RECOGNIZABLE SETS AND WOODIN CARDINALS: COMPUTATION BEYOND THE CONSTRUCTIBLE UNIVERSE

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ABSTRACT. We call a subset of an ordinal λ *recognizable* if it is the unique subset x of λ for which some Turing machine with ordinal time and tape and an ordinal parameter, that halts for all subsets of λ as input, halts with the final state 0. Equivalently, such a set is the unique subset x which satisfies a given Σ_1 formula in $\mathrm{L}[x]$. We further define the *recognizable closure* for subsets of λ by closing under relative recognizability for subsets of λ.

We prove several results about recognizable sets and their variants for other types of machines. Notably, we show the following results from large cardinals.

- Recognizable sets of ordinals appear in the hierarchy of inner models at least up to the level Woodin cardinals, while computable sets are elements of L.
- A subset of a countable ordinal λ is in the recognizable closure for subsets of λ if and only if it is an element of the inner model M^{∞}, which is obtained by iterating the least measure of the least fine structural inner model M_1 with a Woodin cardinal through the ordinals.

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1. INTRODUCTION

Infinitary machine models of computation provide an attractive approach to generalized recursion theory. The first such model, Infinite Time Turing Machines, was introduced by Hamkins and Lewis [HL00].[1] A motivation for considering such machine models is that they capture the notion of an effective procedure in a more general sense than classical Turing machines, thus allowing effective mathematics of the uncountable (see [GHHM13]

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for other approaches on this topic). Such models are usually obtained by extending the working time or the working space of a classical model of computation to the transfinite. The strongest such models considered so far, to our knowledge, are Ordinal Turing Machines (OTMs) and the equivalent Ordinal Register Machines (ORMs). These were defined and studied by Peter Koepke and others [Koe06a, KS08]. It is argued in [Car13b] that OTM-computability adequately expresses the intuitive notion of an idealized computor working in transfinite time and space.

The sets of ordinals which are OTM-computable from ordinal parameters are simply the constructible sets of ordinals. This is rather restrictive and it was asked whether one should study machines that have an extra function allowing them to go outside of L into core models [FW11]. This idea suggests a strengthening of the underlying machine model.

Here we follow a different approach and consider the notion of recognizability. This means that for some initial input, some program will stop with output 1 if the input is the object in question, and stop with output 0 otherwise. It is thus a form of implicit definability. The fact that recognizability is strictly weaker than computability was first noticed for Infinite Time Turing Machines and is called the *lost melody phenomenon* [HL00] (see also [Car13a]). This roughly means that implicit computability (and implicit definability) are different from computability (and definability). Moreover, it should be stressed that recognizability is equivalent to Σ_1-definability in models of the form $L[x]$, where x is the input of the computation (see Lemma 2.3).

The notion of recognizability was independently first considered for OTM-computability by Dawson [Daw09]. He showed that the OTM-computable sets coincide with the recognizable sets, without allowing ordinal parameters. Moreover, he showed that if this was relaxed to allow constructibly countable ordinal parameters, then every recognizable set is still constructible. He further showed that $0^\sharp$ (see e.g. [Kan09, Section 9], [Sch14a, Definition 10.37]), if it exists, is recognizable from some uncountable cardinal, and that adding Cohen reals over L does not add recognizable sets. In fact, an OTM can recognize $0^\sharp$ from the parameter ω_1 [Car13a], although $0^\sharp$ is not constructible.

As for computability, it is natural to study relativized recognizability. Intuitively, an object x is recognizable relative to an object y if this can be used to identify x. This concept is illustrated by our inability to recognize a radioactive stone, while it is possible to recognize a Geiger counter and use this to identify the stone. It is moreover natural to iterate relative recognizability in finitely many steps and thus obtain the *recognizable closure* $\mathcal{C}$ (see Definition 2.6).

We attempt a systematic study of recognizability, its variants, and their relationships to other kinds of implicit definability. Moreover, we study the recognizable closure and particularly its relationship to HOD and other well-known inner models. Thus we address the following questions.

- How does recognizability change with different ordinal parameters?
- How can the recognizable closure be characterized?

The results of this paper show that while recognizability from fixed ordinal parameters is not absolute to generic extensions, the recognizable closure is more stable, assuming the existence of large cardinals.

We now describe some of the results. Since the recognizable sets have not been studied before, we include various fundamental facts, many of which are easy to prove.

It is surprising that there is a close connection between recognizability and the notion of *implicit definability* that was introduced by Hamkins and Leahy in [HL13]. They define a set x of an ordinal α to be *implicitly definable over* L if there is a formula $\varphi(y, \beta)$ and an ordinal γ such that x is the unique subset z of α with $\langle L, \in, z\rangle \models \varphi(z, \gamma)$. We obtain the following equivalence in Theorem 3.12.

Theorem 1.1. A set of ordinals x is constructible from a recognizable set of ordinals if and only if it is constructible from a set of ordinals that is implicitly definable over L.

We will first consider recognizability from fixed parameters. For instance, we show in Lemma 3.2 that without ordinal parameters, the recognizable closure for subsets of ω is equal to $\mathrm{L}_\sigma \cap P(\omega)$, where L_σ is the least Σ_1-elementary substructure of L. We further show in Lemma 3.3 that for countable ordinal parameters, every recognizable set is already computable. Moreover, assuming that $\omega_1 = \omega_1^{\mathrm{L}}$, this characterization implies that relative recognizability relative to countable ordinals coincides with Δ^1_2-reducibility and thus the consequent degree structure is that of the Δ^1_2-degrees.

