarrow_{\mathrm{df}} \quad (\tau_i, \tau_j \notin T_x \wedge i < j) \vee (\tau_i \notin T_x \wedge \tau_j \in T_x) \vee (\tau_i, \tau_j \in T_x \wedge \tau_i <_{\mathrm{KB}} \tau_j).$$

In fact $<_x$ is the union of orderings given by initial segments of x: for $\tau \in {}^{<\omega}\omega$ define

$$i <_\tau j \longleftrightarrow_{\mathrm{df}} i < j < |\tau| \wedge (\tau_i, \tau_j \notin T^{\subseteq}_\tau \wedge i < j) \vee (\tau_i \notin T^{\subseteq}_\tau \wedge \tau_j \in T^{\subseteq}_\tau) \vee (\tau_i, \tau_j \in T^{\subseteq}_\tau \wedge \tau_i <_{\mathrm{KB}} \tau_j).$$

Then:
(i) $<_x = \bigcup_k <_{x \restriction k}$.
(ii) Define T^*_A on $\omega \times \kappa$ by $(\tau, u) \in T^*_A \longleftrightarrow_{\mathrm{df}} \forall i, j < |\tau| (i <_\tau j \leftrightarrow u(i) < u(j))$. Then

$$x \in A \leftrightarrow \exists g \in {}^\omega\kappa((x, g) \in [T^*_A]).$$

In the above $\kappa \geq \omega$ may be any cardinal of course. After the basic definitions were introduced $\mathrm{Det}(\Sigma^0_3)$ (M. Davies [**1**]) was the extent of provable determinacy (indeed provable within Z_2). Remarkably H. Friedman ([**5**]) showed that for Borel Determinacy roughly, order type α iterations of the power set operation (together with some Replacement) would be needed to establish $\mathrm{Det}(\Sigma^0_\alpha)$, thus establishing that $Z \not\vdash \mathrm{Det}(\Delta^1_1)$. This phenomenon would start at Σ^0_4. Paris then showed (using ZFC) $\mathrm{Det}(\Sigma^0_4)$ but there matters languished until

Theorem 3.8 *(Martin* [**18**]*)* $\mathrm{ZF} \vdash \mathrm{Det}(\mathrm{Borel})$ *for Borel subsets of* ω^ω.

Theorem 3.9 *(Martin-Harrington)* $\mathrm{ZF} \vdash \exists j : L \longrightarrow_e L, j \neq \mathrm{id} \longleftrightarrow \mathrm{Det}(\Pi^1_1)$.

Proof: $(\longrightarrow)$ (Martin) We assume the left hand side, and then by the work of Silver, Kunen, we have a c.u.b. set $C \subseteq \omega_1$ of indiscernibles for $(L, \in)$.

Let $A \in \Pi^1_1$. The usual game G_A involves integer moves but has a complicated payoff set. We replace the space ω^ω with a larger one X^ω for some X to be defined, and relate A to some *closed* $A^* \subseteq X^\omega$. By Gale-Stewart, this is determined. We have to show that winning strategies in G_{A^*} can be translated to winning strategies for the same player in G_A.

Definition 3.10 G_{A^*} *is defined between two players I, II playing as follows:*

$$\begin{array}{lccccc} I & (n_0, \xi_0) & (n_1, \xi_1) & (n_2, \xi_2) & \cdots & (n_k, \xi_k) \\ II & m_0 & m_1 & & \cdots & m_k \end{array}$$

together constructing

$$x = ((n_0, \xi_0), m_0,\ (n_1, \xi_1), m_1, (n_2, \xi_2), m_{2,} \cdots (n_k, \xi_k), , \ldots).$$

The *Rules* are that $n_i, m_i \in \omega$, $\xi_i \in \omega_1$. We think of the ξ_i as laying out a function $g : \omega \longrightarrow \omega_1$ with $g(i) = \xi_i$, and the integers yielding $x = (n_0, m_0, n_1, \ldots)$

Winning Conditions: I wins iff $(x, g) \in T^*_A$. g then witnesses that T_x is wellfounded, by giving an order preserving map from $<_x$ into ω_1. Note that the game is closed in the sense that if I loses he does so at some finite stage (by messing up his ordering, which cannot be redeemed at a later stage). Equivalently, *II* is playing into an open set and if she wins, does so at a finite round. Note that T^*_A is defined in L, and so by the Gale-Stewart theorem, is determined in L.

Claim: If σ^* is a winning strategy for I (respectively *II*) in G_{A^*} in L, then there is a winning strategy for *I* (respectively *II*) in G_A.

