Gödel

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> "Gödel's achievement in modern logic is singular and monumental . . . a landmark that will remain visible far in space and time . . . The subject of logic will never again be the same."
>
> (von Neumann)

1 Introduction

Kurt Gödel is claimed by some to be the greatest logician since Aristotle. The ideas that Gödel is associated with in logic are the *Completeness Theorem* which appeared in his 1929 PhD thesis, but more particularly the *Incompleteness Theorem* (actually a pair of theorems) and they both have been crucial in almost all theoretical areas of 20'th century logic since their inception. Whilst Frege's keen insights into the nature of quantifier ensured a great leap forward from the Syllogism, and his attempts to formulate a conception of arithmetic purely as a logical construct, were groundbreaking and influenced the course of the philosophy of mathematics in the late 19'th, and through Russell, in the early 20th century, and determined much of the discourse of that period, ultimately it is Gödel's work that encapsulated the nature of the relationship between *deductive processes* acting on symbolic systems, and the nature of *truth* (or *satisfaction*), or more widely interpreted *meaning* or *semantics* in the Completeness Theorem. It is often said that the much deeper Incompleteness Theorems that came a year later illustrated a limitation of the axiomatic method, and in particular brought to a halt David Hilbert's programme of putting mathematics on a secure, *finitistically* provable, ground of consistency. It may even be true that more ink and paper has been expended on the Incompleteness Theorem and its consequences (imagined or otherwise) than any other theorem in mathematics.

We shall give an introductory account of these two theorems - but one should be aware that Gödel made significant contributions to other areas of logic (notably giving an interpretation of intuitionistic logic in the usual predicate logic) which we do not cover here. Gödel's contributions to set theory (his hierarchy of constructible sets with which he showed the consistency of the Axiom of Choice and the Continuum Hypothesis, one might consider a form of 'ramified' or 'iterated logic') have had almost as much foundational impact in set theory as the logical theorems referred to above have been in all areas of current logic. We do not cover these here either.

2 The Completeness Theorem

We shall describe the Completeness Theorem presently, but to set it in some context we must see what people meant by "Logic".

Russell and Whitehead in the *Principia Mathematica* (PM) had set up a *deductive system* whereby, as in the axiomatic system of Euclid, the concepts of the subject under discussion, here of logic, and then it was hoped, also of arithmetic, could be codified and reduced to a small number of self-evidently true postulates, and rules of inferring from those postulates. What Russell meant by "logic" was perhaps not entirely the same as what has come down to us as a "logic" in the 21st century. A line in a deductive proof of the logical calculus of PM would be an interpreted formula, about something, but in a modern deductive proof need not be so interpreted. However Russell was trying to follow Frege and formulate such a system, ultimately from which it hoped the laws of arithmetic could be derived. This latter aim is not our concern here - merely the idea of a *deductive system* is what is important.

Thus, one might have a collection of axioms, or postulates, P say (which might be PM), and one may derive by applying one or more 'rules of deduction' from a collection R of rules, a particular proposition A, say. Now what exactly a 'proposition' is here, we can ignore (although Russell could not) because all that mattered later was that there was some symbolic, or formal language in which the postulates, the proposition A, and the intermediate propositions $B_1, B_2, \ldots, B_n$ could be written. Here, the point was that B_n was the final proposition A and any B_i was either (i) a postulate from the collection P or followed from one or more earlier propositions on the list, by an application of a deductive rule from R. In modern notation this state of affairs is rendered "$\vdash_P A$."

What is allowed as "deductive rule"? This would have had a different answer to different people in different eras. Bolzano took a "deductive rule" as one that appealed to notions of truth, or something like meaning. For Frege and Russell deductive rules should at least be meaningful and preserve the truth of the consequent A from the truth of the axioms P.

By the the time Gödel emerged as graduate student in Vienna in the later 1920's, Hilbert and Ackermann [9] had codified a system of deduction, the "restricted functional calculus" which emerged from their deliberations on the deductive system of *PM*. This could be applied to sets of postulates in a fixed type of language which we now call a "first order language" and consisted of constant symbols, and symbols of both functional and relational type that might be used to express possible functions and relations amongst *variables* $v_0, v_1, \ldots$. However it is important to note that *what* those variables will be interpreted *as* and what actual functions and relations those functional and relational symbols denote play absolutely no part in this specification whatsoever. The language is to be thought of as merely symbol strings, and the "correct" or "well-formed" strings will constitute the relevant 'formulae' which we shall work with. The rules determining the well-formed formulae are thus purely *syntactic*. The set of derivation or deduction rules for determining which sequences of formulae constituted permissible proofs had emerged from *PM* as we have mentioned, but the question of the correct set, or adequate set of rules had not been settled. Hilbert had remarked that empirical experience with the rules then in use seemed to indicate they were indeed adequate. The question was stated explicitly in [9].

Hilbert had, over the previous decades, embarked on an ambitious scheme of proving the consistency of all of mathematics. The motivation for this came from the disturbances that the discovery by Russell of a 'paradox' in Frege's system - really a fundamental error or inconsistency in Frege's basic conception, and the similar 'paradoxes' in set theory of Cantor, and Burali-Forti. We shall not discuss these here, but only remark that Hilbert had envisaged a thorough-going rethinking of mathematical axioms, and a program for showing the *consistency* of those axioms: namely that one could not prove both A and its negation, $\neg A$, using rules from the given set R. The danger had surfaced that set theory might be inconsistent and whilst all of mathematics could be seen to be developed from set theory how could we be sure that mathematics was safe from contradiction? Hilbert's thinking on this evolved over the first two decades of the 20th century and set out to reassure mathematicians of the logical safety of their field. Hilbert had thus as part of the programme, axiomatised geometry in a clear modern fashion, and had showed that the problem of establishing the consistency of geometry could be reduced to the problem of the consistency of analysis. Hilbert had thus founded the area which later evolved into *proof theory*, being the mathematical study of proofs themselves, which, as indicated above, were to be regarded as finite strings of marks on paper. It was in this arena that the basic question of completeness of a set of rules had arisen.

What would be needed if a deductive system was to be useful was some reassurance that a) deductive rules themselves could not introduce 'falsity' (in short: only true statements could be deduced using the rules from postulates considered to be true - this is the *"soundness"* of the system), and more pertinently b) that any "universally valid" formula could in fact deduced. We have referred already to a) above, and for the system derived from *PM* in [9] this was not hard to show. But what about b) and what does it mean?

