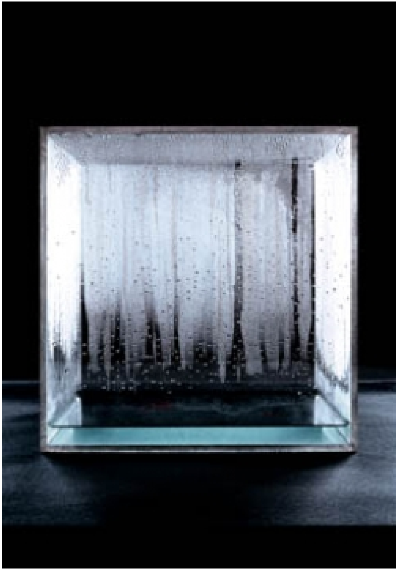
A Condensed History of Condensation

4th European Set Theory Meeting, July 18, 2013

P.D.Welch, University of Bristol



A Condensed History of Condensation

- Part I: Indeed, some history.
- Part II: Some failures of condensation: grading functions
- Part III: Enforcing versions of condensation principles.

Part I: The Gödel Condensation Lemma

Theorem ((ZF))

Let $\langle L_\alpha \mid \alpha \in On \rangle$ *be the constructible hierarchy. Let* $\langle X, \in \rangle \prec \langle L_\alpha, \in \rangle$ *be an elementary substructure. Then* $\langle X, \in \rangle \cong \langle L_\beta, \in \rangle$ *for some* $\beta \leq \alpha$.

Part I: The Gödel Condensation Lemma

Theorem ((ZF))

Let $\langle L_\alpha \mid \alpha \in On \rangle$ *be the constructible hierarchy. Let* $\langle X, \in \rangle \prec \langle L_\alpha, \in \rangle$ *be an elementary substructure. Then* $\langle X, \in \rangle \cong \langle L_\beta, \in \rangle$ *for some* $\beta \leq \alpha$.

• This is fundamental for subsequent work, and for L itself demonstrates GCH, and later, $\diamondsuit$...

Part I: The Gödel Condensation Lemma

Theorem ((ZF))

Let $\langle L_\alpha \mid \alpha \in On\rangle$ *be the constructible hierarchy. Let* $\langle X, \in\rangle \prec \langle L_\alpha, \in\rangle$ *be an elementary substructure. Then* $\langle X, \in\rangle \cong \langle L_\beta, \in\rangle$ *for some* $\beta \leq \alpha$.

• This is fundamental for subsequent work, and for L itself demonstrates GCH, and later, $\diamondsuit$...

• Without some form of condensation fine structural analysis is hopeless; as for example, for general $A \subseteq ON$ condensation for the $(L_\alpha[A] \mid \alpha \in On)$ hierarchy is does not hold.

However...

You might be content with: $M \prec L_\alpha[A]$ implying $(M, \in \cap M^2) \cong (L_\beta[\bar{A}], \in)$ for some $\beta \leq \alpha$ with at least some properties enjoyed by A going down to $\bar{A}$.

You might be content with: $M \prec L_\alpha[A]$ implying $(M, \in \cap M^2) \cong (L_\beta[\bar{A}], \in)$ for some $\beta \leq \alpha$ with at least some properties enjoyed by A going down to $\bar{A}$.

• Eg, preservation of (some) sharps: A might be of the form $A_0 \cup A_0^{\#}$ where $A_0^{\#}$ is some form of sharp for A_0 (with $\alpha > supA_0^{\#}$). Then $\bar{A}$ would be of the form $\bar{A}_0 \cup \bar{A}^{\#}$.

You might be content with: $M \prec L_\alpha[A]$ implying $(M, \in \cap M^2) \cong (L_\beta[\bar{A}], \in)$ for some $\beta \leq \alpha$ with at least some properties enjoyed by A going down to $\bar{A}$.

