Transfinite dynamical models

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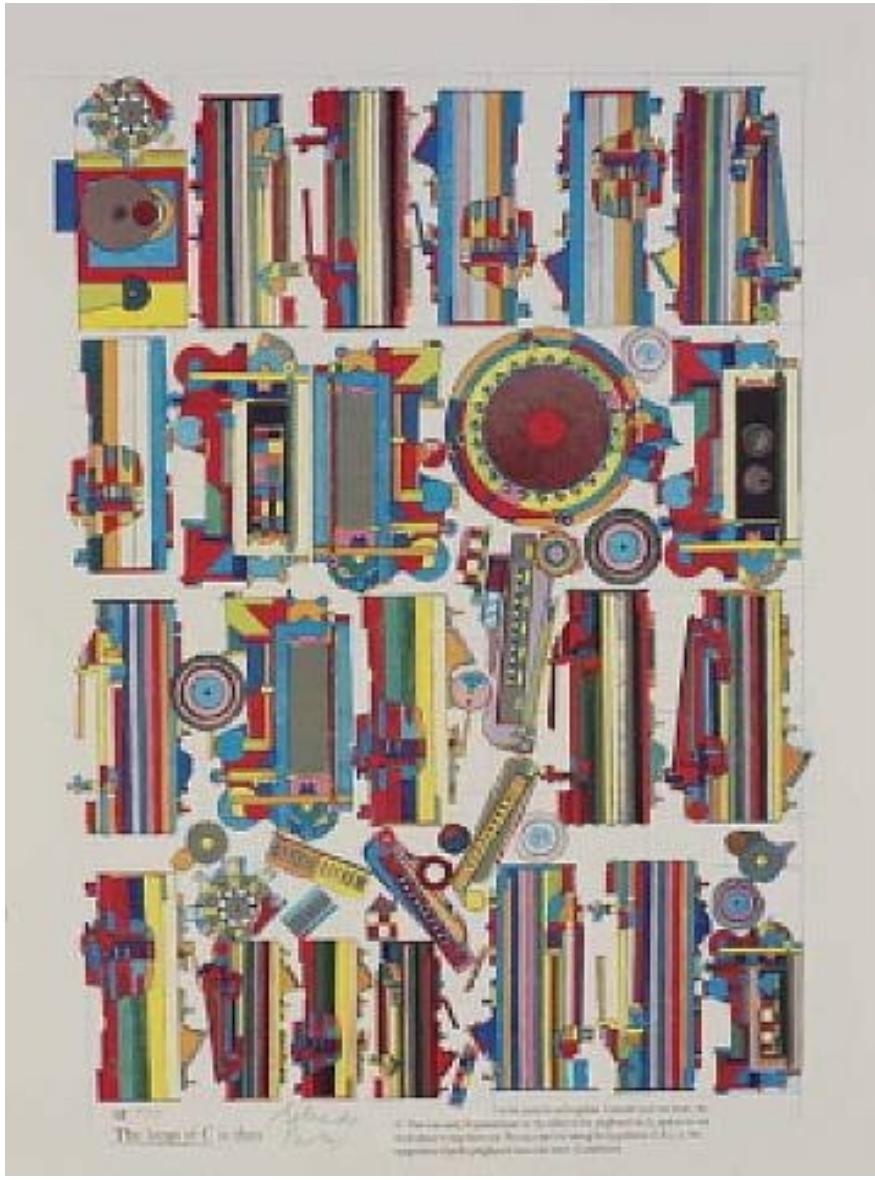

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- HM (*H*ypermachines, the class of all Σ_n-HM's) Friedman-Welch

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- Rudimentary and Primitive Recursion on sets — Gandy, Jensen, Karp
- Set Recursion — Normann, Sacks
- Safe Set Recursion — Beckmann, Buss, Friedman

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- Ideally the 'semi-decidable' sets of integers (and of reals) should form a Spector class.
- When not, then some other proof theoretic feature, or independent representation may be at hand.

Kleene or "Δ^1_1"-recursion

- Rogers called this the "$\aleph_0$-mind": one has machine with a countable tape and work area, and ability to inspect, and manipulate the tape doing countably many operations in a finite time; the oracle version has the ability to query of a set of reals A whether a real that has been computed so far is, or is not, in A.

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• Thus, for computations on integers, the natural "companion structure" here is $L_{\omega_1^{ck}}$ whose reals are precisely the Kleene-decidable sets of integers, the *hyperarithmetic* sets.

• Π^1_1 is the original, first Spector class.

ITTM's[1]

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$$\exists\Sigma(\zeta < \Sigma \wedge L_\zeta \prec_{\Sigma_2} L_\Sigma) \quad - \quad (\zeta \text{ is "}\Sigma_2\text{-extendible"}).$$

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• Indeed this pointclass, Γ_2, is normed and enjoys Uniformisation and Scale properties.