For a well-formed formula B to be *universally valid* would mean that whatever domain of objects the variables were thought to range over, and whatever *interpretation* was given to the relation and functions *symbols* of the language as actual relations and functions on that domain, then the formula B would be seen to be true with that given *meaning*. In modern parlance again, we should say that B is universally valid, or more simply just 'true' in every relevant domain of interpretation. In modern notation, the idea that B is universally valid is written $\models B$.

Given a set of sentences Γ of the language $\mathcal{L}$ one may also define the notion of a formulae B being a *logical consequence* of Γ: in any domain of interpretation in which all of Γ was satisfied or deemed to to be true, then so must also B be true. Denoting this as $\Gamma \models B$ the Completeness Theorem can be stated thus:

Theorem 2.1 *Gödel: The Completeness Theorem (1929)*[4]
For any first order language $\mathcal{L}$ and any sentence B of that language then:

$$\vdash B \quad \Longleftrightarrow \quad \models B.$$

More generally for any further set of sentences Γ from $\mathcal{L}$:

$$\Gamma \vdash B \quad \Longleftrightarrow \quad \Gamma \models B.$$

The ($\Longrightarrow$) direction of both statements represents the *soundness thoorem* of the deductive system's set of rules: if B is provable in the system then B would be universally valid, that is, true in every interpretation. Similarly in the second part, if B were provable from the assumptions Γ, then in any interpretation of $\mathcal{L}$ in which all the sentences of Γ were satisfied, (that is interpreted as true statements), B would also have to be made true in that interpretation. A paraphrase is to regard the rules as being *truth-preserving*. As an example let us take Γ as the set of axioms for group theory in mathematics, with B some assertion in the appropriate language for this theory. The Soundness Theorem states that if B is derivable from the axioms using the rules of inference, then in any structure in which the axioms Γ are true, *i.e.* in any group, B will necessarily be true. This direction of the theorem was essentially known in some form to Hilbert, Ackermann *et al.*

The harder, and novel, part is the implication ($\Longleftarrow$) which sometimes alone is called the *completeness (or* adequacy) *theorem*. Taking the example of Γ the axioms of group theory, if $\Gamma \models B$, which asserts that if in any group B is satisfied, then the conclusion $\Gamma \vdash B$ is read as saying that from the group axioms Γ, the statement B can be *deduced*. This is the sense of 'completeness': the rules of deduction are complete or adequate, they are up to the task of deriving the validities of the system.

If one looks at the proof one can see how it goes beyond a very strictly finitary, syntactically based argument. We shall see that the argument intrinsically involves infinite sets. We give a modern version of the argument due to Henkin, and for the sake of exposition assume first that the set of assumptions Γ is empty and concentrate on the first version. Suppose when trying to prove the ($\Longleftarrow$) direction, we assume that B is not deducible: $\nvdash B$. We wish to show that B is not universally valid and hence we seek an interpretation in which $\neg B$ is satisfied. This suffices, as no interpretation can satisfy both B and $\neg B$. We thus need a structure. In this argument one enlarges the language $\mathcal{L}$ to a language $\mathcal{L}'$ by adding a countable set of new constants $c_0, c_1, \ldots c_n, \ldots$ not in $\mathcal{L}$. One enumerates the formulae of the language as $\varphi_0, \varphi_1, \ldots \varphi_k, \ldots$ taking $\neg B$ as φ_0. One then builds up in an infinite series of steps a collection of sentences $\Delta = \bigcup_{n \in \mathbb{N}} \Delta_n$. At the k'th stage, when defining Δ_k one considers whether $\Delta_{k-1} \nvdash \neg\varphi_k$. If this is the case, then φ_k is added to Δ_{k-1} to obtain Δ_k, which by the case assumption, is not inconsistent. If this is not the case then $\neg\varphi_k$ is so added, and consistency is still maintained. Additionally, if φ_k is to be added and it is of the form $\exists v_m \psi(v_m)$ then a new constant, not yet used so far in the construction, c_r say, is chosen and $\psi(c_r)$ is added as well. These are the two essential features. For our discussion the first is notable: what one does is to make an infinite sequence of choices whether to take φ_k or $\neg\varphi_k$ when building Δ. We are thus picking an infinite branch through an implicitly defined infinite binary branching tree - *binary* since we are making yes/no choices as to whether to add φ_k or $\neg\varphi_k$. To argue that such a branch must exist one appeals to *König's Tree Lemma: any finitely branching infinite tree T must contain an infinite branch or path through T.* The second feature, the choice of constants c_r allows us from the branch, that is the sequence of formulae that make up Δ, to construct a structure whose domain will be built from sets of constants. Because the set of constants is countably infinite, the structure so built will also be countable - this will be important for consideration below. These details and how the language $\mathcal{L}'$ is interpreted in the resulting structure will be suppressed here.

In the case that one starts from a set of sentences Γ and assumes $\Gamma \nvdash B$ and wishes to show $\Gamma \nvDash B$, one needs a structure in which all of $\Gamma \cup \{\neg B\}$ is satisfied. The process is the same as before but starting with Δ_0 as $\Gamma \cup \{\neg B\}$.

The crucial use of the Tree Lemma argument marks the step Gödel took beyond earlier work of Herbrand and Skolem. He saw that use could be made of this infinitary principle to construct a semantic structure. Two immediate corollaries can be drawn from Gödel's argument:

Theorem 2.2 *(The Löwenheim-Skolem Theorem) If* Γ *is a set of sentences in* $\mathcal{L}$ *that is satisfiable in some structure then it is satisfiable in a countable structure.*

The argument here is just the proof of the Completeness Theorem itself: first if Γ is satisfiable in some structure then it is consistent. Now build a structure using the proof of the Completeness Theorem starting from $\Delta_0 = \Gamma$. In the resulting countable structure all of Γ will be true.

This theorem had been proven much earlier by the logicians after whom it is named. Skolem's 1923 argument had come quite close to proving a version of the Completeness Theorem. Later Gödel said:

> The Completeness Theorem, mathematically, is indeed an almost trivial consequence of Skolem 1923. However, the fact is that, at that time, nobody (including Skolem himself) drew this conclusion neither from Skolem 1923 nor, as I did, from similar considerations of his own. This blindness (or prejudice, or whatever you may call it) of logicians is indeed surprising. But I think the explanation is not hard to find. It lies in the widespread lack, at that time, of the required epistemological attitude toward metamathematics and toward non-finitary reasoning ([8]).

The second Corollary is also a simple observation but apparently was unnoticed by Gödel and others for some years before it was explicitly stated.