• Eg, preservation of (some) sharps: A might be of the form $A_0 \cup A_0^\#$ where $A_0^\#$ is some form of sharp for A_0 (with $\alpha > supA_0^\#$). Then $\bar{A}$ would be of the form $\bar{A}_0 \cup \bar{A}^\#$.

We thus have some weak form of "#-condensation." But this can be useful. This method is exploited in many places: for example in the Core Model Induction, a simple sharp can be replaced by an "$M_n^\#$" denoting a sharp for a model with n-Woodin-cardinals-over-A_0, or again for a so-called Q-structure over A_0.

$L^{\#}$: Another successful condensation model

Let $\# : On \longrightarrow \mathcal{P}(On)$ be recursively defined as

$$\#(\alpha) = (\# \restriction \alpha)^{\#}$$

and then $L^{\#} = (L[\#], \in, \#)$. This is the minimal model closed under the $\#$ operation $X \to X^{\#}$.

$L^{\#}$: Another successful condensation model

Let $\# : On \longrightarrow \mathcal{P}(On)$ be recursively defined as

$$\#(\alpha) = (\# \upharpoonright \alpha)^{\#}$$

and then $L^{\#} = (L[\#], \in, \#)$. This is the minimal model closed under the $\#$ operation $X \to X^{\#}$.

• $L^{\#} \models$"$V = L^{\#} + GCH + \diamond + \ldots$.

$L^{\#}$: Another successful condensation model

Let $\# : On \longrightarrow \mathcal{P}(On)$ be recursively defined as

$$\#(\alpha) = (\# \restriction \alpha)^{\#}$$

and then $L^{\#} = (L[\#], \in, \#)$. This is the minimal model closed under the $\#$ operation $X \to X^{\#}$.

- $L^{\#} \models$"$V = L^{\#} + GCH + \diamond + \ldots$.
- Moreover we have $Condens(L^{\#})$:

Let $(X, \in, \# \cap X) \prec (L^{\#}_{\alpha}, \in, \#)$ be an elementary substructure. Then, for some $\beta \leq \alpha$,

$$(X, \in, \# \cap X) \cong (L^{\#}_{\beta}, \in, \#).$$

Jensen Coding

Theorem (Jensen)

Given $(V, \in, A) \models V = L[A] + GCH + A \subseteq On$ *we may define a class forcing* $\mathbb{P}$*, cardinal preserving, with*

$$(V[G], \in) \models \exists r \subseteq \omega\, V = L[r] \wedge A, G \text{ are definable in } L[r].$$

Jensen Coding

Theorem (W)

With similar assumptions on V, A there is a $\mathbb{P}^{DJ}$

$$(V[G], \in) \models \exists r \subseteq \omega\, V = K^{DJ}[r] \wedge A, G \textit{ are definable in } K^{DJ}[r].)$$

Jensen Coding

Theorem (W)

With similar assumptions on V, A *there is a* $\mathbb{P}^{DJ}$

$$(V[G], \in) \models \exists r \subseteq \omega\, V = K^{DJ}[r] \wedge A, G \textit{ are definable in } K^{DJ}[r].)$$

- However the above was deficient in preserving large cardinals.

Jensen Coding

Theorem (W)

With similar assumptions on V, A there is a $\mathbb{P}^{DJ}$

$$(V[G], \in) \models \exists r \subseteq \omega\, V = K^{DJ}[r] \wedge A, G \textit{ are definable in } K^{DJ}[r].)$$

- However the above was deficient in preserving large cardinals.

Theorem (S. Friedman)

With similar assumptions on V, A there is a $\mathbb{P}^{\mu}$

$$(V[G], \in) \models \exists r \subseteq \omega\, V = L^{\mu}[r] \wedge A, G \textit{ are definable in } L^{\mu}[r].$$

Acceptability

Definition

(Acceptability) Let $A \subseteq On$. Then a hierarchy $\langle (L_\alpha[A], \in, A) \mid \alpha < \infty \rangle$ is *acceptable* if, whenever $B \in Def(L_\alpha[A], \in, A) \cap \mathcal{P}(\rho)$ and $B \notin L_\alpha[A]$ then $\exists F \in L_{\alpha+1}[A]$, with $F : \rho \longrightarrow L_\alpha[A]$ which is onto.

