1 Global Reflection Principles

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Abstract. We consider a structural *reflection principle* that seeks to go beyond the traditional reflection principles of Mahlo, Levy, Bernays, *et al.* The latter are all firmly *intra-constructible* that is they produce justifications for only small large cardinals consistent with Gödel's constructible universe L. Our *Global Reflection Principles* by contrast ensure that the universe of set theory has unboundedly many measurable Woodin cardinals. This is a hypothesis which (with variants) is used by current set theorists to establish many absoluteness (and consistency) results concerning the universe V. It is argued that the Principles are justified by appealing to a Cantorian Absolute perspective, together with a mereological view of its classes as *parts*: thus we may distinguish the mathematico-set theoretic part of the realm, the sets, from the classes. As there are no 'super-parts' a hierarchy over and above the universe V does not threaten.

Keywords: Set Theory, Reflection Principles, Large Cardinals, Strong Axioms of Infinity.

Das Absolute kann nur anerkannt, aber nie erkannt, auch nicht annähernd erkannt werden.

Cantor, *Über unendliche lineare Punktmannigfaltigkeiten.* Math. Ann., 1883

To say that the universe of all sets is an unfinished totality does not mean objective undeterminateness, but merely a subjective inability to finish it.

Gödel, in (Wang: "A Logical Journey: From Gödel to Philosophy")

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1 Reflection Principles in Set Theory: introduction

Historically *reflection principles* in set theory have been associated with attempts to say that no one notion, idea, or statement can capture our whole view of the universe $V = \bigcup_{\alpha \in On} V_\alpha$; the motivating idea seemingly that the universe $(V, \in)$ is in some sense ineffable: all attempts to pin it down will fall short. In particular this will be so, once we have formalised our notions, for any formal linguistic expression.

Such reflection principles have usually been formulated in some language (first or higher order) as sentences φ (when interpreted in the appropriate way over V) that hold in $\langle V, \in, \ldots \rangle$, hold in some $\langle V_\beta, \in, \ldots \rangle$ - *sentential* reflection. Here we are initially allowing set objects to be substituted for variables of a purely first order formula to construct a sentence. For formulae of higher orders than one it then becomes a question as to how to reflect second order parameters, and indeed is a genuine problem at third and higher orders. In Section 3 we summarise the history of such principles. As is well known, such principles had been used to a validate the large cardinals known at that time, or at least the plausibility thereof. From a modern perspective such principles are disappointing as they never reveal any arguments for strong axioms that suggest anything beyond those consistent with Gödel's constructible universe L.

However, since Scott's Theorem (itself surprisingly late) that the existence of a measurable cardinal implies that V is not L, and the wide use of axioms implying the existence of much stronger cardinals than measurable, reflection principles have tended to be ignored, (at least by set theorists) as being part of the historical development of the subject and not much more.

This is not entirely true however. Perhaps the most well known attempt was that of Reinhardt (1974a) to give justifications of some ideas (that later were adopted to imply very strong axioms when once re-expressed in an appropriate "set form") using a notion of 'projecting V into idealised realms'.

In Section 4 We suggest a *Global Reflection Principle* to overcome the limitations that these principles are all *intra-constructible*. The concept of a *Woodin cardinal* has become central to the development of set theory in the last few decades. This is not just for the development of determinacy results, but more germane here, for Woodin's remarkable results on the absoluteness of various truths resisting any attempt by Cohen style forcing arguments to falsify them. Many of such theorems use as background assumptions the existence of a proper class of such Woodin cardinals, or even measurable Woodin cardinals. The Global Reflection Principle delivers just such hypotheses.

Thus: we must rise to the challenge to justify a set-theoretic reflection principle that will ensure the existence of large cardinals (or strong axioms of infinity) that are sufficient to deliver the hypotheses needed for modern set theoretical principles.

1.1 Argument

As we shall see that purely syntactic, or linguistic means only deliver intraconstructible principles, we need to go beyond them. We shall need to express the ineffability of V not purely in considerations about $(V,\in)$ alone, but in the general idea of a *set structure* which we view in any case, as extending Martin's concept of "set structure" (Martin, 2012). We thus outline our viewpoint for this section in terms of which we wish to argue for new principles. In summary:

> (I) (*cf.* Martin, 2012) We have no need to 'perceive' in a Gödelian sense, or otherwise 'locate' any mathematical objects in order to understand the concepts involved, and to communicate that understanding.
> Thus: *instantiation* of our concepts is not necessary, but what is needed is uniqueness of the concept up to isomorphism.
>
> (II) We then seek to extend arguments of Martin's along the above lines which he advocated for the 'concept of ω-sequence' and the 'concept of set of x's' to a 'concept of set with absolute infinities'.
>
> (III) We then formulate within this framework *reflection principles* that establish large cardinals beyond those consistent with Gödel's L.

2 Martin on set structures

2.1 The Reality of Concepts: Sets and Classes

Martin (2005) has questioned the level of realism that Gödel, although on occasion expressing this with talk of 'perception of mathematical objects' *etc.*, needs in order to make some of his arguments work, *eg.* , of the analyticity of mathematical truths.

Martin identifies two kinds of sense to 'concept of set'.

> (i) That more nearly akin to pure platonism - the whatever it is that falls under the extension of 'concept of set (of x's)' - that is sets (of x's), and
> (ii) a more general sense of 'concept of set' under which falls concepts of sets, or additionally that of 'set-structures'.

It is this final 'additional version' that I shall want to mostly take here, and go beyond Martin.