Proof of *Claim.* If σ^* is a winning strategy for I he can use it to play in G_A by suppressing the ordinal moves, and clearly wins (both in L and V). Suppose then $\sigma^* \in L$ is, in L, a winning strategy for *II*.

Idea: *II* simulates a run of G_{A^*} by using indiscernibles from $C_0 =_{\rm df} C \cap \omega_1$ as 'typical' ordinal moves for I. She defines a strategy $\sigma : \bigcup_n {}^{2n+1}\omega \longrightarrow \omega$ as follows; fix $n < \omega$ and consider the formula $\varphi(\omega_1, T^*, \sigma^*, \tau, \vec{\xi}, m)$ which defines a term $t(\omega_1, T^*, \sigma^*, \tau, \vec{\xi}) = m$:

$$\vec{\xi} \in {}^{n+1}\omega_1 \wedge \sigma^*(\tau \restriction n+1, \vec{\xi}) \in T^* \wedge \sigma^*(\tau, \vec{\xi}) = m.$$

As $\sigma^*, T^* \in L$ the term t will have a fixed value $m \in \mathbb{N}$ for any $\vec{\xi} \in {}^{n+1}C_0$ she chooses, since the latter are indiscernibles for the formula φ. So she sets (in V where C_0 lives):

$$\sigma(\tau) = m = t(\omega_1, T^*, \sigma^*, \tau, \vec{\xi})\ \textit{for any}\ \vec{\xi} \in {}^{n+1}C_0.$$

Now argue that were $x \in A$ even though *II* followed this strategy, then we'd have that there is an order preserving embedding $g : (\omega, <_x) \longrightarrow (C_0, <)$ (as C_0 is uncountable). However that corresponds to a run of the game G_{A^*} (with ordinal moves delivered by g), where *II* has used σ^*, which was supposed to be winning for her! Contradiction! □

It is an exercise to show that if a G_A for $A \subseteq {}^\omega\omega$ is determined, A is countable, or else contains a perfect subset and hence is the size of the continuum. However in L there is an uncountable Π^1_1 set which has no perfect subset. Conclusion: $\mathrm{Det}(\Pi^1_1)$ is false in L and so ZF $\nvdash \mathrm{Det}(\Pi^1_1)$. In the above we constructed a winning strategy for *II* in V. However the statement "$\exists\sigma[\sigma$is a winning strategy for *II* in $G_A]$" is a strictly Σ^1_3 statement, and such are not in general absolute to L.

We should like to establish $\mathrm{Det}(\Pi^1_n)$ for other $n > 1$ but this requires considerably more ingenuity. The key is to find representations of projective sets as projections of trees enjoying so-called 'homogenity properties' which entail that there are measures (and more) on the trees. This tree representation was implicit in the proofs of $Det(\Pi^1_1)$ of Martin and in Martin-Solovay [**16**] where such trees were used to analyse Π^1_2 sets. However the notion was only made explicit in Kechris [**11**].

Definition 3.11 *(Homogeneous Trees and Sets) Let T be a tree on $\omega \times X$ and $\kappa > \omega$ be a cardinal. (A) Then T is a κ-*homogeneously Suslin tree *iff* $\exists\langle U_\tau \mid \tau \in \mathrm{Seq}\rangle$ *where*

(i) Each U_τ is a κ-complete measure on T_τ;

(ii) For any $\tau \supset \sigma$ U_τ projects to *U_σ: i.e.* $u \in U_\sigma \leftrightarrow \{v \in T_\tau : v \restriction |\sigma| \in u\} \in U_\tau$.

(iii) The U_τ form a countably complete tower*: if $\sigma_i \in \mathrm{Seq}$, $Z_i \in U_{\sigma_i}$ are such that $i < j \longrightarrow \sigma_i \subset \sigma_j$ then there is a $g \in {}^\omega X$ so that: $\forall i(g \restriction i \in Z_i)$.*

*(B) We say $A \subseteq \omega^\omega$ is κ-*homogeneously Suslin *if $A = p[T]$ for a κ-homogeneously Suslin tree T.*

Notice that one way to phrase *(iii)* is to say that if $x \in p[T]$ then $\langle U_{x\restriction i} : i < \omega\rangle$ form a countably complete tower. Although the notion had to be isolated in fact Martin had used essentially the fact that if there exist a measurable cardinal κ then any analytic set is $p[T]$ for a κ-homogenously Suslin tree on $\omega \times \kappa$.