However, for uncountable ordinal parameters, there are more recognizable sets, for instance $0^\sharp$, as is shown in Lemma 3.8. We further show that the recognizability strength may increase with larger ordinal parameters. For instance, 3.9 shows, assuming $\mathrm{V} = \mathrm{L}$, that for every countable ordinal α, there is a real which is recognizable from $\omega_{\alpha+1}$, but not from ω_α.

It is moreover natural to compare the class of recognizable sets with well-known inner models such as L and the class HOD of of all hereditarily ordinal definable sets. Since every set of ordinals in L is recognizable, the question arises whether $\mathrm{V} \neq \mathrm{L}$ implies the existence of non-recognizable set of ordinals. However, we show in Lemma 3.13 that there is a generic extension of L in which every set of ordinals is recognizable. While the recognizable closure is always contained in HOD, we show in Lemma 3.17 that it can be strictly smaller.

While the previous results show that the recognizable sets are highly variable in forcing extensions, we can show that the recognizable hull is more stable, assuming the existence of large cardinals. Let M^∞ denote the inner model that is obtained by iterating the least measure of the least iterable fine structural inner model M_1 with a Woodin cardinal (see [Sch10, Section 5.1], [Ste10, page 1660]) through the ordinals. The following main result is proved in Theorem 4.7 below.

Theorem 1.2. Assuming that there is a Woodin cardinal and a measurable cardinal above it, the recognizable closure for subsets of any countable ordinal α is equal to $P(\alpha)^{M^\infty}$.

The set $P(\omega)^{M^\infty}$ is also known as Q_3 [KMS83] and can be for instance characterized as the maximal countable Π^1_3 set that is downwards closed under Δ^1_3-reducibility [KMS83, p. 202, section II], assuming that projective determinacy holds. Thus the recognizable closure reaches far beyond L.

However, it is our main open problem whether our results can be extended to the recognizable closure for subsets of arbitrary ordinals. A partial answer is given in Theorem 4.8.

Theorem 1.3. It is consistent with the existence of inner models with n Woodin cardinals for each n that every recognizable subset of ω_1 is in M^∞.

Moreover, it is natural to consider the *recognizable hull* $\mathcal{R}$ that is defined as the union of all $\mathrm{L}[x]$, where x is recognizable. This class consists of all sets with a recognizable code. The precise properties of $\mathcal{R}$ are not yet known and in particular, it is not known which axioms and schemes of set theory hold in $\mathcal{R}$. However, we will prove the following result for generic variants $\mathcal{C}_{\mathrm{gen}}$ and $\mathcal{R}_{\mathrm{gen}}$ of the recognizable closure and recognizable hull in Theorem 4.13.

Theorem 1.4. If there is a proper class of Woodin cardinals, then $\mathcal{R}_{\mathrm{gen}} = M^\infty$.

We would further like to mention a connection with a viewpoint in the philosophy of mathematics. Various foundational views consider mathematical objects as objects of an idealized cognitive agent. Such a view on set theory is entertained by Hao Wang

[Wan86] and Philip Kitcher [Kit83] and is also present in various remarks by Gödel. The recognizable closure has a natural interpretation as the range of objects that are recognized by an idealized agent.

This paper is structured as follows. Section 2 introduces Ordinal Turing Machines and basic results about recognizability. Section 3 contains results on recognizability with specific ordinal parameters and on the recognizable closure in generic extensions of L. Section 4 connects recognizable sets with inner models and contains the main results. We would like to thank Gunter Fuchs, Daisuke Ikegami, Vladimir Kanovei, Ralf Schindler and John Steel for discussions related to the topic of this paper and the referees for carefully reading the paper.

2. Definitions and basic facts

Ordinal Turing Machines (OTMs) were introduced independently by Koepke and Dawson, and appeared in [Koe06a] and the latter's unpublished thesis [Daw09], as a further generalization of classical Turing machines to the transfinite, following the infinite time Turing machines (ITTMs) of Hamkins and Kidder. They provide an upper bound on the strength of a reasonable model of transfinite computation (see e.g. [Car13b] for an argument in favor of this claim).

We give a brief description of the model and its computations and refer to [Koe06a] for the definitions. An OTM has a tape whose cells are indexed with the ordinals and runs along an ordinal time axis. At each time, each cell contains either a 0 or a 1. An OTM-program is just an ordinary Turing machine program. A computation state thus consists of the tape content, which is a function $t\colon \mathrm{Ord} \to 2$, the head position, an ordinal, and a program state. There are finitely many states and these are indexed by natural numbers.

At any successor time $\alpha+1$, the computation state is determined from the state at time α in the same way as for an ordinary Turing machine with the supplement that, if the head is currently at a limit position and is now supposed to move to the left, it is set to position 0. At limit times, the content of each cell, the head position and the program state are obtained as the inferior limits of the sequence of earlier cell contents, head positions and program states. A computation stops when it assumes a state for which no further state can be determined from the program. The computation can be given an ordinal parameter α by marking the cell at α with a 1 before the computation starts.

Now a subset X of an ordinal γ is called OTM-computable in the parameter α, if there is an OTM-program P which takes as input the value 1 at α and $\beta < \gamma$ and the value 0 otherwise, stops with 1 on the first cell if $\beta \in X$, and stops with 0 on the first cell if $\beta \notin X$. Moreover, a set X of ordinals is here called OTM-computable if it is OTM-computable from some ordinal.

We fix the following notation. Suppose that P is a program and X is a set of ordinals. Then $P^X \downarrow= y$ means that P stops with output y with the oracle X and $P^X \uparrow$ means that the computation diverges. We further denote a computation of P with the oracle x and the ordinal parameter α by $P^x(\alpha)$. As usual, the *Kronecker symbol* δ_{xy} is defined as $\delta_{xy} = 1$ if $x = y$ and $\delta_{xy} = 0$ otherwise. Moreover, let $x \oplus y := \{2n : n \in x\} \cup \{2n+1 : n \in y\}$ denote the join of $x, y \subseteq \omega$.