Theorem 2.3 *(The Compactness Theorem) Suppose a set of sentences* Γ *of a first order language* $\mathcal{L}$ *is not satisfiable in any structure. Then there is a finite subset* $\Gamma_0 \subseteq \Gamma$ *that is inconsistent.*

The argument here is that if Γ is not satisfiable, then no structure makes Γ true; this implies "$\Gamma \vDash \sigma \wedge \neg\sigma$" (interpretation: any structure that makes Γ true also makes a contradiction true - and the latter can never happen). Hence by Completeness $\Gamma \vdash \sigma \wedge \neg\sigma$. But a proof of $\sigma \wedge \neg\sigma$ from assumptions in Γ is a finite list of formulae, and so can only use a finite set Γ_0 of assumptions from Γ. Thus Γ_0 is a finite inconsistent subset.

What we see here is that the inconsistency of the set Γ can always be localised to a finite subset of sentences responsible for an inconsistency (of course there could be infinitely many non-overlapping inconsistent finite subsets depending on what Γ is).

Returning to the example of general theory above: then many such theories Γ require an infinite set of axioms. If $\Gamma \vDash B$, then we know that in any structure M_1 in which all of Γ is true, then B will be true. The Compactness argument shows that we don't have to give one reason why in another structure M_1 B is true, and another reason for being true in structure M_2: because $\Gamma \vdash B$, a finite subset Γ_0 suffices as assumptions to prove B, and the proof $\Gamma_0 \vdash B$ gives us one single reason as to why in any structure in which Γ is satisfied, B will be true.

2.1 In conclusion

To summarise: we now use the idea of a *logic* more generally as a system comprising three components: a syntactic component - of a language $\mathcal{L}'$ in some form, a deductive component of rules of inference that acts on the formulae of that language, and a third very different semantic component: the notion of a structure or interpretation of the formulae of the language $\mathcal{L}'$. It has become second nature for logicians when studying the plethora of different logics to define such by reference to these components. However it was the Completeness Theorem that showed that, certainly in the case of standard first order logic, the necessary interconnectedness of these concepts for demonstrating the adequacy of the rules of inference.

Given a logic, one of the first questions one asks, is "Is it complete? Is there a Completeness Theorem for it?" For a host of logics, modal logics, logics of use to computer science, ... completeness theorems are provable. For others they are not and the reasons why not are themselves interesting. For so-called *second order logic* where we allow the language to have quantifiers that range not over just individuals but sets of individuals and relations between individuals, the logic is incomplete: there is a deductive calculus for the logic but the lack of a Completeness Theorem renders the logic intractable (apart from special cases or areas such as in finite model theory) for proving useful theorems. It is partly for such reasons that set theorists followed Skolem into formulating set theory and thinking and reasoning in first order logic, because there is a Completeness theorem and all the useful model building tools (such as the Corollaries of the Löwenheim-Skolem Theorem and the Compactness Theorem) that come with it.

Whilst the Theorem is nowadays not seen as at all difficult, Gödel's 1929 result can be seen as distilling out exactly the relationship between the different threads or components of first order logic; by answering Hilbert and Ackermann's question he demonstrated that semantical concepts, concepts of structure, satisfiability and of truth in such structures, had to be brought in to answer the drier questions of whether the rules of the deductive calculus which told only how strings of marks on paper could be manipulated, were sufficient for producing all validities in structures satisfying a set of assumptions. It is not insignificant that the Tree Lemma brought out what would otherwise have remained implicit and hidden: the nature of the argument also required infinite sets. Indeed from our present vantage point we know that the Completeness Theorem *must* use the Tree Lemma: if one assumes the Completeness theorem then we can prove the Tree Lemma from it: they are thus equivalent. But these discoveries were to come much later.

3 Incompleteness

To understand the Incompleteness Theorems we need to discuss further Hilbert's programmatic attempt to put mathematics on a consistent footing, by returning to the position in the late 1920's.

Hilbert had the belief that mathematics could be made secure from possible paradoxes that dogged the early years of set theory, by means of a series of *finitary consistency proofs.* A consistency proof for a theory T stated in a language $\mathcal{L}$ would be some form of proof that one could not have $T \vdash \varphi \wedge \neg\varphi$ for some (or any) formula φ of $\mathcal{L}$. But what should count as a legitimate proof or argument that T was consistent? To be of any value the argument had to use indubitably secure means, that themselves were not open to question. Hence the notion of 'finitary'. What Hilbert meant by this term was never given an absolutely explicit definition by Hilbert or his followers, even though the term was discussed by Hilbert on may occasions. Hilbert divided mathematical thinking into the 'real' and the 'ideal'. The former was essentially the mathematics of number theory, and the subject matter was 'finitist objects' the paradigm being stroke sequences " $||||, \ldots, |$" representing numbers, together with simple repetitive operations performed on them. Hilbert formulated various epistemological constraints on such objects, such as surveyability etc. Operations on them (such as concatenation corresponding to addition) should again be of a simple kind. The truth or otherwise of such finitarily expressed statements was open to inspection as it were.

At a later stage Hilbert and Bernays used the idea of *primitive recursion* (a scheme of building up number functions by simple recursion schemes) and forms of induction that could be expressed in the language of arithmetic $\mathcal{L}_A$ by quantifier free formulae, stating that such should count as finitary. As a deeper discussion of what constituted finitary methods would take us too far afield, we shall let the idea rest with this version.

Ideal mathematics according to Hilbert could prove for us, for example, *quantified* statements in number theory. But if we really were in possession of finitary means for proving the consistency of that piece of idealised mathematics, we could have confidence in the truth of that quantified number theoretical statement, in a way that we could never have otherwise, because of the complexity involved in surveying all the natural numbers required by the quantifiers. Hilbert is sometimes paraphrased as saying that "consistency of a theory yields the existence of the mathematical objects about which it speaks." But his intentions here were more nuanced, and more restricted.

Hilbert had given a thorough-going axiomatisation of geometry in *Foundations of Geometry (1899)*. The consistency of the axioms of higher dimensional geometry could be reduced to that of plane geometry alone, and in turn the latter could be seen to be consistent by interpreting it in analysis, and thus as reducing the problem to that of the consistency of analysis. Given the emergence of the set theoretical paradoxes at the time, Hilbert wanted a proof of the consistency of analysis that was direct and did not involve a reduction using, say Dedekind cuts of sets of reals (as this might be in danger of importing unsafe methods). In his 1900 problems list, at second place he gave this consistency problem for analysis.