Theorem 3.12 *Suppose A is κ-homogeneously Suslin for some κ. Then G_A is determined.*

Proof: Suppose $A = p[T]$ with T κ-homogeneously Suslin on $\omega \times \lambda$ (some $\lambda \geq \kappa$.

Definition 3.13 *G_{A^*} is defined between two players I, II playing as follows:*

I (n_0, ξ_0)	(n_1, ξ_1)	(n_2, ξ_2)	$\cdots$	(n_k, ξ_k)	
II	m_0	m_1	$\cdots$		m_k

together constructing

$$x = ((n_0, \xi_0), m_0,\ (n_1, \xi_1), m_1, (n_2, \xi_2), m_{2,} \cdots).$$

Rules: $n_i, m_i \in \omega$, $\xi_i \in \lambda$. *Winning Conditions:* as before I wins iff $(x, g) \in [T]$ where $g(i) = \xi_i$.

Again this is a closed game, and so there is a winning strategy for one of the players in $L[T]$. Now to show that if *II* has a winning strategy in G_A if she has a winning strategy σ^*in G_{A^*}, we use the ω_1-completeness of the measures rather than indiscernibility.

She defines a strategy $\sigma : \bigcup_n {}^{2n+1}\omega \longrightarrow \omega$ as follows; for $n < \omega$ and then for $\tau \in {}^{2n+1}\omega, u \in^{n+1} \lambda$, define $\overline{\sigma}(\tau, u) = \sigma^*(\tau(0), u(0), \cdots, \tau(2n), u(2n)) \in \omega$.

Define $Z_{\tau,k} =_{\text{df}} \{u \in T(\tau \restriction n+1) : \overline{\sigma}(\tau, u) = k\}$. Since $U_{\tau\restriction n+1}$ is a measure on $T(\tau \restriction n+1)$, by its ω_1-completeness, for precisely one value of k is $Z_{\tau,k} \in U_{\tau\restriction n+1}$. So let that value of k, k_0 say, be the response given by the strategy:

$\sigma(\tau) = k_0$ where k_0 is the unique value of k_0 with $Z_{\tau,k_0} \in U_{\tau\restriction n+1}$.

Again we check that σ is a winning strategy for *II* in G_A. If x is a result of a play with *II* using σ, but nevertheless $x \in A$, then by *(iii)* in the definition of κ-homogeneously Suslin $\exists g \in {}^{\omega}X$ so that for every n $g \restriction n+1 \in Z_{x\restriction 2n+1, \sigma(x\restriction 2n+1)}$. This implies $(x, g) \in [T]$ and is then a losing outcome of the game where *II* in fact uses σ^*, thus contradicting that the latter is a winning strategy for *II*. Hence $x \in A$. □

Hence determinacy would follow if we could establish homogeneity properties for trees.

Definition 3.14 *$A \subseteq \omega^\omega$ is* weakly homogeneously Suslin *if $A = pB$ where $B \subseteq (\omega^\omega)^2$ is homogeneously Suslin.*

In fact this is not the official definition of weakly homogeneously Suslin which defines "*weakly homogenous trees*" and is in terms of towers of measures, but nevertheless has this equivalence. These were also studied by Kechris [**11**] and Martin.

Weakly homogeneously Suslin sets (being the projections of weakly homogeneous trees) can be thought of as generalisations of analytic sets and are of great interest in their own right: we have seen that if there is a measurable cardinal, then Σ^1_2 sets are weakly homogeneously Suslin.

Theorem 3.15 *If a set A is weakly homogeneously Suslin, then it has the regularity properties (Lebesgue measurability, the Baire and Perfect subset properties ...).*

It thus has the *consequences* of being determined, without actually being so. Unfortunately A being weakly homogeneously Suslin does not imply that it is homogeneously Suslin. However one important feature is that they do have complements defined as projections of trees $\tilde{T}$ with the latter definable from their weakly homogenously tree T:

Lemma 3.16 *Let A be $p[T]$ with T weakly homogeneously Suslin. Then there is a tree $\tilde{T}$ with $\omega^\omega \backslash A = p[\tilde{T}]$.*

But on its own this is no help. The breakthrough was:

Theorem 3.17 (Martin-Steel [**17**]) *Suppose that λ is a Woodin cardinal and T is a λ^+ weakly homogeneously Suslin tree. Then for $\gamma < \lambda$, the $\tilde{T}$ above is γ-homogeneously Suslin.*

There is thus some trade off: the completeness of the measures drops. However we now have:

Theorem 3.18 (Martin-Steel) *Suppose λ_0 is a Woodin cardinal, and $\kappa > \lambda_0$ is measurable. Then $\mathrm{Det}(\Pi^1_2)$. Further if $\lambda_{n-1} < \lambda_{n-2} < \cdots < \lambda_0$ are $n-1$ further Woodin cardinals, then $\mathrm{Det}(\Pi^1_{n+1})$. Thus:*
$\mathrm{ZFC} +$ *"there exist infinitely many Woodin cardinals"* $\vdash$ PD.