The main result of [Koe06a], independently obtained by Dawson [Daw09] in his thesis, states that the OTM-computable sets of ordinals coincide with the constructible sets of ordinals. This result and its proof can be relativized in a straightforward manner (see [CS17, Lemma 9]).

Lemma 2.1. Let x and y denote sets of ordinals.

(1) There is a non-halting OTM-program P such that for all $x \subseteq \omega$, P^x enumerates $\mathrm{L}[x]$ in the sense that for any set of ordinals $y \in \mathrm{L}[x]$, the characteristic function of y is written on the tape at some time in the computation.
(2) x is computable by some OTM-program with some ordinal parameter α in the oracle y if and only if $x \in \mathrm{L}[y]$.

Moreover, a set $x \subseteq \omega$ is called ITTM-recognizable if and only if there is a program P such that P stops with output δ_{xy} with the oracle $y \subseteq \omega$ [HL00]. This is generalized to arbitrary sets of ordinals for OTMs in the nect definition. We will also consider a relativized notion of recognizability, similar to the notion studied for ITRMs in [Car14] and for several more machine types in [Car13a].

Definition 2.2. Suppose that x is a subset of an ordinal α and y is a set of ordinals.
(a) x is *recognizable* from y with finitely many ordinal parameters $\gamma_0, \dots, \gamma_n$ if there is an OTM-program P with the parameters y and $\gamma_0, \dots, \gamma_n$ which halts for every subset z of α with δ_{xz} in the first cell.
(b) x is *recognizable* from y without parameters if we can choose $\langle \gamma_0, \dots, \gamma_n \rangle$ as the empty sequence.
(c) x of an ordinal α is *recognizable* from y if it is recognizable from y with some ordinal parameters.

By working with iterated Cantor pairing, we can assume that the parameter is a single ordinal. Therefore we will from now on only consider single parameters. We have the following simple characterisation of recognizability.

Lemma 2.3. A subset x of an ordinal α is recognizable from a set of ordinals y if and only if there is a Σ_1 formula $\varphi(u, v, w)$ and an ordinal β so that x is the unique subset z of α so that $\mathrm{L}[y, z] \models \varphi(y, z, \beta)$.

Proof. Suppose that x is recognizable by a program P_e in the parameters y and β. Then the statement that this program halts on input x with state 1, is a Σ_1 statement true only of x for this pair y, β and is absolute.

Conversely, suppose that $\varphi(y, z, \beta)$ is a Σ_1-formula for which x is the unique solution for z in models of the form $\mathrm{L}[y, z]$. Suppose $\max\{\sup y, \alpha, \beta\} = \tau$. For all ordinals γ and reals y, z, it is well-known, and can be easily shown, that an OTM with the parameters y, z and γ can write a code for $\mathrm{L}_\gamma[y, z]$ on the tape and halt. Therefore, there is a program P_e with parameters y, the V-cardinal $\gamma = |\tau|^+$ and β, that on input $z \subseteq \alpha$ checks if $\mathrm{L}_\gamma[y, z] \models \varphi[y, z, \beta]$. If so, then $z = x$ and it may halt with the state 1. Otherwise $z \neq x$ and it may halt with the state 0. □

It is easy to see that the recognizable sets remain the same if we slightly change the definition, for instance by allowing the program to diverge on inputs other than the recognizable set. Moreover, recognizability is stable under computable equivalence for OTMs with ordinal parameters, and equivalently constructible equivalence.

Lemma 2.4. If x, y are sets of ordinals with $x \in \mathrm{L}[y]$, $y \in \mathrm{L}[x]$ and x is recognizable, then y is recognizable.

Proof. We use the characterisation of recognizable sets in Lemma 2.3. Let $\varphi(v, \alpha) \in \Sigma_1$ have the unique solution x in the parameter α in models of the form $\mathrm{L}[z]$. Let x be the β'th set in $\mathrm{L}[y]$ and y the γ'th set in $\mathrm{L}[x]$. We now check that y is the unique set of ordinals so that $\mathrm{L}[y]$ satisfies the Σ_1 formula which states that the β'th set z in $\mathrm{L}[y]$ satisfies $\varphi(z, \alpha)$ and y is the γ'th set in $\mathrm{L}[z]$. □

Note that in the previous lemma, it is not sufficient to assume that $y \in \mathrm{L}[x]$. For example, the claim fails if $x = 0^\sharp$ and y is a Cohen real over L that is constructible from x.

It is easy to see that recognizability from a fixed ordinal is not absolute between models of set theory. For instance, $0^\sharp$ is recognizable from ω_1, if it exist, but not from any countable ordinal, as we will see below. Therefore it is not recognizable from ω_1^{V} in any generic extension of V in which ω_1^{V} is countable.

A typical phenomenon for infinitary computations is the existence of sets of ordinals which are properly recognizable (i.e. not computable). Following the terminology for Infinite Time Turing Machines in [HL00], we call such sets *lost melodies*.

Definition 2.5. A subset of an ordinal α is called a *lost melody at* α if it is recognizable, but not computable.

We can now define the *recognizable closure*.

Definition 2.6. Suppose that $x, y \subseteq \alpha$.