Definition

(Weak Acceptability) Let $A \subseteq On$. Then a hierarchy $\langle (L_\alpha[A], \in, A) \mid \alpha < \infty \rangle$ is *acceptable* if, whenever $B \in Def(L_\alpha[A], \in, A) \cap \mathcal{P}(\rho)$ and $B \notin L_\alpha[A]$ then $Hull^{L_{\alpha+1}[A]}(\rho) = L_{\alpha+1}[A]$.

Acceptability

Definition

(Acceptability) Let $A \subseteq On$. Then a hierarchy $\langle (L_\alpha[A], \in, A) \mid \alpha < \infty \rangle$ is *acceptable* if, whenever $B \in Def(L_\alpha[A], \in, A) \cap \mathcal{P}(\rho)$ and $B \notin L_\alpha[A]$ then $\exists F \in L_{\alpha+1}[A]$, with $F : \rho \longrightarrow L_\alpha[A]$ which is onto.

Definition

(Weak Acceptability) Let $A \subseteq On$. Then a hierarchy $\langle (L_\alpha[A], \in, A) \mid \alpha < \infty \rangle$ is *acceptable* if, whenever $B \in Def(L_\alpha[A], \in, A) \cap \mathcal{P}(\rho)$ and $B \notin L_\alpha[A]$ then $Hull^{L_{\alpha+1}[A]}(\rho) = L_{\alpha+1}[A]$.

• "(Weak) Acceptability" (for V) means we can find a predicate $A \subseteq On$ so that $L_\alpha[A]$ is an (weak) acceptable hierarchy.

Acceptability

Definition

(Acceptability) Let $A \subseteq On$. Then a hierarchy $\langle (L_\alpha[A], \in, A) \mid \alpha < \infty \rangle$ is *acceptable* if, whenever $B \in Def(L_\alpha[A], \in, A) \cap \mathcal{P}(\rho)$ and $B \notin L_\alpha[A]$ then $\exists F \in L_{\alpha+1}[A]$, with $F : \rho \longrightarrow L_\alpha[A]$ which is onto.

Definition

(Weak Acceptability) Let $A \subseteq On$. Then a hierarchy $\langle (L_\alpha[A], \in, A) \mid \alpha < \infty \rangle$ is *acceptable* if, whenever $B \in Def(L_\alpha[A], \in, A) \cap \mathcal{P}(\rho)$ and $B \notin L_\alpha[A]$ then $Hull^{L_{\alpha+1}[A]}(\rho) = L_{\alpha+1}[A]$.

• "(Weak) Acceptability" (for V) means we can find a predicate $A \subseteq On$ so that $L_\alpha[A]$ is an (weak) acceptable hierarchy.

• Then: Acceptability $\leftrightarrow$ GCH but W. Acceptability $\not\leftrightarrow$ GCH.

In general we build $L[E]$ models with E a sequence of filters, measures, extenders....

Q: How "big" must $L[E]$ be in order for $Condens(L[E])$ to fail?

In general we build $L[E]$ models with E a sequence of filters, measures, extenders....

Q: How "big" must $L[E]$ *be in order for Condens*$(L[E])$ *to fail?*

Answer: Not very big at all. Suppose $L[E]$ has an ω_1-Erdős cardinal κ.

In general we build $L[E]$ models with E a sequence of filters, measures, extenders....

Q: How "big" must $L[E]$ be in order for $Condens(L[E])$ to fail?

Answer: Not very big at all. Suppose $L[E]$ has an ω_1-Erdős cardinal κ.