Proof: Let $A \subseteq \omega^\omega$ be Π^1_2. Then A is the complement of a Σ^1_2 set on ω^ω which is itself the projection of a Π^1_1 set $B \subseteq (\omega^\omega)^2$. B is κ-homogeneously Suslin for some homogeneously tree T, as κ is measurable, and *a fortiori* is also λ_0^+-homogeneously Suslin. By Theorem 3.18 we have a γ-homogeneously Suslin tree $\tilde{T}$ projecting to A with γ-complete measures. By Theorem 3.12 we have G_A is determined, and we are done. The last two sentences follow by repetition of the argument. $\square$

The exact consistency strength of the assumption above is slightly stronger, with the ($\Longleftarrow$) being very involved:

Theorem 3.19 (Woodin)
$\mathrm{Con}(\mathrm{ZFC} +$ *"there exist infinitely many Woodin cardinals"*$) \Longleftrightarrow \mathrm{Con}(\mathrm{ZFC} + \mathrm{AD}^{L(\mathbb{R})})$.

The final conclusion of Theorem 3.18 yields the boundary of the *provability* of PD from large cardinal axioms; this is not a relative consistency result, and the phenomenon of course recurs for any strong axiom(s) of infinity that imply the existence of sufficiently many Woodins: PD holds outright. Set forcing can change the character of the seeming universe *locally*, but, roughly speaking, does not destroy large cardinal properties beyond the rank of the partial order. Consequently if V has a proper class of Woodin cardinals, then PD is not only provable outright but is *absolute* into set forcing extensions. Much more is possible:

Theorem 3.20 (Woodin) *Suppose there is a proper class of Woodin cardinals; then* $\mathrm{Th}(L(\mathbb{R}\})$ *is absolute with respect to set forcing. Thus if $V[G]$ is a set generic extension*

$$(L(\mathbb{R}))^V \equiv (L(\mathbb{R}))^{V[G]}$$

The larger cardinals that we look at next will again all prove PD and in fact $\mathrm{AD}^{L(\mathbb{R})}$ outright. We have seemingly reached a particular stage beyond which these determinacy properties are simply unavoidable.

4 Large Cardinals

We have seen that a measure U on $\mathcal{P}(\kappa)$ in V yields an ultrapower $(\mathrm{Ult}(V,U), E)$ which is wellfounded and hence isomorphic to a transitive inner model $(M, \in)$ of ZFC. The following facts hold:

• $V_{\kappa+1} = (V_{\kappa+1})^M$
• $(j(\kappa)$is measurable$)^M$;
• $U \notin M$ and thus $V_{\kappa+2} \neq (V_{\kappa+2})^M$.
• κ may, or may not, be measurable in M because of the presence in M of some other measure $\overline{U}$). (If κ was the least measurable of V then it can not contain such a $\overline{U}$, by the Scott argument).

Thus given an elementary embedding $j : V \longrightarrow_e M$, the above shows precisely which initial part of V can be expected to be in M in general, namely $V_{\kappa+1}$. Modern set theory now classifies large cardinals (that previously were often argued for "by analogy with ω" or some such), into a hierarchy given by the embedding properties that they enjoy.

Definition 4.1 *A cardinal κ is* α-strong *if there is an embedding $j : V \longrightarrow_e M$ with $V_a = (V_\alpha)^M$, with $\mathrm{cp}(j) = \kappa$, and $j(\kappa) \geq \alpha$.*

Thus a measurable cardinal is $\kappa + 1$ strong. The larger the α, the stronger the embedding, as more of the initial V hierarchy is preserved by the identity map into the inner model M.