Hilbert's thinking was that the logical system of Principia Mathematica was inadequate for their purposes and so developed a new calculus for logical expressions (the ϵ -calculus). In 1924 Ackermann developed an argument for the consistency of analysis, but von Neumann who took a deep interest in the foundations of mathematics at that time (formulating an axiomatic set theory, and the notion of 'von Neumann ordinal' number) saw an error. By the late 1920's Ackermann had developed a new ϵ-substitution method and there was optimism that with this, a new consistency proof for analysis given in this calculus could be given. This was even announced by Hilbert at the 1929 International Congress of Mathematics. But it was to be short-lived.

3.1 Gödel's First Incompleteness Theorem

Gödel worked in Vienna, and quite independently of Hilbert's Göttingen school. He expressed surprise concerning attempts at proving the consistency of analysis.

> It is mysterious why Hilbert wanted to prove directly the consistency of analysis by finitary methods. I saw two distinguishable problems: to prove the consistency of number theory by finitary number theory and to prove the consistency of analysis by number theory . . . Since the domain of finitary number theory was not well-defined, I began by tackling the second half . . . I represented real numbers by predicates in number theory . . . and found that I had to use the concept of truth (for number theory) to verify the axioms of analysis. By an enumeration of symbols, sentences and proofs within the given system, I quickly discovered that the concept of arithmetic truth cannot be defined in arithmetic. If it were possible to define truth in the system itself, we would have something like the liar paradox, showing the system to be inconsistent . . . Note that this argument can be formalized to show the existence of undecidable propositions without giving any individual in-

> stances. (If there were no undecidable propositions, all (and only) true propositions would be provable within the system. But then we would have a contradiction.) . . . In contrast to truth, provability in a given formal system is an explicit combinatorial property of certain sentences of the system, which is formally specifiable by suitable elementary means . . . (in [15])

In September of 1930 there was to be a joint meeting of several academic societies in Königsberg. Various members of the Wienerkreis, Carnap, Feigl, Waismann would speak at the Conference on Epistemology of Exact Sciences. Hilbert would attend and deliver his farewell address as President of the German Mathematical Union. At the first meeting Carnap, Heyting and von Neumann would give hour addresses on logicism, intuitionism and formalism respectively. Gödel was to give a short contributed talk on results relating to his thesis. A few days before Gödel met Carnap in a cafe to discuss the trip and then out of the blue, related to Carnap his theorem concerning incompleteness of systems similar to PM. It seems that although Carnap noted in a memorandum "Gödel 's discovery: incompleteness of the system of PM, difficulty of the consistency proof", he could not have understood exactly what Gödel had achieved. He met Gödel three days later for a further discussion, but a week later in Königsberg, he would still in the ensuing discussions emphasise the role of completeness of a system being an overriding criterion for a formal theory.

Let us suppose a formal system P has sufficient syntax to talk about numerals '0', '1', . . . 'k', . . . (as names for the corresponding actual numbers) as well as symbols for some basic arithmetical operations such as the successor operation of adding one, $x+1$, addition and multiplication in general. The system embodied in Principia Mathematica is such a system. Also the Dedekind-Peano axioms for number theory ("PA") are expressed in such a language and moreover adopt axioms that allow the use of such operations with their normal properties, together with the notion of mathematical induction.

A formal theory P is called *ω-consistent*, if whenever we have that P proves all of $\varphi(\text{'0'}), \varphi(\text{'1'}), \ldots, \varphi(\text{'}k\text{'}), \ldots$ in turn, then it is not the case that P proves $\exists v \neg\varphi(v)$. Because of the infinite hypothesis here, this is a stronger requirement on a theory P than simple consistency alone.

Theorem 3.1 (Gödel: The First Incompleteness Theorem (1930) [5]) *Let P be a formal theory such as that of Principia Mathematica expressed in a suitable language $\mathcal{L}$. If the theory P is ω-consistent, then there is a sentence γ_P of $\mathcal{L}$ such that:*

$$P \nvdash \gamma_P \text{ and } P \nvdash \neg\gamma_P$$

The existence of such a sentence γ_P (a 'Gödel sentence' for the system P) shows that P cannot derive any sentence or its negation whatsoever. The system is 'incomplete' for deciding between γ_P and $\neg\gamma_P$. If the sentence γ_P is purely a number-theoretic statement then the conclusion is that there will be sentences which are presumably true of false of the natural number structure (as the case may be) but the system P is necessarily incapable of deriving either. If some version of 'ideal' mathematics could decide between the two, then that ideal mathematical argument could not be given within the formal system P.

Gödel said nothing about the Incompleteness results during his own talk in Königsberg, and only mentioned them in a rather casual manner during a round-table discussion on the main talks that took place on the last day of the conference. Hilbert was attending the conference and would give his farewell address as President of the German Mathematical Union, but did not attend the session at which Gödel made his remarks. It may well have been the case that, with the exception of von Neumann, no one in the room would have understood Gödel's ideas. However von Neumann realised immediately the import of Gödel's result.

We discuss here the proof of the theorem, which proceeded via a method of encoding numerals, then formulae, then finite sequences of formulae (which might constitute a proof in the system P) all by numbers. This coding has become known as "Gödel coding (or numbering)" and when a numeral 'k' for the number k is inserted into a formula $\varphi(v_0)$, resulting in $\varphi(`k')$, then the latter can be said to be assigning the property expressed by φ to the number k (the which may encode a certain proof, for example). Thus indirectly formulae within the language can talk about properties of other formulae, and properties of sequences of formulae, and so forth, *via* this coding.

How the coding is arranged is rather unimportant as long as certain simplicity criteria are met. It is common to use prime numbers for this. Suppose the language $\mathcal{L}$ is made up from a symbol list: $(\,,\,), 0, S, +, \times, =, \neg, \wedge, \rightarrow, \exists, v_0, v_1, \ldots, v_k, \ldots$ (where S denotes the successor function, and there is an infinite list of variables v_k etc.) To members on the list we respectively assign code numbers: $1, 2, \ldots, 9, 10, 11, 12+k$ (for $k \in \mathbb{N}$). (Hence the variable v_2 receives code number $12 + 2 = 14$.) For the symbol s let $c(s)$ be its code in the above assignment.

A string such as: $\exists v_1(0 = S(v_1))$ expresses (the false) statement that some number's successor is 0. In general a string of symbols $s_1 s_2 \cdots s_k$ from the above list can be coded by a single number:

$$c(s_1 s_2 \cdots s_k) = 2^{c(s_1)}.3^{c(s_2)}.\cdots.p_k^{c(s_k)}$$

where p_m is the m'th prime number.