• Or even less: just need $\alpha < \omega_1 \forall \beta \in [\alpha, \omega_1) \exists I_\beta$ indiscernibles for $\mathfrak{A} = L_\kappa[E]$ with

$$tp_{\mathfrak{A}}(I_\alpha) = tp_{\mathfrak{A}}(I_\beta) \wedge otp(I_\beta) \geq \beta.$$

Because then all the hulls $H_{\mathfrak{A}}(I_\alpha) \cap \omega_1 = H_{\mathfrak{A}}(I_\beta) \cap \omega_1 = \bar{\alpha}$, and there is no $H_{\mathfrak{A}}(I_\beta) \cong M \models$"$|\bar{\alpha}| = \omega$". But *some* $L_\beta[E] \models$"$|\bar{\alpha}| = \omega$" so a tail of the $H(I_\beta)$ are not condensing correctly.

In general we build $L[E]$ models with E a sequence of filters, measures, extenders....

Q: How "big" must $L[E]$ be in order for $Condens(L[E])$ to fail?

Answer: Not very big at all. Suppose $L[E]$ has an ω_1-Erdős cardinal κ.

• Or even less: just need $\alpha < \omega_1 \forall \beta \in [\alpha, \omega_1) \exists I_\beta$ indiscernibles for $\mathfrak{A} = L_\kappa[E]$ with

$$tp_{\mathfrak{A}}(I_\alpha) = tp_{\mathfrak{A}}(I_\beta) \wedge otp(I_\beta) \geq \beta.$$

Because then all the hulls $H_{\mathfrak{A}}(I_\alpha) \cap \omega_1 = H_{\mathfrak{A}}(I_\beta) \cap \omega_1 = \bar{\alpha}$, and there is no $H_{\mathfrak{A}}(I_\beta) \cong M \models$"$|\bar{\alpha}| = \omega$". But *some* $L_\beta[E] \models$"$|\bar{\alpha}| = \omega$" so a tail of the $H(I_\beta)$ are not condensing correctly.

• Define for $\alpha < \omega_1$ $h_0^E(\alpha) =$ least β s.t. $L_{\beta+1}[E] \models$"$|\bar{\alpha}| = \omega$".
Then we only need indiscernibles I s.t. if $H = H_{\mathfrak{A}}(I) \wedge \alpha = H \cap \omega_1$ then $otp(I) \geq h_0^E(\alpha)$.

• In $L[E]$ for *small* κ we shall have condensation: if we take $X \prec L_{\omega_3}[E]$ and then $\pi : X \cong M = L_\beta[\bar{E}]$ then because there are so few M-cardinals by the comparison theory for such levels we must have $L_\beta[\bar{E}] = L_\beta[E]$.

Theorem (Velickovic)

If $L[E]$ is a (sufficiently iterable) model of a Woodin limit of Woodins, then it has no precipitous ideal on ω_1.

Proof: V shows that if the function h_0^E as above dominates the order type of the transitivised countable models (here $(L_\beta[\bar{E}])$, *i.e.* is a "collapsing function," then there are no such ideals. □

Generalising h_0^E

Definition

A *grading* up to $\kappa \in Card \cup \{\infty\}$ is a sequence $\langle h_\alpha \mid \alpha = \nu^+ < \kappa \rangle$ with $h_\alpha : \alpha \longrightarrow \alpha$ s.t. for any $X \prec \langle L_\kappa[A], A, B, \ldots \rangle$

$$sup(X \cap \alpha) < \alpha \rightarrow ot(X \cap On) < h_\alpha(sup(X \cap \alpha)).$$

Generalising h_0^E

Definition

A *grading* up to $\kappa \in Card \cup \{\infty\}$ is a sequence $\langle h_\alpha \mid \alpha = \nu^+ < \kappa \rangle$ with $h_\alpha : \alpha \longrightarrow \alpha$ s.t. for any $X \prec \langle L_\kappa[A], A, B, \ldots \rangle$

$$sup(X \cap \alpha) < \alpha \rightarrow ot(X \cap On) < h_\alpha(sup(X \cap \alpha)).$$

• *Magidor Covering for L* ($\neg 0^\#$)*:*
Every set $X \subseteq On$ *closed under the primitive recursive set functions is a union of countably many sets in L.*

Let us say a filter F that occurs as E_γ, say, on an E sequence, is *ω-closed* if, in $L[E \upharpoonright \gamma, F]$ it is an ω-closed filter.

