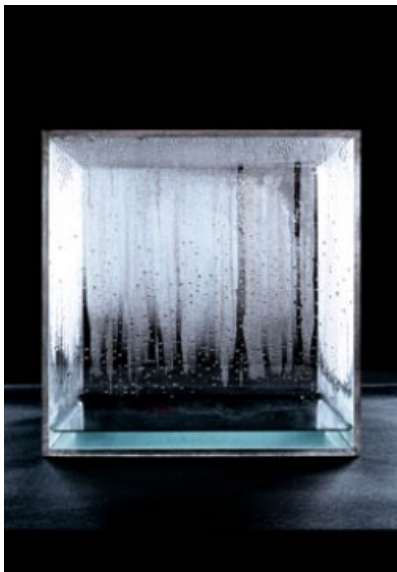


A Condensed History of Condensation

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A Condensed History of Condensation

- Part I: Indeed, some history.
- Part II: Some failures of condensation: grading functions
- Part III: Enforcing versions of condensation principles.

Part I: The Gödel Condensation Lemma

Theorem ((ZF))

Let $\langle L_\alpha \mid \alpha \in On \rangle$ be the constructible hierarchy. Let $\langle X, \in \rangle \prec \langle L_\alpha, \in \rangle$ be an elementary substructure. Then $\langle X, \in \rangle \cong \langle L_\beta, \in \rangle$ for some $\beta \leq \alpha$.

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- This is fundamental for subsequent work, and for L itself demonstrates GCH, and later, \diamond ...
- Without some form of condensation fine structural analysis is hopeless; as for example, for general $A \subseteq ON$ condensation for the $(L_\alpha[A] \mid \alpha \in On)$ hierarchy is does not hold.

However...

You might be content with: $M \prec L_\alpha[A]$ implying $(M, \in \cap M^2) \cong (L_\beta[\bar{A}], \in)$
for some $\beta \leq \alpha$ with at least some properties enjoyed by A going down to \bar{A} .

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- Eg, preservation of (some) sharps: A might be of the form $A_0 \cup A_0^\#$ where $A_0^\#$ is some form of sharp for A_0 (with $\alpha > \text{sup}A_0^\#$). Then \bar{A} would be of the form $\bar{A}_0 \cup \bar{A}^\#$.

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We thus have some weak form of “#-condensation.” But this can be useful. This method is exploited in many places: for example in the Core Model Induction, a simple sharp can be replaced by an “ $M_n^\#$ ” denoting a sharp for a model with n -Woodin-cardinals-over- A_0 , or again for a so-called Q -structure over A_0 .

$L^\#$: Another successful condensation model

Let $\# : On \rightarrow \mathcal{P}(On)$ be recursively defined as

$$\#(\alpha) = (\# \upharpoonright \alpha)^\#$$

and then $L^\# = (L[\#], \in, \#)$. This is the minimal model closed under the $\#$ operation $X \rightarrow X^\#$.

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- $L^\# \models "V = L^\# + GCH + \diamond + \dots"$
- Moreover we have $Condens(L^\#)$:

Let $(X, \in, \# \cap X) \prec (L_\alpha^\#, \in, \#)$ be an elementary substructure. Then, for some $\beta \leq \alpha$,

$$(X, \in, \# \cap X) \cong (L_\beta^\#, \in, \#).$$

Jensen Coding

Theorem (Jensen)

Given $(V, \in, A) \models V = L[A] + GCH + A \subseteq$ On we may define a class forcing \mathbb{P} , cardinal preserving, with

$$(V[G], \in) \models \exists r \subseteq \omega V = L[r] \wedge A, G \text{ are definable in } L[r].$$

Jensen Coding

Theorem (W)

With similar assumptions on V, A there is a \mathbb{P}^{DJ}

$$(V[G], \epsilon) \models \exists r \subseteq \omega \ V = K^{DJ}[r] \wedge A, G \text{ are definable in } K^{DJ}[r].)$$

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Theorem (S. Friedman)

With similar assumptions on V, A there is a \mathbb{P}^μ

$$(V[G], \epsilon) \models \exists r \subseteq \omega \ V = L^\mu[r] \wedge A, G \text{ are definable in } L^\mu[r].$$

Acceptability

Definition

(Acceptability) Let $A \subseteq On$. Then a hierarchy $\langle (L_\alpha[A], \in, A) \mid \alpha < \infty \rangle$ is *acceptable* if, whenever $B \in Def(L_\alpha[A], \in, A) \cap \mathcal{P}(\rho)$ and $B \notin L_\alpha[A]$ then $\exists F \in L_{\alpha+1}[A]$, with $F : \rho \rightarrow L_\alpha[A]$ which is onto.