Given a number, by computing its prime factors we can ascertain whether it is a) a code number of a proof, or b) of a formula, or c) of just a single symbol, and moreover, which symbol, formula, or proof, in a completely algorithmic fashion. The *number* 2 has the *numeral term* or *name*: S(S(0)) from $\mathcal{L}$ and we abbreviate the latter as '2'.

We shall write 'φ' for the numeral of the code number of the formula φ. 'φ' functions thus as a *name* for φ. Then the formula $0 = S(0)$ has as code: $c(0 = S(0)) = 2^3.3^7.5^4.7^1.11^3.13^2$. Suppose the latter value is k say, then '$0 = S(0)$' is 'k'. The efficiency of the coding system is entirely irrelevant: we only require the simple algorithmicity of the coding processes that map the syntax of the language in a (1-1) fashion into $\mathbb{N}$ in a recoverable fashion. Those codes are then named by the appropriate numerals which are terms in $\mathcal{L}$.

A further minimal requirement on the formal system P is that it be sufficiently strong to be able to *represent* predicates or properties of natural numbers which are definable over the standard structure. By this is meant the following: let $\varphi(v_0, \ldots, v_k)$ be any formula. *P represents* φ if for all natural numbers $n_0, \ldots, n_k$ both of the following hold:

$$\mathbb{N} \models \varphi[n_0, \ldots, n_k] \Rightarrow P \vdash \varphi(`n_0', \ldots, `n_k');$$

$$\mathbb{N} \not\models \varphi[n_0, \ldots, n_k] \Rightarrow P \nvdash \varphi(`n_0', \ldots, `n_k').$$

One may show that any primitive recursive (p.r.) predicate, as mentioned above, can be represented in a system such as PM. Gödel then developed a series of lemmas that showed that operations on syntax could be mimicked by p.r. operations on their code numbers. For example, there is a p.r. predicate $R_\wedge(v_0, v_1, v_2)$ so that if χ is $\varphi \wedge \psi$, then in P we may prove $R_\wedge(`\varphi', \psi', `\chi')$. This is merely a reflection of the fact that we can calculate a code number for χ once those for φ and ψ are given. Ultimately though, the fact that $R_\wedge(`\varphi', `\psi', `\chi')$ may hold is because of certain arithmetical relationships between the code numbers, not because of any 'meaning' associated to those numbers (which we attribute to them because they code particular formulae).

Similarly if we have three formulae ϕ, ψ, χ and ψ happens to be the formula $\phi \rightarrow \chi$, then we could view this triple if it occurred in a list of formulae which may or may not constitute a derivation in P, that they stand in for a correct application of Modus Ponens on the first two formulae yielding the third. Gödel shows that the relation $P_{MP}(v_0, v_1, v_2)$ which holds of a triple of numbers as above if indeed that third follows by application to the first two in the appropriate order, is primitive recursive: then we have:

$$\mathbb{N} \models P_{MP}[c(\varphi), c(\varphi \rightarrow \chi), c(\chi)], \text{ and hence } P \vdash P_{MP}(`\varphi', `\varphi \rightarrow \chi', `\chi').$$

In short, syntactic operations, the checking of formulae for correct formation, and substitution of terms for variables, etc., up to the concept of the checking of a number that codes a list of formulae whether that list constitutes a correct proof of the last formula of the list, these are all p.r. relations of the code numbers concerned. For the last then Gödel constructs a p.r. predicate $\mathrm{Prf}(v_0, v_1)$ which is intended to represent in P (in the above sense)

"v_0 is the code number of a proof of the last formula which has code number v_1."

If then, k is the code number of a correct proof in P of the last formula σ of the proof, then the number relation $\mathrm{Prf}[k, c(\sigma)]$ holds, and indeed is itself a relation between numbers, moreover it is provable in P because P can represent any p.r. predicate: $P \vdash \mathrm{Prf}(`k', `\sigma')$. Again, to repeat, the fact that $\mathrm{Prf}(x, y)$ holds just says something about a particular numerical relationship between x and y, which either holds (or does not) irrespective of the interpretation we may put on it in terms of formulae, correct proofs, etc., etc.

Having done this, $\exists v_0 \mathrm{Prf}(v_0, v_1)$ is naturally interpreted as "There is a proof of the formula with code number v_1." This existential statement we abbreviate $\mathrm{Prov}(v_1)$. This final relation, due to the existential quantifier turns out not now to be p.r., but this does not matter.

Lemma 3.2 (The diagonal lemma) *Given any formula $\varphi(v_0)$ of the number theoretic language we may find a sentence θ so that $P \vdash \theta \leftrightarrow \varphi(`\theta')$.*

Proof: We let s_e be the string with code e, thus $c(s_e) = e$. Define the function $r : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ by:

(1) $r(e, n) = c(\forall v_1(v_1 = `e' \longrightarrow s_n))$.

Then $r(e, n)$ is p.r. being a composition of simple p.r. functions, indeed just multiplications involving some primes, the number n, and the codes for the symbols $S, \forall, =, \to$ *etc.* occurring in the string $\forall v_1(v_1 = `e' \longrightarrow s_n)$.

This entails *inter alia*, that r is representable in P. We now define a *diagonal function* $d : \mathbb{N} \longrightarrow \mathbb{N}$ by $d(e) = r(e, e)$. Then d is p.r. and so representable too: there is a formula $D(v_1, v_0)$ that represents the graph of d as above. Moreover it can be shown:

(2) $P \vdash \forall v_0(D(`n', v_0) \longleftrightarrow v_0 = `d(n)')$.

Given our φ let $\psi(v_1)$ be $\exists v_0(D(v_1, v_0) \wedge \varphi(v_0))$. Let $h = c(\psi(v_1))$. Then, to spell it out: s_h is $\psi(v_1)$. Let θ be $\forall v_1(v_1 = `h' \longrightarrow s_h)$. Then by our definition of s_h:

$$P \vdash \theta \longleftrightarrow \forall v_1(v_1 = `h' \longrightarrow \psi(v_1))$$

by logic: $\longleftrightarrow \psi(`h')$

Using (2), Def of ψ: $\longleftrightarrow \varphi(`d(h)')$

But $`\theta'$ is $`d(h)'$, so we are done. Q.E.D.

The lemma, despite its construction, is less mysterious than it seems: it is just a fixed point construction. Indeed there is nothing terribly particular about the choice of θ: one may show for each choice of φ that there are in fact infinitely many different formulae θ' satisfying the lemma: the argument has just provided one of them. One should note however that the proof is completely constructive (and can be run in intuitionistic logic): a θ is given, together with a proof that it satisfies the biconditional.