Let us say a filter F that occurs as E_γ, say, on an E sequence, is *ω-closed* if, in $L[E \restriction \gamma, F]$ it is an ω-closed filter.

• *Magidor Covering for K (Assume $\neg 0^{pistol}$ and no ω-closed filter on E): Every set $X \subseteq On$ closed under the primitive recursive-in-E set functions is a union of countably many sets in the core model K.*

Theorem

(No $IM(Woodin)$ & no ω-closed F on E)
(i) $MCL(L[E])$
(ii) $\exists\langle h_\alpha \mid \omega < \alpha = \nu^+ < \infty\rangle$ a grading up to On.

Condensation principles

We suppose we have a hierarchy $\mathfrak{M} = \langle M_\alpha | \alpha \in On \rangle$ with $M = \bigcup_{\alpha<\infty} M_\alpha$ an IM of *ZFC* which is a continuous chain: (i) $Trans(M_\alpha)$; (ii) $\alpha < \beta \to M_\alpha \in M_\beta$; (iii) $Lim(\lambda) \to M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$; (iv) $\theta \in Card^{\mathfrak{M}} \to (H_\theta = M_\theta)^{\mathfrak{M}}$.

Definition

$\mathfrak{B} \prec (M_\alpha, \langle M_\beta | \beta < \alpha \rangle, \ldots)$ *condenses* if for some $\gamma \leq \beta$:

$$\mathfrak{B}_0 = (B, \langle M_\beta \mid \beta \in B \rangle) \cong (M_\gamma, \langle M_\beta | \beta < \gamma \rangle).$$

Definition (Strong Condensation [at κ])

We require of the hierarchy that for all α [$\alpha \leq \kappa$] there exists an expansion in a countable language

$\mathfrak{A} = (M_\alpha, \langle M_\beta | \beta < \alpha \rangle, \ldots)$, so that any $\mathfrak{B} \prec \mathfrak{A}$ condenses.

Definition (Local Club Condensation [up to κ])

We require that $\forall \alpha$ [$\forall \alpha \leq \kappa$] if $|\alpha| > \omega \wedge \mathfrak{A} = (M_\alpha, \langle M_\beta | \beta < \alpha \rangle, \ldots)$, then there is a continuous chain $\langle \mathfrak{B}_\gamma \rangle_{\gamma < |\alpha|}$ of condensing substructures, with $\gamma \subseteq B$, $|B_\gamma| = |\gamma|$ and $\bigcup_{\gamma<|\alpha|} B_\gamma = M_\alpha$.

First results

Lemma (Wu, Friedman-Holy)

If $\langle M_\alpha \rangle_{\alpha \in On}$ *satisfies LCC, and* $(\tau \in Card \wedge \kappa = \tau^+)^M$, $cf(\tau) > \omega$
$\mathfrak{B} \prec (M_\kappa, \langle M_\beta | \beta < \kappa \rangle) \wedge B \cap \tau \in \tau$, *then* $\mathfrak{B}$ *condenses.*

Corollary (Wu)

($V = M$) LCC up to ω_2 *implies SC at* ω_2.

Enforcing condensation

Theorem (Wu, Friedman-Holy)

Assume GCH. There is a cardinal preserving iterated forcing of length $\aleph_2$ to add a SC at ω_2. Hence: $Con(ZFC) \Longrightarrow Con(ZFC + SC(\omega_2))$

Enforcing condensation

Theorem (Wu, Friedman-Holy)

Assume GCH. There is a cardinal preserving iterated forcing of length $\aleph_2$ to add a SC at ω_2. *Hence:* $Con(ZFC) \Longrightarrow Con(ZFC + SC(\omega_2))$

Q. Is there a set forcing to add a SC at ω_3?

Enforcing *LCC* globally

Theorem (Friedman-Holy)

Assume GCH. There is a class forcing that adds LCC and preserves large cardinals up to and including ω-superstrongs.