His primary point is perhaps plainly put: the example of Axiom of Extensionality: it does not say what a set is, on this it is silent, it only prescribes what it means for any two sets to be equal. The concept of set alone does not determine what it is for an object to be a set (as he states in Martin, 2012). The notion is objective: we understand it, talk about it, as no doubt they do on some other planet with discretely individuated intelligences. (It is not for nothing that we engrave on steel plates pictures of Pythagoras' theorem and place them on the moon, or send them out on Voyager 23.)

However first:

> A concept of set expressed by axioms such as comprehension axioms cannot put any constraint on which objects count as sets and which do not. Such axioms put constraints on the isomorphism type of a set theoretic structure ... a concept of set could count as concept of set in my [indirect *(ii)*] sense even if it determined completely what objects count as sets and what counts as the membership relation. A concept of this sort would have at most one instance: it would allow at most one structure to count as a set-theoretic universe ...

What is ultimately at play here is the point Martin wishes to make that *instantiation* of a concept for mathematics (or set theory) is not needed: what we require is *uniqueness* (up to isomorphism) in order to make sense and understand concepts. At the end of the quotation he says that it *might* have only one structure instantiating it, but this need not be the case. It might not have any instantiations. (Who seriously believes there is a well ordering in our own physical manifold that there is well ordering order type ω_{23}?) He reads Gödel as primarily not needing instantiation in many crucial places: for example, he notes that neither it nor perception of objects plays any significant role in Gödel's justifications of **strong axioms of infinity.** In short instantiation is not needed either in mathematics or in set theory. Hence: this is closer to an eliminative structuralist viewpoint.

Basic concepts and their properties

He (Martin, 2012) discusses in the following terms two basic concepts: the *concept of an ω-sequence* and the *concept of set*. He identifies three properties a basic concept may have:

1. First order completeness: the concept determines truth values for all first order statements.
2. Full determinateness: the concept fully determines what any instantiation would be like.
3. Categoricity.

An analysis of the notion of $\mathbb{N}$, as to whether the notion is *first-order complete, categorical, or perhaps fully determinate*. He claims that instantiation of the natural numbers is not needed for number theory: what we require is *categoricity*, although mathematicians probably think that $\mathbb{N}$ is in fact fully determinate. He identifies *In-*

formal Axioms schemes of PA with which one can, informally, prove the categoricity of $\mathbb{N}$.

Turning to set theory, we have only glimmerings of what goes on when considering subsets of $V_{\omega+1}$: is the Continuum Hypothesis true? Is every definable subset of the plane definably uniformisable? So we are hopelessly far from first order completeness. (However, when considering subsets of V_ω we are in a better position. We now know that adding the assumption of Projective Determinacy to analysis, or to the theory of hereditarily countable sets give us as complete a picture of HC as PA does for $V_\omega = \mathrm{HF}$. (See the discussion in Woodin, 2001.))

Martin identifies four components of the modern iterative concept of set.

(1) the concept of natural number
(2) concept of 'set of x's'
(3) concept of transfinite iteration
(4) concept of absolute infinity.

Corresponding to the informal axioms for Peano Arithmetic, we may then ask: which informal axioms are implied by the concept of set?

(i) *If a and b have the same members, then $a = b$. (Extensionality)*
(ii) *For any property P, there is a set whose members are those x's that have P (Informal Comprehension).*

The Comprehension scheme is called informal since "property" is not specified in generality. However any worries can be dispelled since it will be clear that the few instances we shall use are clear cases of properties.

Martin seeks to further soothe any worries that we need to specify what objects sets are in order to 'fully understand' the concept. He will ignore whatever structural constraints one may put on what sets actually are, other than the structural constraints of (i) and (ii).

Theorem 1. *(Essentially Zermelo) Informal Axioms (i) and (ii) are categorical: if $(\mathfrak{V}_1, \in_1)$, $(\mathfrak{V}_2, \in_2)$ are two structures satisfying (1) and (2) with the same x's, then with each set of x's $b \in_1 \mathfrak{V}_1$ we associate a set of x's $\pi(b) \in_2 \mathfrak{V}_2$.*

Proof: Let P be the property of being an x such that $x \in_1 b$. By the Informal Comprehension Scheme there is a $c \in_2 \mathfrak{V}_2$ such that

$$\forall x[x \in_2 c \leftrightarrow P(x)]$$

Q.E.D.

This is the basis of Zermelo's proof that any two models of ZFC (without *urelemente)* of the same ordinal height are isomorphic. Similarly for any two $\mathfrak{V}_1 = (V_1, \in_1)$, $\mathfrak{V}_2 = (V_2, \in_2)$ obtained by iterating the V_α function throughout all the absolute infinity of ordinals, we have an isomorphism

$$\pi : (V_1, \in_1) \cong (V_2, \in_2).$$

Note for later that we see that $\pi \restriction \mathrm{On}^{\mathfrak{V}_1} : \mathrm{On}^{\mathfrak{V}_1} \cong \mathrm{On}^{\mathfrak{V}_2}$.

2.2 From set structures to set structures with classes

We shall want to extend the application of Martin's analysis to a broader territory - no doubt broader than he might like. The components of the modern iterative concept of set (1)-(4) above lead, in our view inexorably, to classes associated to absolute infinities in the sense that Cantor recognised and spoke of. We continue our informal development of the theory of sets, just as any mathematician would develop her theory of some mathematical construct. The absolute infinity of *On* we regard as just that: it is determinate what constitutes an ordinal (*à la* von Neumann: a transitive set well ordered by $\in$), and *On* is the class of such, and is an absolute infinity as Cantor realised, and Burali-Forti showed (albeit half unknowingly it seems, see van Heijenoort, 1967). The notion is not regarded as indefinitely extensible: if we posit beyond *ZFC* stronger axioms of infinity, such as measurable cardinals, we enrich our view of V, but we do not somehow 'add more ordinals'.