We can immediately derive a corollary which is often referred to as that "truth is undefinable." What this means is that there is no formula of the language $\tau(v_1)$ for which we have that for any k: $\mathbb{N} \models \tau(`k')$ if and only if $\mathbb{N} \models \sigma$ where $c(\sigma) = k$. For, assume the axioms of P are true in $\mathbb{N}$. Suppose τ were such formula, thence we should have by applying the Diagonal Lemma to the formula $\neg\tau(v_1)$, and then Soundness Theorem, that there is some sentence θ:

$$\mathbb{N} \models \theta \leftrightarrow \neg\tau(`\theta'). \qquad (**)$$

However this θ is like a liar sentence: for if it is true in $\mathbb{N}$, then so is $\neg\tau(`\theta')$; but by the assumption on $\tau$ then we also have $\mathbb{N} \models \tau(`\theta')$. But by Soundness then $\neg\tau(`\theta')$ is also true of the natural numbers, which is absurd! Hence $\neg\theta$ is true in $\mathbb{N}$; but this immediately leads to the same form of contradiction. Hence there is no such formula $\tau(v_1)$. We have shown that the set of arithmetical truths is not arithmetically definable:

Corollary 3.3 (Tarski: The Undefinability of Truth) *There is no formula $\tau(v_1)$ of $\mathcal{L}$ for which we have that for any k: $P \vdash \tau(`k')$ if and only if $\mathbb{N} \models \sigma$ where $c(\sigma) = k$.*

This theorem, usually attributed to Tarski, is easier to establish than the Incompleteness Theorem to come, and seems to have been also known to Gödel (see the quotation at the beginning of this section). Gödel seems to have come to the realisation that an Incompleteness Theorem would be provable precisely because provability within a formal system such as P was, unlike truth, representable within P. That was the key.

In the proof of Theorem 3.1 we shall apply it with $\neg\exists v_o \mathrm{Prf}(v_0, v_1)$ as φ. This yields

$$P \vdash \gamma \leftrightarrow \neg\exists v_o \mathrm{Prf}(v_0, `\gamma') \qquad (*)$$

We emphasise once more that the diagonal lemma says nothing about truth or meaning or satisfaction in the structure $\mathbb{N}$: it says something only about provability of certain formulae in the formal system P, formulae which express certain equivalences between sentences and formulae containing certain numeral terms. And that holds for the expression $(*)$ too.

Proof of Theorem 3.1

Suppose γ is as at $(*)$ above. Suppose for a contradiction that $P \vdash \gamma$. Let n be the code number of such a proof. Combining that proof with $(*)$ yields that $P \vdash \neg\exists v_o \mathrm{Prf}(v_0, `\gamma')$. However $n$ is after all a code of a proof of $\gamma$ and as $P$ represents Prf, then $P \vdash \mathrm{Prf}(`n', `\gamma`)$. The conclusions of the last two sentences imply that P is inconsistent. This is a contradiction. Hence $P \nvdash \gamma$.

Now from the last statement we conclude that no natural number n is a code of a proof of γ in P. Hence, as P represents Prf, we have for all n: $P \vdash \neg Prf(`n', `\gamma')$. The assumption of ω-consistency now requires that $P \nvdash \neg\gamma$ Q.E.D.

Note that the assumption of ω-consistency of P is only deployed in the second part of the argument to show $P \nvdash \neg\gamma$. Rosser later showed how to reduce this assumption to that of ordinary consistency by the clever trick of applying the Diagonal Lemma to the formula
"$\forall v_0(\neg Prf(v_0, v_1) \vee \exists v_2$
(v_2*is the code of a shorter proof than* v_0 of the formula $\neg v_1$))."

Another remark: it is often asserted that the Incompleteness Theorem states that "there are true sentences (in arithmetic, or in a formal theory, or in ...) that are not provable". This is not a strictly accurate account of the theorem: the theorem itself mentions only deduction in formal theories and says nothing about truth. However the Gödel sentence γ is indeed true if we assume the consistency of P: by assuming the consistency of P we concluded that $P \nvdash \gamma$, that is $\neg\exists v_o \mathrm{Prf}(v_0, `\gamma')$ is true, which of course is γ itself. We thus have "$Con(P) \Rightarrow \gamma$." But note this is not (yet) an argument *within* the deductive system P.

Yet another remark: the use of the self-referential Gödel sentence γ that asserts its own unprovability sometimes leads to the impression that all undecidable statements unprovable in such a theory as PM must of necessity have some degree of self-reference. However this is false. We comment on this again below. Similarly we do not refer to *the* Gödel sentence γ for PM, since one can show there are infinitely many such.

3.2 The Second Incompleteness Theorem

Von Neumann left the room in Königsberg realising the import of what Gödel had achieved. He may have been the only person to do so: Hans Hahn, Gödel 's thesis supervisor and who was present, made no mention of the Incompleteness results. Neither the transcript of the session, nor the subsequent summary prepared by Reichenbach for publication made any mention even of Gödel's participation. Although Gödel attended Hilbert's lecture, the two never met (or corresponded later), and the Viennese party then returned home. If von Neumann approached Hilbert whilst at the meeting to appraise him of the results, then it was not recorded.

Von Neumann shortly realised that more could be obtained by these methods. We can express the consistency of the formal system P by the assertion that from P we cannot prove a contradiction, $0 = 1$ say. We thus let $Con(P)$ be the sentence $\neg\exists v_0\, Prf(v_0, `0 = 1\text{'})$.

THEOREM 3.4 (GÖDEL SECOND INCOMPLETENESS THEOREM) *Let P be a formal system as above. Then* $P \nvdash Con(P)$.

Proof: The essence of the argument is that we may formalise the argument of "$Con(P) \Rightarrow \gamma$" at the end of the last section in number theory, and so in a system such as P. We should thus have shown

$$P \vdash Con(P) \rightarrow \gamma. \qquad (**)$$

We know from the First Incompleteness Theorem that $P \nvdash \gamma$. Hence $P \nvdash Con(P)$.
Q.E.D.

Von Neumann realised that the something akin to the Second Incompleteness Theorem would follow by the same methods Gödel had used for the First, and in th November after Königsburg, wrote to Gödel. However Gödel had himself already realised this and submitted the Second Theorem for publication in October. Of course the above is extremely sketchy: the devil then is in the detail of how to formalise within the theory P, the inference above from $Con(P)$ to $\neg\exists v_o \text{Prf}(v_0, `\gamma\text{'})$, we thus need to show "$P \vdash Con(P)$ implies $P \vdash \neg\exists v_o \text{Prf}(v_0, `\gamma\text{'})$" *within* P itself. Gödel did not publish these rather lengthy details himself, they were first worked out by Hilbert & Bernays in 1939 ([11]).