(a) x is an element of the recognizable closure $\mathcal{C}_\alpha^\beta(y)$ of y for subsets of α in the parameter β if there is a sequence $\langle x_i \mid i \leq n\rangle$ of subset of α with $x_0 = x$ and $x_n = y$ such that x_i is recognizable from x_{i+1} with the parameter β for all $i < n$.
(b) x is an element of the recognizable closure $\mathcal{C}_\alpha(y)$ of y for subsets of α if $x \in \mathcal{C}_\alpha^\beta(y)$ for some β.
(c) x is an element of the recognizable closure $\mathcal{C}(y)$ of y if $x \in \mathcal{C}_\alpha(y)$ for some α.

Moreover, we will omit y if it is empty.

We note that the recognizable closure $\mathcal{C}$ is closed under constructibility and in particular under joins. Moreover, the iteration in the definition of the recognizable closure is necessary, since relativized recognizability is in general not transitive by Lemma 3.16 below. However, two iteration steps suffice by the following argument.

Lemma 2.7. Suppose that $x \in \mathcal{C}(z)$. Then there is some y that is recognizable from z such that x is computable from y.

Proof. Suppose that $\vec{x} = \langle x_i \mid i \leq n\rangle$ witnesses that $x \in \mathcal{C}_\alpha(z)$ as in Definition 2.6 for some ordinal α. Moreover, suppose that y is the join of $\vec{x}$. It is easy to see that y is recognizable by checking that x_i is recognized from x_{i+1} for each $i < n$, and clearly x is computable from y. □

It is worthwhile to note that the recognizable closure $\mathcal{C}$ has the following absoluteness property.

Lemma 2.8. $\mathcal{C}$ is a Σ^1_2-elementary substructure of V.

Proof. Suppose that $x \subseteq \omega$, $x \in \mathcal{C}$ and $\varphi(x)$ is a Σ^1_2-formula which holds in V. Since $\mathrm{L}[x] \subseteq \mathcal{C}$, $\varphi(x)$ holds in $\mathcal{C}$. If $x \in \mathcal{C}$ and $\varphi(x)$ is a Σ^1_2-formula which holds in $\mathcal{C}$. Then $\varphi(x)$ holds in V by Π^1_1-absoluteness between $\mathrm{L}[x, y]$ and V for all $y \subseteq \omega$. □

If we assume the existence of inner models with large cardinals, we can further obtain the following stronger absoluteness. In the next proof, a *Dodd-Jensen x-mouse* is a fully iterable model $(\mathrm{J}_\alpha[U], \in, U)$ of ZFC^-, where U is a $\mathrm{J}_\alpha[x, U]$-normal ultrafilter on the largest cardinal in $\mathrm{J}_\alpha[x, U]$ and there are no measurable cardinals below $\mathrm{crit}(U)$ in $\mathrm{J}_\alpha[x, U]$. Moreover, $0^\dagger$ denotes the least structure of the same form for $x = \emptyset$, but with a measurable cardinal below $\mathrm{crit}(U)$.

Lemma 2.9. Suppose that $x^\sharp$ exists for every $x \subseteq \omega$, but $0^\dagger$ does not exist. Then $\mathcal{C}$ is a Σ^1_3-elementary substructure of V.

Proof. Jensen's proof of the Σ^1_3-absoluteness of K^{DJ} (see [DJK81]) shows that the following statement holds for any Π^1_2-predicate $\varphi(x, y)$: if there is an x such that $\varphi(x, y)$ holds, $y \in \mathrm{K}$ and $(x \oplus y)^\sharp$ exists, then there is such an $x \in \mathrm{L}(M, y)$, where M is the least Dodd-Jensen mouse in H_{ω_1} with $y \in M \notin \mathrm{L}[x,y]$. Let M be the τ'th mouse in the canonical well-order of Dodd-Jensen mice. Since it is easy to see that we can recognize a canonical code c for the τ'th mouse M by checking its iterability, and the order type of the set of its predecessors in $\mathrm{L}[c]$, this implies that $\mathcal{C}$ is Σ^1_3-elementary in V. □

The previous lemma suggests the question how far the large cardinal hypothesis can be relaxed and in particular, whether it is sufficient to assume that there is no inner model with a Woodin cardinal.

We can further consider the core model $\mathrm{K} = \mathrm{K}^{\mathrm{MOZ}}$ that is constructed with the assumption that there is no mouse with a measure of order 1, see [Zem02].

Lemma 2.10. If there is no inner model with infinitely many measurable cardinals, then ${}^{<\mathrm{Ord}}\mathrm{Ord} \cap \mathrm{K} \subseteq \mathcal{C}$.

Proof. Let $\mathrm{K} = \mathrm{L}[E^{\mathrm{K}}] = \mathrm{L}[E]$. Firstly suppose there are measurable cardinals in K and let them and their successors be $\kappa_0 < \lambda_0 = \kappa_0^+ < \cdots < \kappa_n < \lambda_n = \kappa_n^+$. In this case define $M = \mathrm{K}|\lambda_n$ and let M be undefined if K has no measurables. Then, if M is defined, any mouse $N \models$"$E^N_{\lambda_i}$ is a normal measure on κ_i for all $i \leq n$" must satisfy $N = M$ by standard comparison arguments. Let $c \subseteq \lambda_n$ be the $\mathrm{L}[E|\lambda_n]$-least code for M. Then c is recognizable from $\vec{\kappa}, \vec{\lambda}$. Building on this, suppose that γ is any cardinal of V (additionally with $\gamma > \lambda_n$, if K has measurables, and λ_n is defined). If $X \subseteq \tau < \gamma$ is any set of ordinals in K, then again in the canonical wellorder of mice $<^*$ restricted to H_γ, X is the η'th set in the δ'th mouse in $<^*$ (in the δ'th mouse in $<^*$ above M, if M is defined). This then suffices as in the proof of the last lemma. □

We will see in Lemma 4.11 below that the assumption in the last result is necessary.