V itself is, then, also one of many 'absolutely infinite' or 'inconsistent multiplicities' $V, \mathrm{On}, \mathrm{Card}, \ldots$, the class of all singletons, $\ldots$ that Cantor recognised. Let us imagine that $\mathscr{C}$ is the collection of all such. The nature of $\mathscr{C}$ is admittedly somewhat ineffable, but we can agree with him that as absolute and final knowledge cannot be obtained by V alone, neither can we find such for $\langle V, \in, \mathscr{C} \rangle$. Just as Martin used the idea of 'property' in Informal Comprehension, without specifying exactly what that constituted, as he would only need to use this in a small number of uncontroversial cases, so we shall think about $\mathscr{C}$, and shall see that we only need to consider a small part of $\mathscr{C}$. Later we shall consider formalizations of this.

We may, if we wish, tell a *mereological* story about the whole mathematical universe: we think of sets as the sole representatives of *mathematical* objects. The class of sets thus forms the realm of, or the arena for, mathematical discourse. Thus strictly, the class of sets is not a mathematical object. However absolute infinities are *parts* of the whole realm of that discourse. (One may wish to add to this sets as well as parts of the realm, but we shall just identify such parts with the sets themselves.) We take $\mathscr{C}$ to be the collection of all absolute infinities (or non-set parts, or proper classes if you will). $\mathscr{C}$ then, contains the possible parts of V.

The usual danger is that we may risk having to build super-classes, and more such above those; in other words a hierarchy that looks too similar to the V_α hierarchy. However this is to extrapolate the use of *mathematical* methods of set formation - principally power set and recursion along the ordinals. The power set axiom is a mathematical operation and not a warrant for a power class operation, even if we could form a definite domain of quantification. In our pre-formalised thinking, we bite the Cantorian bullet and admit two types of objects to our notion of set-structure: V together with its absolute infinities.

In Welch and Horstein, 2016 we discuss further the status of such classes. In particular the possible use of plural quantification as a means of establishing the 'existence' of such classes. However we reject that view, principally for reasons to come: we need actual entities of some kind in $\mathscr{C}$ not just something that is ultimately a linguistic construct: "some sets such that ..."

2.3 Isomorphism again

Theorem 2. *If we have two structures of sets $\mathfrak{V}_i = (V_i, \in_i)$ $(i = 1,2)$ satisfying Martin's (i) and (ii) above, let $\pi : (V_1, \in_1) \to (V_2, \in_2)$ as above be an isomorphism. Now assume we have collections of classes $\mathscr{C}_i$. Then π extends to an isomorphism:*

$$\pi : (V_1, \in_1, \mathscr{C}_1) \cong (V_2, \in_2, \mathscr{C}_2).$$

Proof: Let $(V_1)_\alpha$ be the set of $\mathfrak{V}_1$-sets of rank α in the sense of $\mathfrak{V}_1$ *etc.* It suffices to show for every class $X \subseteq V_1$ (thus $X \in \mathscr{C}_1$) there is a $Y \subseteq V_2$ with $\pi(X \cap (V_1)_\alpha) = Y \cap (V_2)_\beta$ where $\alpha \in_1 \mathrm{On}^{\mathfrak{V}_1}$ and $\beta \in_2 \mathrm{On}^{\mathfrak{V}_2}$ with $\pi(\alpha) = \beta$, and conversely - since then we may define $\pi(X) = \bigcup_{\alpha \in_1 \mathrm{On}^{\mathfrak{V}_1}} \pi(X \cap (V_1)_\alpha)$.

Q.E.D.

Here we are taking the 'informal union' of the sets of the form $\pi(X \cap (V_1)_\alpha)$. However we are not declaring this union to be a 'set' or any such, so no formal axiom is needed. This is unproblematic as it is simply taking a union (or fusion if you will) of the classes $\pi(X \cap (V_1)_\alpha)$ and thus is a class of V_2.

3 Reflection

We now very briefly survey some of the history of reflection principles, before discussing some more recent work.

3.1 Historical Reflection

First order reflection is unproblematic: any one syntactic instant of it for a formula in $\mathscr{L}_{\dot\in}$, the language of first order set theory alone, is provable in ZF:

(R_0) : *For any* $\varphi(v_0, \ldots, v_n) \in \mathscr{L}_{\dot\in}$

$$ZF \vdash \forall\alpha\exists\beta > \alpha\forall\vec{x} \in V_\beta[\varphi(\vec{x}) \leftrightarrow \varphi(\vec{x})^{V_\beta}].$$

Indeed by formalising a Σ_n-Satisfaction predicate we have (still in ZF, here "c.u.b." abbreviates "closed and unbounded"):

For each n there is a term c_n for a c.u.b. class of ordinals, so that so that for any $\varphi \in \mathrm{Fml}_{\Sigma_n}$:

$$\mathrm{ZF} \vdash \forall \beta \in c_n \forall \vec{x} \in V_\beta [\varphi(\vec{x}) \leftrightarrow \varphi(\vec{x})^{V_\beta}].$$

Informally we write for each n: $\forall \beta \in c_n : (V_\beta, \in) \prec_{\Sigma_n} (V, \in)$. More formally we can define first order satisfaction for Σ_n formulae as a two place predicate $\mathrm{Sat}_n(\ulcorner \varphi \urcorner, x)$ and have that for $\beta \in c_{n+1}$ that

$$\forall \ulcorner \varphi \urcorner \in \Sigma_n, \forall x \in V_\beta (V_\beta, \in) \models \mathrm{Sat}_n(\ulcorner \varphi \urcorner, x) \leftrightarrow (V, \in) \models \mathrm{Sat}_n(\ulcorner \varphi \urcorner, x)).$$

We thus have reflection of $(V, \in, \mathrm{Sat}_n)$ to $(V_\beta, \in, \mathrm{Sat}_n)$ for such β. As is well known, we can not do this for all n simultaneously.