4 THE SEQUEL

There are many points of interest and possibilities for elaboration in these theorems, and hence the extensive academic literature on them. Gödel left deliberately vague what he meant by 'formal system'. He said at the time that it was not clear what a formal system was or how it could be delineated. He stated his theorems as being true in the system of PM and for "related systems". It was clear that a similar system that had sufficient strength to prove the arithmetical facts needed in the coding and deduction processes would do. Hence the theorems were more general than had they been restricted to just PM. It was left to Alan Turing [14] five years later to give a mathematical definition of 'computable' that could be used to demarcate what a formal system was: a set of axioms and rules that could be 'recognised' by a Turing machine, and so that a programmed machine could decide whether a derivation in the system was correct. In the intervening period Gödel had speculated on what an 'effectively given' formal system could subsist, and rejected proposals from Church for such. However he recognised that what Turing proposed was definitive:

> "When I first published my paper about undecidable propositions the result could not be pronounced in this generality, because for the notions of mechanical procedure and of formal system no mathematically satisfactory definition had been given at that time . . . The essential point is to define what a procedure is."
> "That this really is the correct definition of mechanical computability was established beyond any doubt by Turing." ([6])

We give now a more modern statement of the First Incompleteness Theorem.

THEOREM 4.1 (GÖDEL; FIRST INCOMPLETENESS THEOREM) *Let P be a computable set of axioms for number theory that contain the axioms PA. Then if P is consistent, it must be incomplete: there is a sentence* γ_P *so that* $P \nvdash \gamma_P$ *and* $P \nvdash \neg\gamma_P$.

4.1 Consequences for Hilbert's program.

The most dramatic consequences of the theorems were for Hilbert's program of establishing the consistency of mathematics, and in particular focussing on arithmetic, by 'finitary means'. As we have discussed above finitary methods were to be of a restricted kind: the writing of, and operations on, finite strings of marks on paper, and using intuitive reasoning that "includes recursion and intuitive induction for finite existing totalities," (Hilbert in a 1922 lecture). However finitary reasoning was also left somewhat vague, but clearly the usual arithmetical operations on numbers (and this for him included exponentiation) counted as finitary. [10] seems to have settled on primitive recursive arithmetic, *PRA*, which allows the definition of functions by primitive recursion schemes, and induction on quantifier free formulae. If this constituted the 'finitary means' of Hilbert, then indeed the Incompleteness Theorems dealt a death blow to this program. Von Neumann thought so, and Weyl, writing in his 1943 obituary of Hilbert described it as a "catastrophe." Gödel was initially more circumspect: he did not consider at that time that it had been argued that all methods of a 'finitary' nature could be formalised in, say, *PM*. At a meeting of the Vienna Circle in January 1931 he said that he thought that von Neumann's assertion that all finitary means could be effected in one formal system (and thus the Incompleteness Theorems should have the devastating effect on Hilbert's programme) was the weak point in von Neumann's argumentation. Again in his 1931 paper Gödel wrote that there might be finitary proofs that could not be written in a formal system such as PM. It was hard to bring forward convincing arguments in either direction at this point: there was no clear notion of 'formal system' - this had to wait until 1936 for Turing - and there was also something of a confusion about intuitionistic logic: both von Neumann and Gödel thought that intuitionistic logic could count as finitary reasoning. However this turned not to be the case: in 1933 Gödel showed that classical arithmetic could be interpreted in the intuitionistic version (known as Heyting Arithmetic HA and which only uses intuitionistic axioms of logic and rules of inference), thus ruling out the idea of using intuitionistic logic to help codify finitary reasoning, since the consistency of HA alone would now give the consistency of PA.

However by 1933 ([7]) he had changed his mind and acknowledged that all finitary reasoning could indeed be formalised in the axiom system of *Peano Arithmetic* ("PA") which in particular allowed mathematical induction for formulae with quantifiers. He later remarked in several places (as in the quotation above) that Turing's precise definition of a formal system convinced him that the Incompleteness Theorems refuted Hilbert's programme.

4.2 Salvaging Hilbert's program

It was not recorded when precisely Hilbert he learnt of the Incompleteness results is not recorded. Bernays, when he had previously suggested to Hilbert that after all a completeness proof might not be possible, reported that Hilbert reacted with anger, as he ddi to the results themselves. Nevertheless attempts were made to recover as much of the program as was consistent with the Incompleteness Theorems. Bernays (who also in correspondence to Gödel indicated that he was also not convinced that all finitary reasoning could be captured by a single system) in particular sought to discover modes of reasoning that could count as finitary but avoid being captured by the formal systems of the kind Gödel discussed. Hilbert and Bernays soon afterwards reacted positively by trying to see what could be done. By 1931 Hilbert was suggesting that an 'ω-rule' might be deployed where from an infinite set of deductions proving $P(0), P(1), \ldots, P(n), \ldots$ one would be allowed to infer $\forall k P(k)$ might be permissible as a form of reasoning. It is unknown whether Hilbert was reacting directly to the hypothesis of ω-consistency in the first version of the First Incompleteness Theorem, since indeed the displayed Gödel sentence was a pure $\forall$ sentence which would be "proved" if the ω-rule were allowed. But the rule itself was not perceived as being finitary.

However the major and most striking advance here came from G. Gentzen who showed that the consistency of Peano Arithmetic could be established after all, if one allowed inductions along well orderings up to the first "epsilon number" (ϵ_0 - the first fixed point of the ordinal exponentiation function $\alpha \to \omega^\alpha$). Clearly these are not finitary operations in any strict sense, but nevertheless Gentzen's work opened a whole area of logical investigation of formal systems involving such transfinite inductions - thus opening the area of proof theory and 'ordinal notation systems.' (Gentzen also had the result exactly right: inductions bounded below ϵ_0 would not have sufficed.)

4.3 After Incompleteness

Had the Hilbert Program succeeded, it would be showing that ideal mathematics could be reduced to finitary 'real' mathematics: the consistency of a piece of ideal mathematics could be shown using just real, finitistic mathematical methods. A *relativized Hilbert program* seeks to reduce an area of classical mathematics to some theory, necessarily stronger than finitary mathematics. Feferman has argued that most of mathematics needed for physics, for example, can be reduced to *predicative* systems which can be proof-theoretically characterised using ordinal notation systems albeit longer than ϵ_0, but still of a small or manageable length.