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The Σ_n-*hypermachines*[2] successfully lift all the above from Σ_2 to Σ_n:

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• Thus every real in the least model of second order number theory is HM-computable.

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Conjecture $\supset \Sigma^0_3$ sets of integers are FITTM-computable.

ITRM's

• (Koepke-Miller) We take a standard N-register machine with entries $m \in \mathbb{N}$ and define a transfinite action on it - again a *Liminf* construction, for limit times λ we set

$$R_i(\lambda) = Liminf^*_{\alpha \to \lambda} R_i(\alpha)$$

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However we have weak universality: the companion structure is $L_{\omega^{ck}_{\omega}}$ - the first limit of admissibles, and all reals here are ITRM-computable.

Infinite Time Blum-Shub-Smale Machines: IBSM

• A standard BSS machine consists of
(i) a finite number N of registers which hold standard reals $r \in \mathbb{R}$;
(ii) a program of a finite flow diagram with algebraic operations or conditional jumps to be performed at nodes.

A typical instruction is of the form
$(i, \varphi, j, k, l) \in \omega \times \{f \mid f : \mathbb{R}^N \to \mathbb{R}^N, f \text{ rational } \} \times 2 \times \omega \times \omega$

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We allow the machine to run transfinitely by requiring at limit stages λ:
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Question: What is the strength of such a machine?

Continuous IBSM machines

Koepke-Seyfferth[3] require additionally *continuity* in (b) (o.w. the computation crashes).

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They show:

Theorem (Koepke-Seyfferth)

Any CIBSM computation either crashes or halts in $< \omega^\omega$ steps, or is for ever looping.

Fact (K,S)

(i) Standard TM's can be simulated by IBSM's.
(ii) The standard halting problem is computable by a CIBSM: there is a CIBSM program P which halts with a real whose binary expansion is the characteristic of $0'$.

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Q.1 Does this model correspond to a recursional theory as listed earlier?
Q.2 It is natural to conjecture that any set of integers in L_{ω^ω} can be computed, and that CIBSM's are weakly universal.

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Indeed, CIBSM's are weakly universal

- We wish to see that a code for every course of an CIBSM-computation is also CIBSM-computable.

- Note that we can consider any CIBSM operating with integer, or recursive input, with recursive parameters, as being run inside the model L_{ω^k} for some sufficiently large $k < \omega$, as everything is very absolute.

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Fact: Let $r \in \mathcal{P}(\mathbb{N}) \cap L_{\omega^\omega}$. Then $\exists \alpha < \omega^\omega (r \leq_1 h_\alpha)$ where we define:

$$h_0 = \mathbb{N}; \quad h_{\beta+1} = h_\beta \cap (h_\beta)'; \quad Lim(\lambda) \to h_\lambda = \bigcap_{\alpha<\lambda} h_\alpha.$$

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• This shows that the 'companion-structure' to CIBSM computing is L_{ω^ω}.

Is there a recursion theory for CIBSM?

- The answer here is yes: *Safe Recursive Set Functions*[4]

Theorem

The least safe recursively closed set M_0 with $\omega \in M_0$ is L_{ω^ω}.

Conclusion: Since codes for sets in M_0 are computable by *CIBSM*'s, we can view this machine model as a computational vehicle for this theory of (set)-recursion.

[4] of *Beckmann, Buss, Friedman* to appear

What else can these machines do? What is L_{ω^ω} good for?

• Note: This is the first level beyond ω of the L and J hierarchies of constructible sets, where $J_\alpha = L_\alpha$.

[5] *This CIE2012 conference volume*

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• Example: *The isomorphism problem for tree-automatic wellfounded trees (on $2^{<\omega}$) is $\Delta^0_{\omega^\omega}$ complete* (by Kartzow, Liu, Lohrey) [5]. As a corollary of weak universality, for any two such trees which have rank $< \omega^k$ say) there is a CIBSM-machine checking for an isomorphism.

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Broadening our viewpoint

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Suppose $f : \mathbb{N}^n \longrightarrow \mathbb{N}^n$. *One may think of* f *acting on the points of an n-dimensional lattice torus where we identify '*∞*' (here* ω*) with 0. To let this act transfinitely we just require that at limit times* λ *the coordinates be the Liminf** *of their previous values.*

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• The existence of a Zero set (those points that eventually sink to the origin) for all such f's is a statement equivalent to Π^1_1-CA_0.

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• This again can be coded by an oracle $Z \subseteq \mathbb{N}$ but now taking a coordinate-wise $Liminf^*$ operation can be problematic:

(i) The $Liminf$ values may not be rational;

(ii) $Liminf$, even when rational, is a Π_3 calculation.

A rational variant

• Suppose now $f : (\mathbb{Q} \cap [0,1])^n \longrightarrow (\mathbb{Q} \cap [0,1])^n$ and consider iterates of f on this *rational torus*.