Reaching back to ideas of P. Mahlo, using *normal functions* and their fixed points we have:

(F) *Any (definable) normal function* $F : \mathrm{On} \to \mathrm{On}$ *has a regular fixed point.*

Such fixed points are inaccessible cardinals (and indeed strong limits of limits of such). A reflection principle is related to this. We let Inacc denote the class of strongly inaccessible cardinals.

(R_1) : For any $\varphi(v_0, \ldots, v_n) \in \mathscr{L}_{\dot\in}$

$$\forall \alpha \exists \beta > \alpha (\beta \in \mathrm{Inacc} \wedge \forall \vec{x} \in V_\beta [\varphi(\vec{x}) \leftrightarrow \varphi(\vec{x})^{V_\beta}]).$$

Then it easy to see that $(F) \Longleftrightarrow (R_1)$. One then may ask for (F) itself to reflect down from normal functions definable over V to *all* normal functions $f : \kappa \longrightarrow \kappa$ for $\kappa \in \mathrm{Inacc}$. Then, if $(V_\kappa, \in)$ witnesses (F), in this extended sense for all such functions f, κ is called a *Mahlo* cardinal. Being Mahlo is a second order property; Bernays investigated such in more generality, and one may then end up with *indescribability properties* that in general a Π^1_m sentence about second order parameters may reflect down to V_κ. Here φ is a first order statement about the illustrated upper case second order variables F_i and parameter P, and possible further parameters $\vec{x} \in V_\kappa$.

$$\forall F_1 \exists F_2 \cdots Q F_m (\varphi(\vec{F}, \vec{x}, P)) \leftrightarrow (\forall F_1 \exists F_2 \cdots Q F_m (\varphi(\vec{F}, \vec{x}, P \cap V_\kappa))^{V_\kappa}.$$

One observation at this point is that first order satisfaction is second order, indeed Π^1_1-definable, consequently if we have Π^1_1-reflection in the above sense, we may reflect $(V, \in, \mathrm{Sat})$ down to arbitrarily large $(V_\beta, \in, \mathrm{Sat})$ (where Sat codes all of first order satisfaction simultaneously). Reflection of first order satisfaction is not strong.

The localisation of this idea to a cardinal λ in place of V itself (just as a Mahlo cardinal is the localised version of a concept considering class functions over V) then yields the notion of a Π^1_m-*indescribable* cardinal. We thus arrive at indescribable cardinals, indeed not just second order, but of any n'th order: Π^{n-1}_m-*indescribable cardinals.*

This can be extended to transfinite orders. There are difficulties with 3rd order parameters as Reinhardt observed. This again is tracing the lines of the following thought of Gödel:

> *The Universe of sets cannot be uniquely characterized (i.e. distinguished from all of its initial segments) by any internal structural property of the membership relation in it, which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal number.*
>
> (Wang - 1996)

In a different direction Reinhardt studied Ackermann's set theory, Ackermann, 1956 and his thesis Reinhardt, 1967 (published in Reinhardt, 1970) established its conservativity over ZF.

From Ackermann, 1956 (p.337): "We must require from already defined sets that they are determined and well-differentiated, thus the conditions of a collection [to be a set] only turn on that it must be sufficiently sharply delimited what belongs to a collection and what does not belong to it. However now the concept of set is thoroughly open." For Ackermann a well-defined and differentiated condition for an object to be a set cannot include in its definition assertions of the form "such and such a set is in V" since the latter is not a sufficiently sharply determined concept. Thus the definition of a set x cannot include clauses of the form '$y \in V$' although the latter is a *bona fide* expression of the formal language.

Reinhardt interpreted this as realising in some form ideas about the *projection* of the universe V into some virtual realm. The language for this theory includes a symbol V to denote the 'constructed' sets, but objects such as $\{V\}, \{\{V\}\}$ *etc.* are permissible (and such are classes for Ackermann). The Cantorian ordinals form Ω and V is V_Ω, but we imagine further set-like objects. We posit ZF for V and the axiom $V = V_\Omega$. The following $(S2)$ schema asserts that any first order sentence θ true in V of sets in V, is true in the 'projected universe' and conversely.

$$(\forall x, y \in V)(\theta^V(x,y) \leftrightarrow \theta(x,y)) \qquad (S2)$$

This system was called "ZA". The extension of a formula θ in V is thus $\{x \in V \mid \theta(x)\}$ and is merely the restriction of that in the projected or virtual universe where it is of course $\{x \mid \theta(x)\}$.