Definition 4.2 *A cardinal κ is* strong *if it is α-strong for all α.*

(Note the order of quantifiers: for every α there is an embedding j, depending on α ...) One may wonder about the first order formalizability of the above notions in ZF. However just as the first statement of the existence of a measure on $\mathcal{P}(\kappa)$ is equivalent to the existence of a class embedding j (which we might formulate in ZFC$_{\dot{j}}$) it is possible to give an *extender representation* of such embeddings. Thus: for an α-strong embedding j we may find a generalisation of a measure on $\mathcal{P}(\kappa)$ called an extender which we may think of as given by a *sequence* of measures $\langle E_a : a \in [\alpha]^{<\omega} \rangle$ with each E_a itself a measure on $\mathcal{P}([\kappa]^{|a|})$.

Given an α-strong embedding $j; V \longrightarrow N$ we define an *α-extender at κ*generalising what we did for measures.

$$X \in E_a \leftrightarrow_{\mathrm{df}} X \in \mathcal{P}([\kappa]^{|a|}) \wedge a \in j(X).$$

The sequence $\mathcal{E} = \langle E_a : a \in [\alpha]^{<\omega} \rangle$ then has satisfactory coherence properties, in fact enough so that we can define an *extender ultrapower* $(\mathrm{Ult}(V, \mathcal{E}), E)$ from it. In the situation described, this ultrapower has a wellfounded E-relation, and is again isomorphic to some $(M, \in)$ which in general is not necessarily equal to $(N, \in)$ but which can be elementarily embedded back into $(N, \in)$ and we thus shall have:

$$\begin{tikzcd}
j : (V, \in) \arrow[r] \arrow[d, dashed, "\pi"'] & (N, \in) \\
(Ult(V, \mathcal{E}), E) \arrow[ur, "k"'] &
\end{tikzcd}$$

It is possible to view the $\mathrm{Ult}(V, \mathcal{E})$ as a direct limit of the 'ordinary' ultrapowers by the measures E_a. It is part of the flexibility of the approach that this is inessential though.

Having thus generalised the notion of measure ultrapower to extender ultrapower we can use these to give us first order formulations of α-strong etc. A simplified statement is:

Lemma 4.3 *Let α be a strong limit cardinal; then κ is α-strong iff there is an α-extender sequence $\mathcal{E} = \langle E_a : a \in [\alpha]^{<\omega} \rangle$ at κ, with $V_{\kappa+\alpha} \subseteq \mathrm{Ult}(V, \mathcal{E}) \wedge j(\kappa) > \alpha$.*

The notion of κ being strong is then also first order (although involving a quantifier over On). One may note the following easily proven fact: κ strong implies that $V_\kappa \prec_{\Sigma_2} V$.

Definition 4.4 *κ is* superstrong *if there is $j : V \longrightarrow_e M$ with $V_{j(\kappa)} \subseteq M$.*

Note that α-strong only asked for $V_\alpha \subseteq M$ whilst $j(\kappa) > \alpha$. This seemingly innocuous extension is in fact a powerful strengthening. Again it has a first order formalisation. We proceed to a definition of Woodin cardinal. First we define a strengthening of the concept of strong.

Definition 4.5 *Let $A \subseteq V$. We say that κ is A-*strong in *V if for every α there are M, $B \subseteq V$ with $IM(M)$ and an $\mathcal{L}_{\dot{\in},\dot{A}}$ - elementary embedding*

$$j : \langle V, A\rangle \longrightarrow_e \langle M, B\rangle \textit{ such that } \mathrm{cp}(j) = \kappa, V_\alpha \subseteq M \textit{ and } V_\alpha \cap A = V_\alpha \cap B.$$

This is not a first order formalisation, but now consider an inaccessible λ and relativise the notion from V down to V_λ

Theorem 4.6 *An inaccessible cardinal λ is called* Woodin *if for every $A \subseteq V_\lambda$ there is a $\kappa < \lambda$ which is A-strong in V_λ.*

There are many other equivalent formulations (*cf.* Kanamori [**10**] Sect 26.). A Woodin cardinal is necessarily Mahlo, but may fail to be weakly compact. It turned out that this is precisely the right concept to analyse the various determinacy properties. Subsequently it turned out to be precisely the right concept to gauge a whole host of other set theoretical phenomena, and thus it has become one of the central notions of modern set theory.

Lemma 4.7 *(i) If κ is superstrong then it is a Woodin limit of Woodins.*
(ii) If λ is Woodin then ("there are arbitrarily large strong cardinals")$^{V_\lambda}$.

A particular constellation of cardinals is also of interest for determinacy of infinite games played with reals, rather than integers. The assertion "$\mathrm{AD}_\mathbb{R}$" is that for every $\mathcal{A} \subseteq {}^\omega\mathbb{R}$, the game $G_\mathcal{A}$ is determined, where in the game now, players play a complete real number at each round instead of a single integer. The following conjecture emerged.