3. Recognizable sets

In this section, we consider among others the question what is the influence of the ordinal parameters on the recognizability strength. We first consider various infinite time machines without ordinal parameters.

3.1. **Fixed parameters.** Recognizability is a general concept associated with models of infinitary computation that has been studied for Infinite Time Register Machines (ITRMs) [CFK+10] and Infinite Time Turing Machines (ITTMs) [HL00]. The definition of recognizability for ITRMs and ITTMs is completely analogous to Definition 2.2 (see [HL00], [Car14]). We denote by $\mathcal{C}^M$ the recognizable closure for subsets of ω for a machine model M without parameters.

The notion of recognizable closure is remarkably stable, as we will see that the recognizable closure for the above machines without ordinal parameters is the same. Moreover, the relation that x is in the recognizable closure of y for these machines without ordinal parameters coincides with Δ^1_2-reducibility. Thus the recognizable closure for OTMs with ordinal parameters is a generalization of Δ^1_2-reducibility in two ways, first by admitting arbitrary ordinal parameters, second by admitting arbitrary sets of ordinals.

We need the following notions to calculate the recognizable closure for the above machines.

Definition 3.1. Suppose that $x \subseteq \omega$ and α is an ordinal.

(a) The ordinal α is Σ_1-*fixed* if there is some Σ_1-statement φ in the language of set theory such that α is minimal with $\mathrm{L}_\alpha \models \varphi$. Moreover, let σ denote the supremum of the Σ_1-fixed ordinals.

(b) The ordinal α is Σ_1-*stable (in x)* if $\mathrm{L}_\alpha \prec_{\Sigma_1} \mathrm{L}$ ($\mathrm{L}_\alpha[x] \prec_{\Sigma_1} \mathrm{L}[x]$). Moreover, let $\sigma(\alpha)$ ($\sigma(\alpha, x)$) denote the least Σ_1-stable (in x) ordinal $\tau > \alpha$.

Note that α being Σ_1-fixed implies that α is Σ_1-definable. As the halting of an OTM-program without an ordinal parameters is a Σ_1-statement, and an OTM can search for the ordinal α fixed by some sentence φ as in (i), σ is equal to the supremum of such the halting times [CS17, Lemma 7].

Lemma 3.2. $\mathcal{C}_\omega = \mathcal{C}_\omega^{\mathrm{ITTM}} = \mathcal{C}_\omega^{\mathrm{ITRM}} = \mathrm{L}_\sigma \cap P(\omega)$.

Proof. We first argue that all elements of $\mathcal{C}_\omega^{\mathrm{ITRM}}$, $\mathcal{C}_\omega^{\mathrm{ITTM}}$ and $\mathcal{C}^{\mathrm{OTM}}$ are elements of L_σ. Suppose that P is a program for one of these machine types. The statement that there is some $x \subseteq \omega$ such that $P^x \downarrow = 1$ is a Σ_1-statement. This is absolute between L and V by Shoenfield absoluteness and computations are absolute between transitive models of ZFC. If P recognizes some real number x, then $\mathrm{L} \models \exists y P^y \downarrow = 1$ and hence $x \in \mathrm{L}$. Since $\exists y P^y \downarrow = 1$ is a Σ_1-statement, we have $x \in \mathrm{L}_\sigma$. Since L_σ is admissible, $\mathcal{C}_\omega^M \subseteq \mathrm{L}_\sigma$ for these machine models M by the proof of Lemma 2.7.

Suppose that $x \subseteq \omega$ and $x \in \mathrm{L}_\sigma$. There is some Σ_1-fixed ordinal $\alpha < \sigma$ such that $x \in \mathrm{L}_\alpha$. Suppose that φ is a Σ_1-statement and α is minimal with $\mathrm{L}_\alpha \vDash \varphi$. Then L_α is its Σ_1-hull, hence its Σ_1-projectum is ω. Then there is a surjection of ω onto L_α in $\mathrm{L}_{\alpha+1}$ by acceptability. Let $c \subseteq \omega$ denote the L-least code for L_α. Then c is recognizable by any of these machines by checking whether for the least β such that φ holds in L_β, the oracle z is the L-least code for L_β (see [CFK⁺10]). Moreover x is Turing computable from c and hence $x \in \mathcal{C}_\omega^M$. □

We note that for the weaker version of ITRMs that was introduced in [Koe06b], now called unresetting or weak Infinite Time Register Machines (wITRMs), we have $\mathcal{C}_\omega^{\mathrm{wITRM}} = \mathrm{L}_{\omega_1^{\mathrm{CK}}} \cap P(\omega)$. This follows from the fact that recognizability and computability coincide for wITRMs [Car13a] and the main result of [Koe06b] shows that the wITRM-computable reals coincide with the hyperarithmetical reals.

The recognizable closure is closely connected with Δ^1_2-degrees. For subsets x and y of ω, we say that x is *Δ^1_2-reducible to y* ($x \leq_{\Delta^1_2} y$) if x is Δ^1_2 in y. The class Δ^1_2 can be relativized to countable ordinal parameters α, letting x be $\Delta^1_2(y, \alpha)$ if there is a Δ^1_2-definition in real parameters y, z which defines x as a subset of ω for all codes z of α. We will further use the same notion for other classes and for sets of reals.

The equivalence of the first two conditions in the following lemma without parameters was independently proved in [Daw09].

Lemma 3.3. Suppose that x, y are subsets of ω and α is a countable ordinal. The following conditions are equivalent.