Reinhardt (1974b) also studied and formalised a view of Shoenfield (1967) that thought of V as the actually existing sets, but that there is some 'formal extension' of the ordinals, and a class of 'imaginable sets' U. This came about by studying a thesis of Shoenfield that had shades of Ackermann set theory (Reinhardt, 1974b p.6):

If C is a collection of stages [ranks], and if we can imagine *a situation in which all stages in P have been completed, then there is to be a stage s after all the stages in C.*

After some further considerations Reinhardt formalises this as having "imaginable sets" and at the same time real existing sets, with the latter elements of V. (Ackermann would have called the imagined sets 'classes' and Shoenfield's V is Ackermann's class of all sets.) V cannot be a real existing set, and the property P such that $x \in V \leftrightarrow P(x)$ cannot be said to exist either. However as a consequence one has the slogan:

If $X \subseteq Y$ and Y is imaginable, and suppose X is definable using only $\vec{x} \in V$ as parameters then $X \in V$.
Reinhardt in Reinhardt, 1974a argued for a formally projectible realm V_λ for some '$\lambda >> \mathrm{On}$' with a corresponding $V_{\lambda'}$ related to V_λ *via* an embedding j:

$$j : V_\lambda \longrightarrow V_{\lambda'} \; ; \; j(\mathrm{On}) = \mathrm{On}', \mathrm{crit}(j) = \mathrm{On}\,.$$

We finish this subsection with just one more quote of Gödel that is often referred to when discussing reflection in the set/class theoretic setting, and can be seen as motivating reflection properties in general.

> *All the principles for setting up the axioms of set theory should be reducible to Ackermann's principle: The Absolute is unknowable. The strength of this principle increases as we get stronger and stronger systems of set theory. The other principles are only heuristic principles. Hence, the central principle is the reflection principle, which presumably will be understood better as our experience increases. Meanwhile, it helps to separate out more specific principles which either give some additional information or are not yet seen clearly to be derivable from the reflection principle as we understand it now."* (Wang - 1996).

3.2 Strengthening Reflection Principles

Tait considered in Tait, 2005 some strengthened reflection principles that were of order higher than two. As already alluded, Reinhardt had observed that third order with third order parameters would be false. Tait considered how to use the higher order languages with some syntactical restrictions. Koellner showed (Koellner, 2009) that those that were consistent were all weaker than $\kappa(\omega)$ (an ω-*Erdős* cardinal). The latter cardinal is of only modest size, and is also consistent with V being L.

We are thus left with the thesis that all *syntactically based* Reflection Principles to date are all consistent with a view of the universe as being L the constructible one: they are *intra-constructible*.

The moral that may be adduced is that if we need stronger reflection principles: those that generalise Montague-Levy in terms of enlarging our set/class-theoretic language are not up to the task of providing any justification for the large cardinals needed for modern set theory.

Why ask for stronger reflection principles?

Theorem 3 (Woodin)**.** *Suppose there is a proper class of Woodin cardinals. Then* $Th(L(\mathbb{R}))$ *is immune to change by set forcing.*

This supposition is now ubiquitous in modern set theory. The above is just singled out as a dramatic instance of how large cardinals affect the universe V. One might have cited the equally dramatic results of Martin-Steel that infinitely many Woodin cardinals implies determinacy for projectively defined sets of reals. If there is a proper class of Woodin cardinals, then again Cohen's set forcing techniques cannot change that fact. Projective determinacy implies a host of results about the projective sets: their Lebesgue measurability, their having the *perfect subset property*, implying that every uncountable such set has size the full continuum; that projectively defined subsets of the plane $\mathbb{R} \times \mathbb{R}$ can be *uniformized* by a function with a projective graph; the non-existence of a Banach-Tarskian paradoxical decomposition of the sphere into projective pieces. The list can be extended, but the point is made: large cardinals at the level of Woodin cardinals *prove* such results to hold. These are not relative consistency proofs of one theory relative to another given by the forcing method, they are facts about the universe of mathematical discourse, V, which must hold if those requisite large cardinals exist.

In order to obtain such large cardinals, we need therefore a means of thinking about, or justifying, or obtaining *extra-constructible* reflection principles. We therefore define a *Global Reflection Principle* (GRP).

4 III Global Reflection Principles

Instead of 'formally projecting V' *à la Reinhardt* let us turn it around and generalise to obtain a strong reflection principle. Let us take a naive Cantorian (and non-Zermelian) stance: we bite the bullet of the necessity of the seeming existence of two types of objects: the *sets* of mathematical discourse and the *absolute infinities* (or *proper classes*) that Cantor, Burali-Forti, Russell and others early on saw the need for. We let $\mathscr{C}$ denote the informal collection of such absolute infinities, without worrying about a formal definition delineating these objects. We shall see that we shall only be concerned with principles that require $\mathscr{C}$ to contain a small number of such proper classes, where a 'small number' only means set-many.

We then consider the constellation $(V, \in, \mathscr{C})$ of the universe of sets V obtained in the usual mathematical fashion of iterating the power set operation as above, and writing membership $\in$ in the usual sense: $x \in y$ and $x, y \in Y$ for a Y in $\mathscr{C}$. As this indicates here, we shall be considering the usual first order language for set theory $\mathscr{L}$, but augmented by second order variables $X, Y, Z, \ldots, X_n, \ldots$ which appear atomically as statements of the "$x \in Y$", "$x \notin y$". Equality between classes X, Y is given extensionally: $X = Y \leftrightarrow \forall z(z \in X \leftrightarrow z \in Y)$ *etc.* However there are no second order quantifiers. (In any case, we have left the domain $\mathscr{C}$ of any such putative quantification vague.) But we shall allow *bounded quantifiers* as $\exists x \in Y(\cdots)$, $\forall x \in Y(\cdots)$ into our Δ_0 classified formulae of our extended language $\mathscr{L}^+$.

Just as earlier reflection principles expressed the ineffability of V by asserting principles that reflect the whole realm to a small part of itself, we shall do the same, whilst reflecting at the same time some satisfaction. One may note that the truth or otherwise of $\varphi(x,X)^{(V,\in,\mathscr{C})}$ for an $X \in \mathscr{C}$ and $\Delta_0\, \varphi$ is only dependent on a limited number of other classes in $\mathscr{C}$, depending on a few other class parameters derived from sub-formulae, and rudimentary combinations of such. The pathway is thus open to consider reflection from the whole universe of $(V,\in,\mathscr{C})$ with its classes to some smaller set-sized domain. We shall want to reflect ordinary first order expression of $\mathscr{L}$ about objects x which are members of the domain, but also some collection of, or possibly all, statements from $\mathscr{L}^+$ involving classes. Clearly 'class many' statements are not going to be reflectable.