Conjecture (The "$\mathrm{AD}_\mathbb{R}$ hypothesis") The consistency strength of $\mathrm{AD}_\mathbb{R}$ is that of a cardinal μ that is simultaneously a limit of infinitely many Woodins $\lambda_n < \lambda_{n+1} \cdots < \mu$ and of μ-strong cardinals $\kappa_n < \lambda_n < \kappa_{n+1}$.

The hypothesis is also interesting since (given a mild strengthening of AD), the $AD_\mathbb{R}$-hypothesis is equivalent to the assertion that every set of reals is Suslin. The following is another connection between the worlds of large cardinals and determinacy of games.

Theorem 4.8 (Woodin ($\leftarrow$), Neeman-Steel ($\rightarrow$))

$$Con(ZF + AD_\mathbb{R}) \leftrightarrow Con(ZFC + AD_\mathbb{R}\textit{-hypothesis})$$

We continue our cataloguing of some more large cardinals through elementary embeddings.

Definition 4.9 *(i) A cardinal κ is α-*supercompact *if there is a $j : V \longrightarrow_e M$ with ${}^\alpha M \subseteq M$.*
(ii) κ is supercompact if it is α-supercompact for all α.

An embedding arising from a measurable cardinal κ in general only implies the closure of M under κ sequences, so this definition only starts to have bite once $\alpha > \kappa$. The closure under all α-sequences is again a considerable strengthening over the "strong" hierarchy of principles. It masks also a telling difference in the kind of embeddings: up to this point the embeddings j have always been "continuous at κ^+" meaning that $\sup(j``\kappa^+) = j(\kappa^+)$. If j is a 2^κ-supercompact embedding, this fails with '$<$' occurring. Whilst looking a rather technical difference this in fact introduces a wide variety of new phenomena. Current inner model theory has a target of producing a good fine structural inner model of a supercompact cardinal, but this target is not yet met. It can do so for strong cardinals, Woodin cardinal, Woodin limit of Woodin cardinals, ..., but although we know what the inner models should look like for example, for measurably Woodin cardinals, we are unable to prove they exist. The chief difficulty being the inability to prove that sufficiently many ultrapowers of the models are wellfounded in order for their construction to get off the ground. This has been dubbed the "iterability problem". Continuing onwards we come the *extendible* cardinals of Reinhardt (see [**27**]):

Definition 4.10 *(i) A cardinal κ is α-*extendible *if there are λ, j with $j : V_{\kappa+\alpha} \longrightarrow_e V_{\lambda+\alpha} \wedge \mathrm{cp}(j) = \kappa$.*
(ii) κ is extendible *if it is α-extendible for all α.*

Even 1-extendibility is a strong concept:

Lemma 4.11 *If κ is 1-extendible, then it is superstrong (and there are many such below it.)*

Lemma 4.12 *If κ is extendible, it is supercompact; (ii) κ extendible implies $V_\kappa \prec_{\Sigma_3} V$.*

Ascending further:

Definition 4.13 *A cardinal κ is* huge *if there is $j : V \longrightarrow_e M$ with $\mathrm{cp}(j) = \kappa \wedge {}^{j(\kappa)}M \subseteq M$.*

However if we try to maximise the extendibility properties we run into inconsistency:

Theorem 4.14 (Kunen [**13**]) $(\mathrm{ZFC}_{\dot{j}})$
There is no non-trivial $\mathcal{L}_{\dot{\in}}$-elementary embedding $j : V \longrightarrow_e V$.

It is unknown whether AC is necessary for this theorem. The proof of Kunen's theorem actually yields a direct ZFC result:

Theorem 4.15 (Kunen) (ZFC) *There is no non-trivial elementary embedding $j : V_{\lambda+2} \longrightarrow_e V_{\lambda+2}$.*

The "2" is an essential artefact of the argument. That there may be a non-trivial $j : V_{\lambda+1} \longrightarrow_e V_{\lambda+1}$ is not known to be inconsistent; if this is to be the case, then $\kappa_0 = \mathrm{cp}(j) < \lambda$ and it can be shown that λ has cofinality ω being $\sup\{\kappa_0, j(\kappa_0), jj(k_0), \ldots\}$. There are now several proofs of Kunen's theorem - see Kanamori, [**10**].

For such $V_{\lambda+1}$ the model $L(V_{\lambda+1})$ and its possible elementary embeddings has become an object of intense study, and is likely to be significance.

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