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- “(Weak) Acceptability” (for V) means we can find a predicate $A \subseteq On$ so that $L_\alpha[A]$ is an (weak) acceptable hierarchy.
- Then: Acceptability $\leftrightarrow GCH$ but W. Acceptability $\not\leftrightarrow GCH$.

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• Or even less: just need $\alpha < \omega_1 \forall \beta \in [\alpha, \omega_1) \exists I_\beta$ indiscernibles for $\mathfrak{A} = L_\kappa[E]$ with

$$tp_{\mathfrak{A}}(I_\alpha) = tp_{\mathfrak{A}}(I_\beta) \wedge otp(I_\beta) \geq \beta.$$

Because then all the hulls $H_{\mathfrak{A}}(I_\alpha) \cap \omega_1 = H_{\mathfrak{A}}(I_\beta) \cap \omega_1 = \bar{\alpha}$, and there is no $H_{\mathfrak{A}}(I_\beta) \cong M \models “|\bar{\alpha}| = \omega”$. But *some* $L_\beta[E] \models “|\bar{\alpha}| = \omega”$ so a tail of the $H(I_\beta)$ are not condensing correctly.

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• Define for $\alpha < \omega_1$ $h_0^E(\alpha) = \text{least } \beta \text{ s.t. } L_{\beta+1}[E] \models “|\bar{\alpha}| = \omega”$.

Then we only need indiscernibles I s.t. if $H = H_{\mathfrak{A}}(I) \wedge \alpha = H \cap \omega_1$ then $otp(I) \geq h_0^E(\alpha)$.

- In $L[E]$ for *small* κ we shall have condensation: if we take $X \prec L_{\omega_3}[E]$ and then $\pi : X \cong M = L_\beta[\bar{E}]$ then because there are so few M -cardinals by the comparison theory for such levels we must have $L_\beta[\bar{E}] = L_\beta[E]$.

Theorem (Velickovic)

If $L[E]$ is a (sufficiently iterable) model of a Woodin limit of Woodins, then it has no precipitous ideal on ω_1 .

Proof: V shows that if the function h_0^E as above dominates the order type of the transitivised countable models (here $(L_\beta[\bar{E}])$, *i.e.* is a “collapsing function,” then there are no such ideals. □

Generalising h_0^E

Definition

A *grading* up to $\kappa \in \text{Card} \cup \{\infty\}$ is a sequence $\langle h_\alpha \mid \alpha = \nu^+ < \kappa \rangle$ with $h_\alpha : \alpha \rightarrow \alpha$ s.t. for any $X \prec \langle L_\kappa[A], A, B, \dots \rangle$

$$\text{sup}(X \cap \alpha) < \alpha \rightarrow \text{ot}(X \cap \text{On}) < h_\alpha(\text{sup}(X \cap \alpha)).$$

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• *Magidor Covering for L ($\neg 0^\#$):*

Every set $X \subseteq \text{On}$ closed under the primitive recursive set functions is a union of countably many sets in L .

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• *Magidor Covering for K (Assume $\neg 0^{pistol}$ and no ω -closed filter on E):*
Every set $X \subseteq On$ closed under the primitive recursive-in- E set functions is a union of countably many sets in the core model K .

Theorem

(No IM(Woodin) & no ω -closed F on E)

(i) $MCL(L[E])$

(ii) $\exists \langle h_\alpha \mid \omega < \alpha = \nu^+ < \infty \rangle$ a grading up to On .