The domain, the "V" of this small realm, should be set sized, and indeed we should speak of a small number of "classes" for it, as possible interpretations for the variables $X_i \ldots$

So motivated by earlier reflection principles, we choose the small realm to be a 'typical normal domain' V_κ, and we seek some form of satisfaction preserving reflection of $(V,\in,\mathscr{C})$ together with some collection of its classes, to some $(V_\kappa,\in,D)$ with $D \subseteq V_{\kappa+1}$, which we regard as a collection of classes D over V_κ. We may obtain such smaller structures by considering *set-sized substructures* of $(V,\in,\mathscr{C})$, $(V',\in,\mathscr{C}') \prec (V,\in,\mathscr{C})$ with V' equal to some V_κ and consider its isomorphic transitive image $(V',\in,\mathscr{C}') \cong (V_\kappa,\in,D)$ say. We shall want that $D = V_{\kappa+1}$ but first we discuss the kind of elementarity we are pursuing.

A class $X \subset V_\kappa$, with X from D, is the reflection of something satisfied in the realm $(V,\in,\mathscr{C})$ by one of the parts, or classes, of the whole realm, some $\tilde{X}$ say. What should $\tilde{X}$ be? If even truth of atomic statements of the form "$y \in X$" are to be faithfully reflected then it should be a part of the whole (V) that extends X, *i.e.* $\tilde{X} \cap V_\kappa$ should be the same as X. This then is no different from the reflection of second order parameters in the earlier principles. In order then to have simultaneous preservation of Δ_0 formulae of our language between $(V_\kappa,\in,D)$ and $(V,\in,\mathscr{C})$, we only need that $\mathscr{C}$ contain extensions $\tilde{X}$ for every $X \in D$. We are not yet requiring it (although we shall do shortly), but to have *full* or *'global'* downwards reflection, we ask that $D = V_{\kappa+1}$. The implicit posit here is that there is some choice of some κ to fix a domain, and, for every X in the given D, there is some choice of $\tilde{X} \in \mathscr{C}$ with $\tilde{X} \cap V_\kappa = X$. If these choices can be made then we have a Δ_0-reflection property between $(V,\in,\mathscr{C})$ down to $(V_\kappa,\in,D)$. (Indeed making such choices is just another way of stating that we shall take a substructure $(V',\in,\mathscr{C}') \prec (V,\in,\mathscr{C})$ as above.) Some elementary observations can be made: for example, as $On \in \mathscr{C}$ we'd better have $On \cap \kappa = \kappa$ in D, and thus $\bar{X} = On$ has to be our choice of extension for $X = (On \cap \kappa)$ as $\tilde{\kappa}$, if we are to have Δ_0

reflection.

Having sorted out this version of reflection for Δ_0-Satisfaction we may naturally require that our choices of $\tilde{X} \in \mathscr{C}$ with $\tilde{X} \cap V_\kappa = X$, for $X \in D$, can further sustain reflection of satisfaction for the Σ_1 formulae, the Σ_n formulae, or indeed the whole language $\mathscr{L}^+$. These desiderata are fulfilled in the following definition.

Definition 4 (GRP). *The* Global Reflection Property *holds if the universe* $(V, \in, C)$ *admits of a set-sized* $\mathscr{L}^+$*-elementary substructure,* $(V', \in, \mathscr{C}') \prec (V, \in, \mathscr{C})$ *with* $V' = V_\kappa$ *for some* κ *and which is isomorphic to a structure of exactly the same kind, namely* $(V_\kappa, \in, V_{\kappa+1})$.

As discussed, by 'elementary' we mean first order elementary in the usual language of set theory $\mathscr{L}_\in$ but we allow the addition of second order predicate symbols $P_0, P_1, \ldots,$ $P_n, \ldots$.

4.1 Reflection Increased: a spectrum

We step back somewhat and consider properties in a spectrum of increasing strength leading up to GRP.

Definition 5. *[Partial Global Reflection] The* Partial Global Reflection Property *holds if the universe* $(V, \in, \mathscr{C})$ *admits of a set-sized* $\mathscr{L}^+$*-elementary substructure,* $(V', \in, \mathscr{C}') \prec (V, \in, \mathscr{C})$ *with* $V' = V_\kappa$ *for some* κ *and* $(V', \in, \mathscr{C}') \cong (V_\kappa, \in, D)$ *with* $D \subseteq V_{\kappa+1}$.

We can immediately rewrite this in a way familiar to set theorists[1].

Definition 6. *Let* $(V', \in, \mathscr{C}'), D, V_\kappa$ *satisfy the last definition. We define* j *to be the inverse of the isomorphism* $(V', \in, \mathscr{C}') \cong (V_\kappa, \in, D)$ *and write:*

$$j : (V_\kappa, \in, D) \longrightarrow_{\mathscr{L}^+} (V, \in, \mathscr{C}).$$

Then, to be specific, note that $j \upharpoonright V_\kappa$ is simply the identity function: $j(x) = x$ for any $x \in V_\kappa$.