Enforcing *LCC* globally

Theorem (Friedman-Holy)

Assume GCH. There is a class forcing that adds LCC and preserves large cardinals up to and including ω-superstrongs.

Theorem (Friedman-Holy)

(GCH) It is possible to force to obtain LCC and Weak Acceptability simultaneously whilst preserving cofinalities and large cardinals such as ω-superstrong cardinals.

Enforcing *LCC* globally

Theorem (Friedman-Holy)

Assume GCH. There is a class forcing that adds LCC and preserves large cardinals up to and including ω-superstrongs.

Theorem (Friedman-Holy)

(GCH) It is possible to force to obtain LCC and Weak Acceptability simultaneously whilst preserving cofinalities and large cardinals such as ω-superstrong cardinals.

- Objective here:

Theorem (Friedman-Holy)

If V is a proper extension of a model M satisfying Local Club Condensation, Weak acceptability, square on the singular cardinals, $\square_\lambda$ for every singular λ and PFA($\mathfrak{c}^+$-linked), then there is a Σ^2_1-indescribable gap $[\kappa, \kappa^+)$ in M.

Q. To what extent does SC (or LCC) capture true fine structural features?

Q. To what extent does SC (or LCC) capture true fine structural features?

Theorem (Wu)

Let κ be a Mahlo limit of measurable cardinals. Then the forcing to collapse κ to $\aleph_2$ to add a $SC(\omega_2)$ sequence can be modified to also ensure $\neg\Box_{\omega_1}$. Hence:

$$Con(ZFC + \exists\kappa \textit{ Mahlo, and a stationary limit of measurable cardinals }) \\ \implies Con(ZFC + SC(\omega_2) + \neg\Box_{\omega_1}) \; .$$

Q. To what extent does SC (or LCC) capture true fine structural features?

Theorem (Wu)

Let κ be a Mahlo limit of measurable cardinals. Then the forcing to collapse κ to $\aleph_2$ to add a $SC(\omega_2)$ sequence can be modified to also ensure $\neg\Box_{\omega_1}$. Hence:

$$Con(ZFC + \exists\kappa \text{ Mahlo, and a stationary limit of measurable cardinals }) \\ \implies Con(ZFC + SC(\omega_2) + \neg\Box_{\omega_1}) .$$

• (PDW) The hypothesis here can be weakened to a Mahlo limit of ω-Erdős cardinals, thus rendering the hypothesis consistent with $V = L$.

Q: Can this be weakened to κ a Mahlo alone?

Q. To what extent does SC (or LCC) capture true fine structural features?

Theorem (Wu)

Let κ be a Mahlo limit of measurable cardinals. Then the forcing to collapse κ to $\aleph_2$ to add a $SC(\omega_2)$ sequence can be modified to also ensure $\neg\Box_{\omega_1}$. Hence:

$$Con(ZFC + \exists\kappa \text{ Mahlo, and a stationary limit of measurable cardinals }) \implies Con(ZFC + SC(\omega_2) + \neg\Box_{\omega_1}).$$

• (PDW) The hypothesis here can be weakened to a Mahlo limit of ω-Erdős cardinals, thus rendering the hypothesis consistent with $V = L$.

Q: Can this be weakened to κ a Mahlo alone?

Theorem (Holy...)

Let κ be a Mahlo limit of measurable cardinals. Let $\omega_1 \leq \lambda = \nu^+$. Then the forcing to collapse κ to λ^+ whilst adding an $LCC(\kappa = \lambda^+)$ sequence can be modified to also ensure $\neg\Box_\lambda$.

Thank you