As the Second Incompleteness Theorem had shown, given a formal theory T_0 (such as *PA*) we have that $T_0 \nvdash Con(T_0)$. But we may add $Con(T_0)$ as a new axiom itself to T_0 thereby obtaining a somewhat stronger deductive theory. (It is not that the Theorem casts any doubt on the theory T_0 or its consistency, it is only that it demonstrates the impossibility of formalising a proof of that consistency within T_0.)

Thus we may set:

$$T_1 : T_0 + \mathrm{Con}(T_0)$$

the thinking is that since we accept *PA* and believe that its axioms are true of the natural number structure, we should also accept that *PA* is consistent. (Whilst a T_0, if consistent, neither proves $Con(T_0)$ nor its negation, it would be presumably perverse to claim that $\neg Con(T_0)$ is the correct choice of the two to make here.)

Continuing, we may define:

$$T_{k+1} : T_k + \mathrm{Con}(T_k) \text{ for } k < \omega, \text{ and then: } \quad T_\omega = \bigcup_{k<\omega} T_k.$$

Having collected all these theories together as T_ω, we might continue:

$$T_{\omega+1} = T_\omega + \mathrm{Con}(T_\omega) \textit{ etc.}$$

We thus obtain a transfinite hierarchy of theories. What can one in general prove from a theory in this sequence? Turing called these theories "Ordinal Logics" and was the first to investigate the question as to what extent such a sequence could be considered complete:

Question: Can it be that for any problem, or arithmetical statement A there might be an ordinal α so that T_α proves A or $\neg A$? And if so can this lead to new knowledge of arithmetical facts?

Such a question is necessarily somewhat vaguely put, but anyone who has considered the Second Incompleteness Theorem comes around to asking similar or related questions as these. The difficulty with answering this, is that much has been swept under the carpet by talking rather loosely of 'T_α' for $\alpha \geq \omega$. This is a subtle matter, but there seems no really meaningful way to arrive at further arithmetical truths. Such iterated consistency theories have been much studied by Feferman and his school (see Franzen [2]).

Lastly we consider the question of Gödel sentences themselves. Much has been studied and written on this theme alone. However the use of the diagonal lemma leading to the self-referential nature of such a Gödel sentence gives a contrived feeling to the sentence. (One should also beware the fact that not any fixed point of a formula $\varphi(v_1)$ is necessarily stating that it says of itself that it satisfies φ.) Could there be propositions that were more genuinely mathematical statements, and were not decided by *PA*? Gödel's methods produced only sentences of the diagonal kind, and the problem was remarkably difficult. Some 40 years were to pass before the first example was found by Paris and Harrington (concerning so-called Ramsey-like partition principles). Since then many more examples of mathematically interesting sentences independent of *PA* have been discovered.

References

[1] J. Dawson. *Logical Dilemmas: the Life and Work of Kurt Gödel.* A.K. Peters, Wellesley, Mass., 1997.

[2] T. Franzen. *Inexhaustibility: A non-exhaustive treatment*, volume 16 of *Lecture Notes in Logic*. ASL/A.K.Peters, 2004. 20

[3] T. Franzen. *Gödels Theorem: an incomplete guide to its use and abuse*. A.K.Peters, 2005.

[4] K. Gödel. Die Vollständigkeit der Axiome del logischen Funkionkalkuls. *Monatsheft für Mathematik und Physik*, 37(transl. in [?]):349–360, 1930. 4

[5] K. Gödel. Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme. *Monatsheft für Mathematik und Physik*, 38(transl. in [?]):173–198, 1931. 10

[6] K. Gödel. In S. Feferman, J. W. Dawson Jr., S. C. Kleene, G. H. Moore, R. M. Solovay, and J. van Heijenoort, editors, *Collected Works Vol III: Unpublished Essays and Lectures*, pages 166–168. Oxford University Press, 1995. 17

[7] K. Gödel. The present situation in the foundation of mathematics. In S. Feferman, J. W. Dawson Jr., S. C. Kleene, G. H. Moore, R. M. Solovay, and J. van Heijenoort, editors, *Collected Works Vol III: Unpublished Essays and Lectures*. Oxford University Press, 1995. 18

[8] K. Gödel. In S. Feferman, J. W. Dawson Jr., S. C. Kleene, G. H. Moore, R. M. Solovay, and J. van Heijenoort, editors, *Collected Works Vol V: Unpublished Essays and Lectures*, pages 304–323. Oxford University Press, 2005. 6

[9] D. Hilbert and W. Ackerman. *Grundzüge der theoretischen Logik*. Springer-Verlag, Berlin, 1929. 3, 4

[10] D. Hilbert and P. Bernays. *Grundlagen der Mathematik*, volume I. Springer-Verlag, Berlin, 1934. 17

[11] D. Hilbert and P. Bernays. *Grundlagen der Mathematik*, volume II. Springer-Verlag, Berlin, 1939. 16

[12] J. Kennedy. Kurt Gödel. *Stanford Encyclopedia of Philosophy*, ¡http://plato.stanford.edu/entries/goedel/¿, 2011.

[13] R. Smullyan. *Gödel's Incompleteness Theorems*. O.U.P., Oxford, New York, 1993.

[14] A.M. Turing. On Computable Numbers with an application to the *Entscheidungsproblem*. *Proceedings of the London Mathematical Society*, 42(2), 1936. 16

[15] H. Wang. *A Logical Journey: From Godel to Philosophy*. MIT Press, Boston, USA, 1996. 10

[16] R. Zach. Hilbert's program. *Stanford Encyclopedia of Philosophy*, ¡http://plato.stanford.edu/entries/hilbert-program/¿, 2003.

Further Reading:

References

[1] J. Dawson. *Logical Dilemmas: the Life and Work of Kurt Gödel.* A.K. Peters, Wellesley, Mass., 1997.

[2] T. Franzen. *Inexhaustibility: A non-exhaustive treatment*, volume 16 of *Lecture Notes in Logic*. ASL/A.K.Peters, 2004. 20

[3] T. Franzen. *Gdel's Theorem: An Incomplete Guide to Its Use and Abuse*. A.K.Peters, 2005.

[4] J. Kennedy. Kurt Gödel. *Stanford Encyclopedia of Philosophy,* , <http://plato.stanford.edu/entries/goedel/>, 2011.

[5] R. Smullyan. *Gdel's Incompleteness Theorems*. O.U.P., Oxford, New York, 1993.

[6] R. Zach. Hilbert's program. *Stanford Encyclopedia of Philosophy,* ,
<http://plato.stanford.edu/entries/hilbert-program/>, 2003.