[1] Readers of Welch and Horstein, 2016 will note that the primary definition of Global Reflection (Axiom 3) there is based on Def. 6 (with $D = V_{\kappa+1}$). Conversations with Sam Roberts incline us to the view, that, although making no difference mathematically, for presentational purposes it is perhaps nicer to define the *range* of j first (as in Def. 5) and then j^{-1} as its transitivisation collapse map. We warmly thank him for these discussions. He has defined (Roberts, preprint, 2016) wide ranging principles extending *GRP* by defining certain particular substructures of V with its classes, and certain satisfaction predicates.

To spell it out further: j must preserve as follows for φ in $\mathscr{L}^+$, $x \in V_\kappa$, $X \subseteq V_\kappa, X \in D$:

$$\varphi(x,X)^{(V_\kappa,\in,D)} \leftrightarrow \varphi(j(x),j(X))^{(V,\in,\mathscr{C})}$$

but $j(x) = x$ so $$\leftrightarrow \varphi(x,j(X))^{(V,\in,\mathscr{C})}.$$

Note that we shall have $(V_\kappa,\in) \prec_{\mathscr{L}} (V,\in)$, but of course the latter is not a strong property.

The strength of these reflection principles is entirely signified by the richness of D. We may impose closure conditions on D. The intra-constructible reflection principles such as those from indescribability, could be obtained from principles such as these, that is when $D \subsetneq \mathscr{P}(\kappa)^L$.

If $D \supseteq \mathscr{P}(\kappa)^L$ then this will deliver to us $0^\sharp$: define U on $\mathscr{P}(\kappa) \cap L$ by

$$X \in U \leftrightarrow X \in L \wedge \kappa \in j(X).$$

By the Σ_1 reflection property of j, standard arguments show that this is a normal L-ultrafilter and we may form $\tilde{j} : (L,\in) \to_e Ult(L,U) \cong (L,\in)$ with $\tilde{j}(\kappa) > \kappa$. The existence of such an embedding from L to L contradicts $V = L$ and is often referred to as the 'least' large cardinal property, or strong axiom of infinity. We thus should have our first extra-constructible principle.

Similarly for other definable inner models, M say in place of L, then requiring $D \supseteq \mathscr{P}(\kappa)^M$ will reveal that there is a $j : M \to_e M$, and so forth.

The natural limit of the above is when we require $D = V_{\kappa+1}$. There is a step difference here: $(V_\kappa,\in,V_{\kappa+1})$ we consider as a normal domain with *all* of its parts, and we assert the property that anything satisfied in it by a set $x \in V_\kappa$ and a class $X \subset V_\kappa$, is the reflection of something satisfied in the realm $(V,\in,\mathscr{C})$ by x and by one of the parts of V, $\tilde{X} \in \mathscr{C}$ with $\tilde{X} \cap V_\kappa = X$. We thus have that $(V_\kappa,\in,V_{\kappa+1})$ is a simulacrum of $(V,\in,\mathscr{C})$. The j above then maps as follows:

$$j : (V_\kappa,\in,V_{\kappa+1}) \longrightarrow_e (V,\in,\mathscr{C})$$

with j fully $\mathscr{L}^+$-elementary.

If we assume the axiom of choice, AC, then $(V,\in,\mathscr{C})$ will be a model of Global Choice (since if $R \subseteq V_\kappa^2$ is a well ordering of V_κ then $j(R)$ is a wellorder of V). (It would be entirely within keeping of other reflection principles if we were to ask that there be unboundedly many such κ in On for which there is such a j, but we don't pursue that here.) Just as above for L, define a field of classes U on $\mathscr{P}(\kappa)$ by

$$X \in U \leftrightarrow \kappa \in j(X)$$

As $\mathscr{P}(\kappa) \subseteq V_{\kappa+1} \subseteq \mathrm{dom}(j)$ by Σ_1-elementarity (in j), this is an ultrafilter on $\mathscr{P}(\kappa)$. Standard arguments show that U is a normal measure on κ, and thus κ is a measurable cardinal. This again shows that $V \neq L$ (by an old theorem of Scott, 1961).

But then:

$$\begin{aligned} \forall\alpha < \kappa \langle V, \in\rangle &\models \text{“}\exists\kappa > \alpha(\kappa \textit{ a measurable cardinal})\text{”} && \Longrightarrow \\ \langle V_\kappa, \in\rangle &\models \text{“}\forall\alpha\exists\lambda > \alpha(\lambda \textit{ a measurable cardinal})\text{”} && \Longrightarrow \\ \langle V, \in\rangle &\models \text{“\textit{There are unboundedly many measurable cardinals}”.} \end{aligned}$$

Known methods improve this to:

Theorem 7 (GRP). *$(V, \in) \models \forall\alpha\exists\lambda > \alpha(\lambda$ a measurable Woodin cardinal).*

Consequently by results of Martin-Steel, Woodin GRP this implies:

a) Projective Determinacy (PD) and $(AD)^{L(\mathbb{R})}$.

b) (Woodin) $\mathrm{Th}(L(\mathbb{R}))$ is fixed: no set forcing notion can change $\mathrm{Th}(L(\mathbb{R}))$, and in particular the truth value of any sentence about reals in the language of analysis, thereby including PD.

One may extend the above and reason that there is a proper class of Shelah cardinals. Although it is easy to see that

Lemma 8. *If $Con(ZFC + \exists\lambda(\lambda$ a 1-extendible cardinal $))$ then $Con(NBG + GRP)$*

originally GRP was thought of as obtained from a weakening of the notion of another downward reflecting cardinal, a *sub-compact cardinal*:

Definition 9. *μ is a* subcompact cardinal *iff*
$\forall A \subseteq H_{\mu^+} \exists\kappa < \mu\exists B \subseteq H_{\kappa^+}, \exists j \neq id, j \restriction H_\kappa = id$, so that:

$$j : (H_{\kappa^+}, \in, B) \longrightarrow_e (H_{\mu^+}, \in, A).$$

This is a third order principle (over H_μ). By dropping the A, B component, we arrive at the notion of $\mu = On$ being subcompact, or in other words, something like (GRP).