(a) x is computable from y and α.
(b) x is recognizable from y and α.
(c) $x \in \mathcal{C}_\omega^\alpha(y)$.
(d) $x \in \mathrm{L}_{\sigma(y,\alpha)}[y]$.

Proof. The condition (a) implies (b). Moreover (b) implies (c) by the proof of Lemma 2.7. This also shows that (c) implies that x is $\Delta_1(y, \alpha)$. It follows from the proof of [Jec03, Lemma 25.25] that this implies that x is $\Delta^1_2(y, \alpha)$ and it is easy to see that this implies that $\{x\}$ is $\Sigma^1_2(y, \alpha)$.

Now suppose that $\{x\}$ is defined by the Σ^1_2-formula $\varphi(x, y, u)$, where u is an arbitrary code for α. Suppose that G is $\mathrm{Col}(\omega, \alpha)$-generic over V and that u is a relation on ω in $\mathrm{L}[G]$ which is isomorphic to α. Let $\psi(z, y, v)$ denote the statement that there is a relation v on ω such that v is isomorphic to u and $\varphi(z, y, v)$ holds.

Claim. In $\mathrm{V}[G]$ the real x is the unique real z with $\psi(z, y, u)$.

Proof. Let $\chi(y, v)$ denote the statement that there is a real $z \neq x$ and a relation w on ω such that w is isomorphic to v and $\varphi(z, y, w)$ holds. Suppose that v is a code for α in V.

The statement $\chi(y,v)$ is a Σ^1_2 statement in y,v which is false in V by the uniqueness of x. Hence $\chi(y,v)$ is false in V$[G]$ by Shoenfield absoluteness. Hence x is the unique real z with $\psi(z,y,v)$ in V$[G]$. Since the truth value of $\psi(z,y,v)$ is equal to that of $\psi(z,y,u)$ if v is isomorphic to u, x is the unique real z with $\psi(z,y,u)$ in V$[G]$. □

Since $u \in \mathrm{L}[y,G]$, there is a real z in $L[y,G]$ with $\psi(z,y,u)$ by Shoenfield absoluteness. Then $z = x$ by the previous claim. Suppose that H is $\mathrm{Col}(\omega,\alpha)$-generic over V$[G]$. Then $x \in \mathrm{L}[y,G] \cap \mathrm{L}[y,H] = \mathrm{L}[y]$. The Σ^1_2-statement $\varphi(x,y,\alpha)$ is equivalent to a Σ_1-statement $\theta(x,y,\alpha)$ by the proof of [Jec03, Lemma 25.25]. Suppose that $\beta > \alpha$ is least such that there is a real z in $\mathrm{L}_\beta[y]$ such that $\theta(z,y,\alpha)$ holds in $\mathrm{L}_\beta[y]$. Then $\beta < \sigma(y,\alpha)$ and hence $x \in \mathrm{L}_{\sigma(y,\alpha)}[y]$. This implies (d).

If (d) holds, then there is a Σ_1-formula $\theta(z,y,\alpha)$ such that $x \in \mathrm{L}_\beta$ and $\beta > \alpha$ is least such that $\theta(x,y,\alpha)$ holds in L_β. Hence x is computable from y,α. This implies (a). □

In the previous lemma, it is necessary to assume that α is countable, since we will see in the next section that recognizability from ω_1 does not imply constructibility, if $0^\sharp$ exists. Moreover, the previous proof shows that we further have the equivalent conditions x is $\Delta_1(y,\alpha)$, x is $\Delta^1_2(y,\alpha)$ and $\{x\}$ is $\Sigma^1_2(y,\alpha)$. Thus OTM-reducibility without parameters coincides with Δ^1_2-reducibility. It follows that computability and recognizability generalize Δ^1_2-reducibility by allowing arbitrary ordinal parameters instead of only countable ordinals and by allowing arbitrary sets of ordinals. Moreover, Lemma 3.3 has the following immediate generalization to uncountable ordinals.

Lemma 3.4. Suppose that x,y are subsets of ω and α is an ordinal. The following conditions are equivalent.

(a) x is computable from y and α.
(b) x is recognizable from y and α in any generic extension in which α is countable.
(c) $x \in \mathcal{C}^\alpha_\omega(y)$ in any $\mathrm{Col}(\omega,\alpha)$-generic extension.
(d) $x \in \mathrm{L}_{\sigma(y,\alpha)}[y]$.

Moreover, the previous lemmas have the following consequences for L. We will write $x \leq_{\mathrm{OTM}} y$ if x is OTM-computable from y without ordinal parameters.

Lemma 3.5. If $\mathrm{V} = \mathrm{L}$ and $x,y \subseteq \omega$, then $x \leq_{\mathrm{OTM}} y$ holds if and only if $\sigma(x) \leq \sigma(y)$.

Proof. Our assumption implies that $\mathrm{L}_{\sigma(z)}[z] = \mathrm{L}_{\sigma(z)}$ for any z. If $x \leq_{\mathrm{OTM}} y$, then $x \in \mathrm{L}_{\sigma(y)}[y]$ by Lemma 3.3. Hence x is Σ_1-definable in $\mathrm{L}_{\sigma(y)}[y] = \mathrm{L}_{\sigma(y)}$ and hence $\sigma(x) \leq \sigma(y)$. If $\sigma(x) \leq \sigma(y)$, then $x \in \mathrm{L}_{\sigma(y)}[y]$ and hence x is OTM-computable from y by Lemma 3.2. □

Lemma 3.6. Suppose that $\mathrm{V} = \mathrm{L}$ and $x,y \subseteq \omega$. If x is recognizable from y and α, then x is computable from y and α.