Condensation principles

We suppose we have a hierarchy $\mathfrak{M} = \langle M_\alpha \mid \alpha \in On \rangle$ with $M = \bigcup_{\alpha < \infty} M_\alpha$ an IM of ZFC which is a continuous chain: (i) $Trans(M_\alpha)$; (ii) $\alpha < \beta \rightarrow M_\alpha \in M_\beta$; (iii) $Lim(\lambda) \rightarrow M_\lambda = \bigcup_{\alpha < \lambda} M_\alpha$; (iv) $\theta \in Card^{\mathfrak{M}} \rightarrow (H_\theta = M_\theta)^{\mathfrak{M}}$.

Definition

$\mathfrak{B} \prec (M_\alpha, \langle M_\beta \mid \beta < \alpha \rangle, \dots)$ condenses if for some $\gamma \leq \beta$:

$$\mathfrak{B}_0 = (B, \langle M_\beta \mid \beta \in B \rangle) \cong (M_\gamma, \langle M_\beta \mid \beta < \gamma \rangle).$$

Definition (Strong Condensation [at κ])

We require of the hierarchy that for all α [$\alpha \leq \kappa$] there exists an expansion in a countable language

$\mathfrak{A} = (M_\alpha, \langle M_\beta \mid \beta < \alpha \rangle, \dots)$, so that any $\mathfrak{B} \prec \mathfrak{A}$ condenses.

Definition (Local Club Condensation [up to κ])

We require that $\forall \alpha$ [$\forall \alpha \leq \kappa$] if $|\alpha| > \omega \wedge \mathfrak{A} = (M_\alpha, \langle M_\beta \mid \beta < \alpha \rangle, \dots)$, then there is a continuous chain $\langle \mathfrak{B}_\gamma \rangle_{\gamma < |\alpha|}$ of condensing substructures, with $\gamma \subseteq B$, $|B_\gamma| = |\gamma|$ and $\bigcup_{\gamma < |\alpha|} B_\gamma = M_\alpha$.

First results

Lemma (Wu, Friedman-Holy)

If $\langle M_\alpha \rangle_{\alpha \in \mathcal{O}_n}$ satisfies LCC, and $(\tau \in \text{Card} \wedge \kappa = \tau^+)^M$, $\text{cf}(\tau) > \omega$
 $\mathfrak{B} \prec (M_\kappa, \langle M_\beta \mid \beta < \kappa \rangle) \wedge B \cap \tau \in \tau$, then \mathfrak{B} condenses.

Corollary (Wu)

$(V = M)$ LCC up to ω_2 implies SC at ω_2 .

Enforcing condensation

Theorem (Wu, Friedman-Holy)

Assume GCH. There is a cardinal preserving iterated forcing of length \aleph_2 to add a SC at ω_2 . Hence: $\text{Con}(\text{ZFC}) \implies \text{Con}(\text{ZFC} + \text{SC}(\omega_2))$

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Q. Is there a set forcing to add a SC at ω_3 ?

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- Objective here:

Theorem (Friedman-Holy)

If V is a proper extension of a model M satisfying Local Club Condensation, Weak acceptability, square on the singular cardinals, \square_λ for every singular λ and $PFA(\mathfrak{c}^+$ -linked), then there is a Σ_1^2 -indescribable gap $[\kappa, \kappa^+)$ in M .

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Theorem (Wu)

Let κ be a Mahlo limit of measurable cardinals. Then the forcing to collapse κ to \aleph_2 to add a $SC(\omega_2)$ sequence can be modified to also ensure $\neg \square_{\omega_1}$.

Hence:

$$\begin{aligned} & \text{Con}(\text{ZFC} + \exists \kappa \text{ Mahlo, and a stationary limit of measurable cardinals}) \\ & \implies \text{Con}(\text{ZFC} + SC(\omega_2) + \neg \square_{\omega_1}). \end{aligned}$$

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Theorem (Holy...)

Let κ be a Mahlo limit of measurable cardinals. Let $\omega_1 \leq \lambda = \nu^+$. Then the forcing to collapse κ to λ^+ whilst adding an $LCC(\kappa = \lambda^+)$ sequence can be modified to also ensure $\neg \square_\lambda$.

Thank you