• This again can be coded by an oracle $Z \subseteq \mathbb{N}$ but now taking a coordinate-wise $Liminf^*$ operation can be problematic:

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Iterates of f, if always defined until γ, where γ is the least $\Pi_3(f)$-reflecting admissible ordinal will be defined, (and be looping) for ever.

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• Q. Is this best possible?

The real thing

• Suppose now $f : (\mathbb{R} \cap [0,1])^n \longrightarrow (\mathbb{R} \cap [0,1])^n$ is continuous and consider iterates of f on this the *real torus*.

Now this kind of iteration requires reals to be carried along, *i.e.* sets of rationals; again the *Liminf** operation is a $\Pi_3(f)$ operation.

Definition

$\zeta \in On$ is f*-extendible* if $L_\zeta[f] \prec_{\Sigma_2} L_\Sigma[f]$ for some $\Sigma > \zeta$.
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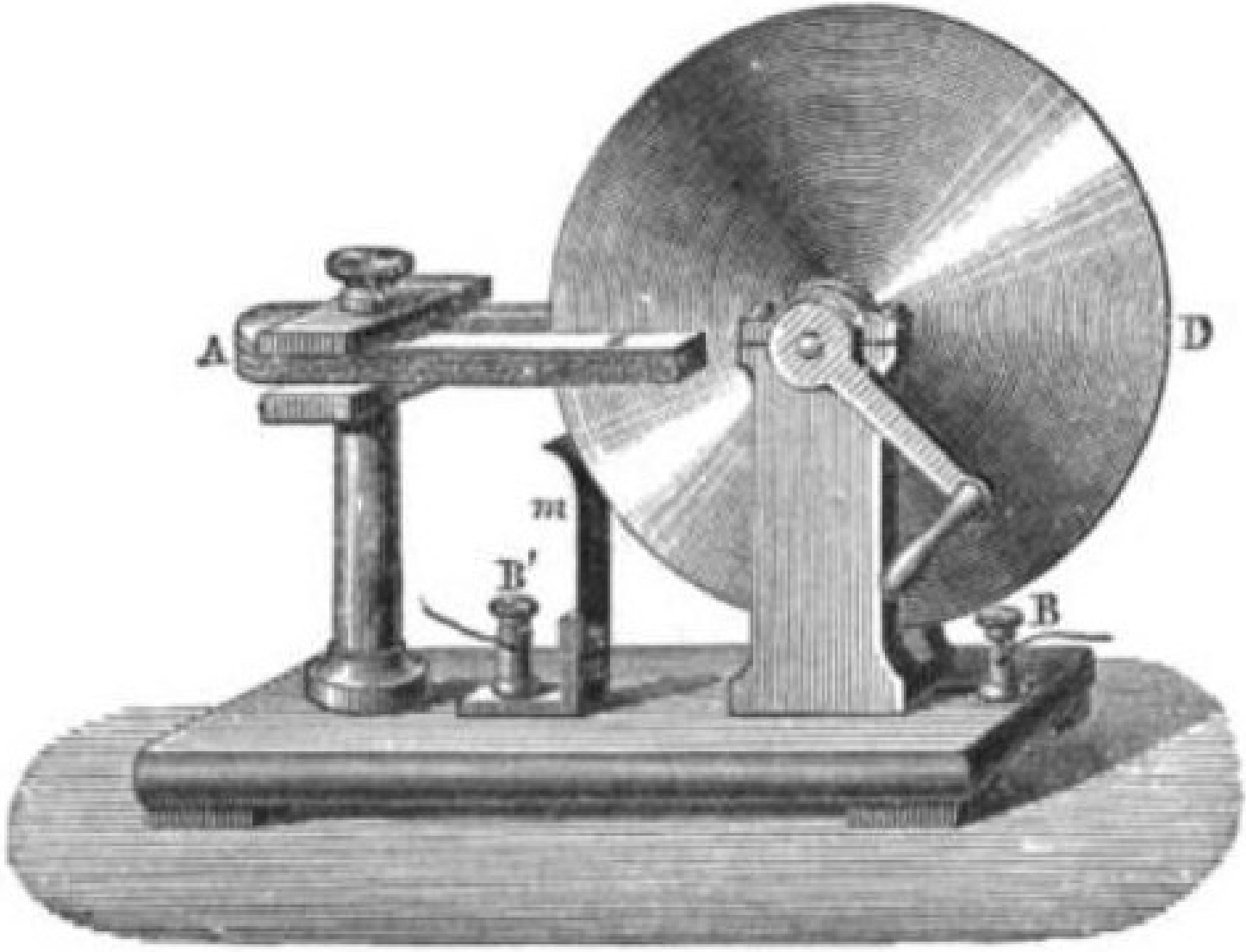
Then to determine whether, say iterations of f on starting real x go through $(0,0,\ldots,0)$ can be determined by an iteration of length $\zeta(f \oplus x)$.
But:

Lemma

$$\Pi^1_3\text{- CA}_0 \vdash \text{“ For every } f \textit{ an } f\textit{-extendible exists”}.$$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