Proof. Suppose that P recognizes x from y and α. We enumerate L as in Theorem 2.1. Whenever a new real number z appears on the tape, we run $Q^{y,z}(\alpha)$ and return z when $Q^{y,z}(\alpha) = 1$. This computes x from y and α. □

Let η denote the least Σ_2-stable ordinal, i.e. the least ordinal η with $\mathrm{L}_\eta \prec_{\Sigma_2} \mathrm{L}$. It may occur to the reader whether there is a characterization of the reals in L_η similar to Lemma 3.3. To this end, in analogy with [HL00], we define a subset x of γ to be *eventually writable* if there is an OTM-program P with empty input such that from some time onwards, the contents of the initial segment of the tape of length γ is x. It is then easy to show the following result.

Lemma 3.7. The eventually writable subsets of ω are exactly the subsets of ω in L_η.

3.2. **Comparing the strength for different parameters.** A first observation is that $0^\sharp$ is recognizable from ω_1 by [Car13a, Theorem 4.2], if it exists, while it is not recognizable from any countable ordinal.

Lemma 3.8. If $x \subseteq \omega$ and $x^\sharp$ exists, then $x^\sharp$ is recognizable from x. Moreover $x^\sharp$ is recognizable from x and α if and only if $\alpha \geq \omega_1$.

Proof. If $\alpha \geq \omega_1$, then the relation $x^\sharp = y$ is Π^1_2 and hence absolute between $\mathrm{L}_\alpha[x,y]$ and V [Jec03, Theorem 25.20]. Given a real y, we can check the definition of $x^\sharp$ for y in $\mathrm{L}_\alpha[x,y]$. This shows that $x^\sharp$ is recognizable from x and α. Moreover, since $x^\sharp$ exists and hence ω_1 is inaccessible in $\mathrm{L}[x]$, Lemma 3.3 shows that $x^\sharp$ is not recognizable from x and any countable ordinal. □

We further observe that the recognizability strength does not depend monotonically on the parameter.

Lemma 3.9. (1) There is a constructible real x that is recognizable from some $\alpha < \omega_1^{\mathrm{L}}$ but not from ω_1^{L}.
(2) For any $\alpha < \omega_1^{\mathrm{L}}$, there is a real x that is recognizable from $\omega_{\alpha+1}^{\mathrm{L}}$ but not from $\omega_\alpha^{\mathrm{L}}$.

Proof. For the first claim, let $\tau < \omega_1^L$ be the strict supremum of L-ranks of reals recognizable in the parameter ω_1^{L}. Moreover, let c be the τ'th real with respect to $<_L$. Then c is recognizable in τ but not in ω_1^{L}. For the second claim, let τ_α be the strict supremum of the L-ranks of reals recognizable in the parameter $\omega_\alpha^{\mathrm{L}}$. Let x_α be the τ_α'th real with respect to $<_{\mathrm{L}}$. Then each of ω_α, τ_α and x_α is recognizable in $\omega_{\alpha+1}$. □

Another natural example of this phenomenon is the α-jump J^α i.e. the set of natural numbers i for which the i'th program recognizes a real number in the parameter α. One can show that in L, J^α is recognizable in $\omega_{\alpha+1}$, but not in ω_α.

However, the conclusion of the previous lemma fails if $x^\sharp$ exists for every real x.

Lemma 3.10. Suppose that $x^\sharp$ exists for every real x. Suppose that κ and λ are uncountable cardinals. If P recognizes x from κ, then P recognizes x from λ.

Proof. As κ and λ are indiscernible over $\mathrm{L}[x]$, for each state i, $P^x(\kappa)$ halts with final state i if and only if $P^x(\kappa)$ halts with final state i. □

3.3. **Arbitrary parameters.** In this section, we consider the recognizable closure with arbitrary ordinal parameters. We first give an alternative definition of the recognizable closure by considering the following notion of implicitly definable sets from [HL13].

Definition 3.11. If α is an ordinal and x is a subset of α, then x is *implicitly definable over* L if there is a formula $\varphi(z,\beta)$ and an ordinal β such that x is the unique subset z of α with $\langle \mathrm{L}, \in, z\rangle \vDash \varphi(z,\beta)$.

Theorem 3.12. Suppose that α is an ordinal and x is a subset of α. Then the following conditions are equivalent.

(a) x is constructible from a recognizable subset y of some ordinal β.
(b) if and only if x is constructible from some subset y of an ordinal β that is implicitly definable over L.

Proof. Suppose that x is constructible from a recognizable subset y of some ordinal β. Let φ be a Σ_1-formula that states that there is a computation which halts with end state 1 for the input y.

Claim. There is a set A of ordinals such that A is implicitly definable over L and $\mathrm{L}_\gamma[y]$ is constructible from A.

Proof. We consider the Gödel functions in [Jec03, Definition 13.6] with three additional functions $H_0(u,v) = \langle u,v\rangle$, $H_1(u) = u$ and $H_2(u) = u \cap y$. These functions ensure that for any transitive set u, the set of images of all $x \in u$ and $x, y \in u$ is again transitive. Instead of the L-hierarchy over y, we consider the hierarchy of sets $\mathrm{M}_\alpha[y]$, where $\mathrm{M}_{\alpha+1}[y]$ is defined by closing under these functions. This hierarchy induces a canonical wellorder on $\mathrm{L}[y]$.