5 Philosophical Reflections

In Welch and Horstein, 2016 we discuss the possible interpretations in the light of current philosophical positions regarding classes. We regard mathematics as taking place within $(V, \in)$ - using sets as representations of mathematical conceptual objects. (‘Large’ categories indeed need some special consideration as definable classes. However we might point out that for any *theorem* that depends on, or builds on such large categories we may invoke the Montague-Levy Reflection theorem and bring the result down to one within ZF. (Even in one's wildest moments, no set theorist seriously thought that Wiles's theorem really required a proper class of Grothendieck universes.)

It is the author's view that fully second (or higher order) methods should be eschewed as these require a domain of quantification: the class of all classes at the very least. We do not wish to quantify over the collection $\mathscr{C}$ in any universal fashion (as we are unconvinced of the coherence of such an act); although we do wish to say that certain elements of $\mathscr{C}$ stand in certain relations to each other (and to a set-sized domain of sets). We just want the ability to say that such and such is a sufficiently rich substructure of $(V, \in, \mathscr{C})$. The richness is expressed in the form of elementarity in $\mathscr{L}^+$. Thus we do not wish to collect the classes together to form a new level V_{On+1}, and then regard them simply as some sets we forgot to add on to the top of V. We do not wish to form $\{V\}$.

One might think of the members of the class $\mathscr{C}$ as *pluralities.* The plural reading may allow us to talk of the such-and-such things without committing ourselves to the existence of entities. We rejected that view as it ties the notion too closely to linguistic concerns. We wish to have entities to go with V to be able to assert a substructure of V together with (some of) those entities in $\mathscr{C}$. Those entities must be in the range of our j.

We can however use the language of mereology to talk about classes as the *parts* of V. This is similar to David Lewis's interpretation of sets (Lewis, 1991). For Lewis the parts of sets are obtained from the singleton function and the process of mereological fusion. Here classes have subclasses as their parts. A set-sized part of a class we identify with the corresponding set. Indeed an instance of Comprehension can be interpreted as declaring that part of set through a defining clause, is a set. Unlike Lewis, we take the sets and so V as given, and use the part/whole relation to discuss the absolute infinities and V's parts. An absolute infinity then becomes the fusion of all of its parts.

There are thus either the set-sized parts which we identify with the sets themselves, or the parts that are proper-class sized. (As noted GRP implies Global Choice and so for any latter part there is a bijection with a proper class of ordinals.)

The reflection principle embodied in GRP is one involving quantification over sets, our mathematical objects (although classes are allowed unquantified). It is possible to have a strengthened GRP requiring the set-sized substructure of $(V', \in, \mathscr{C}') \prec (V, \in, \mathscr{C})$ to be Σ^1_∞-elementary in the full second order language $\mathscr{L}^2_\in$. This would indeed be taking quantification over classes and stepping out into higher orders. In this case after taking an isomorphism to a set structure $(V_\kappa, \in, V_{\kappa+1})$ we shall have: *There is* $\kappa \in \mathrm{On}$, *there is* $j \neq \mathrm{id}$, $\mathrm{crit}(j) = \kappa$ *with*

$$j : (V_\kappa, \in, V_{\kappa+1}) \longrightarrow_{\Sigma^1_\infty} (V, \in, \mathscr{C}).$$

This we may call *mereological reflection*: we fully reflect on the parts of V by allowing now quantification over parts. By sticking to "mathematical reflection" we have a principle consistent relative to *ZFC* with a 1-extendible cardinal.

We can go further and using Global Choice to define Σ^1_n-Skolem Functions, define full second order satisfaction, *Sat*. We might in turn add that to our language, call it $\mathscr{L}^2_{Sat}$ and get a yet stronger principle, GRP^+ say, with

$$(V', \in, \mathscr{C}', Sat') \prec (V, \in, \mathscr{C}, Sat)$$

now required to be an $\mathscr{L}^2_{Sat}$-elementary substructure.

Just as a sample of what one could obtain then, we see that V can be filtrated as an *On*-length system of embeddings of GRP type:

Proposition 10 (GRP^+)**.** *There is a commuting system* $\langle \kappa_\alpha, j_{\alpha\beta} \rangle_{\alpha \leq \beta \in \mathrm{On}}$ *of embeddings* $j_{\alpha\beta} : (V_{\kappa_\alpha}, \in, V_{\kappa_\alpha+1}) \longrightarrow_{\Sigma^0_1} (V_{\kappa_\beta}, \in, V_{\kappa_{\beta+1}})$ *with, each* $j_{\alpha\beta}$*,* $\alpha < \beta$*, witnessing the simple* GRP *at the 'universe'* V_{κ_β} *with its parts* $V_{\kappa_{\beta+1}}$*. Thus each* $j_\alpha \restriction \kappa_\alpha = \mathrm{id} \restriction \kappa_\alpha$*, and* $j(\kappa_\alpha) = \kappa_\beta$*. Moreover for* $\alpha \in \mathrm{On}$*, there are maps*

$$j_\alpha : (V_{\kappa_\alpha}, \in, V_{\kappa_\alpha+1}) \longrightarrow_{\Sigma^0_1} (V, \in, \mathscr{C})$$

also witnessing GRP *in the universe.*

It is not hard to derive the consistency of such a system from a subcompact cardinal (Welch, 2016).

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