To code the levels of this hierarchy, we can construct a strictly increasing sequence $\langle \delta_\alpha \mid \alpha < \gamma\rangle$ such that for each α with $\alpha + 1 < \gamma$, there is a canonical bijection between the interval $[\delta_\alpha, \delta_{\alpha+1})$ and all applications of the Gödel functions to the sets coded by ordinals below δ_α. Note that elements of $\mathrm{M}_\gamma[y]$ have more than one representation. We now consider $\langle \delta_\alpha \mid \alpha < \gamma\rangle$, the pointwise images under the canonical bijection of the equality and element relations of $\mathrm{M}_\gamma[y]$ and the pointwise image of the canonical wellorder of $\mathrm{M}_\gamma[y]$ in $\delta = \sup_{\alpha<\gamma} \delta_\alpha$. Let A code these sets via Gödel pairing.

We can now consider the implicit definition of A over L which states that the structure coded by A follows the definition of the hierarchy and that $\varphi(z, \beta)$ holds in the structure coded by A. □

The remaining implication is easy to see. □

We note that the above proof also shows the equivalence with the conditions that x is constructible from some set of ordinals y with the following properties. The first property states that y is the unique subset z of some ordinal γ with $\mathrm{L}[z] \vDash \varphi(y)$, where φ is a formula with an ordinal parameter. The second condition states that there is a formula in L in the logic built from the atomic formulas $\alpha \in x$ for all ordinals γ by negations, infinitary conjunctions and disjunctions, such that y is the unique subset z of some ordinal γ such that $\varphi(z)$ holds, where φ is a again a formula with an ordinal parameter. Moreover, the proof shows that every subset of an ordinal which is implicitly definable over L is recognizable, but the converse is open.

We mentioned $0^\sharp$ as an example of a lost melody above. The next result shows that large cardinal assumptions are not necessary for the existence of lost melodies.

Theorem 3.13. There is a set-generic extension of L by a real such that in the extension, every set of ordinals is recognizable.

Proof. There is a c.c.c. subforcing of Sacks forcing in L which adds a Π^1_2-definable minimal real x over L [Jen70]. Since x is Π^1_2-definable, it is recognizable from ω_1 by Shoenfield absoluteness, as in the proof of Theorem 3.8. Clearly, every constructible real is computable and hence recognizable. By minimality we have $x \in \mathrm{L}[y]$ for every real $y \in \mathrm{L}[x] \setminus \mathrm{L}$. Hence all non-constructible elements of the generic extension are constructibly equivalent to each other and in particular to x. Since x is recognizable, it follows from Theorem 2.4 that every real $y \in \mathrm{L}[x]$ is recognizable.

We now argue that the Jensen real is minimal. Jensen [Jen70, Lemma 11] showed that the Jensen real is minimal for reals. We would like to thank Vladimir Kanovei for the following argument.

Claim. Suppose that G is Jensen generic over L. Suppose that $X \subseteq \kappa$ is a set of ordinals in $\mathrm{L}[G] \setminus \mathrm{L}$. Then there is a real y such that $\mathrm{L}[X] = \mathrm{L}[y]$.

Proof. Let $\mathbb{J}$ denote Jensen forcing. Suppose that σ is a $\mathbb{J}$-name for a set of ordinals, and $1_{\mathbb{J}} \Vdash \sigma \notin \mathrm{L}$. Then for any pair S, T of conditions in $\mathbb{J}$, there is a pair of conditions $\bar{S}, \bar{T}$ with $\bar{S} \leq S$, $\bar{T} \leq T$ and an ordinal $\alpha_{S,T}$ such that $\bar{S} \Vdash \alpha_{S,T} \in \sigma$, while $\bar{T} \Vdash \alpha_{S,T} \notin \sigma$. Suppose that A is a maximal antichain in $\mathbb{J}^2$. The forcing $\mathbb{J}^2$ is c.c.c. and in fact $\mathbb{J}^n$ is c.c.c. for any n by [Jen70, Lemma 6]. Hence A is countable. Let B denote the set of $\alpha_{S,T}$ for all $(S,T) \in A$.

Then for any pair S, T of conditions in $\mathbb{J}$, there is a pair $\bar{S}, \bar{T}$ with $\bar{S} \leq S$ and $\bar{T} \leq T$ and an ordinal $\alpha \in B$ such that $\bar{S} \Vdash \alpha \in \sigma$, while $\bar{T} \Vdash \alpha \notin \sigma$. Let τ denote a $\mathbb{J}$-name for $\sigma^G \cap B$, where G is $\mathbb{J}$-generic over L. Since B is countable in L, there is a real $y \in \mathrm{L}[G]$ such that $\mathrm{L}[\tau^G] = \mathrm{L}[y]$ and hence $y \notin \mathrm{L}$. Since the Jensen real is minimal for reals, $\mathrm{L}[x] = \mathrm{L}[y]$. Since $\tau^G \in \mathrm{L}[\sigma^G]$, we have $\mathrm{L}[\sigma^G] = \mathrm{L}[x]$. □

This completes the proof. □

It is undecidable in ZFC whether there are lost melodies for OTMs by Theorem 3.13. We will now show that moreover, relativized recognizability is not necessarily transitive. To this end, we show that homogeneous forcing does not add recognizable sets, which was proved indepently in [Daw09] for Cohen forcing.

Definition 3.14. A forcing $\mathbb{P}$ is *homogeneous* if for all conditions $p, q \in \mathbb{P}$, there is an automorphism $\pi \colon \mathbb{P} \to \mathbb{P}$ with $p \parallel \pi(q)$ (i.e. p and $\pi(q)$ are compatible).

Lemma 3.15. We work in ZF. Suppose that $\mathbb{P}$ is a homogenous forcing and G is $\mathbb{P}$-generic over V. Suppose that $\mu \in \mathrm{Ord}$ and x is a recognizable subset of μ in $\mathrm{V}[G]$. If x is recognizable, then $x \in \mathrm{V}$.

Proof. Suppose that $p \in \mathbb{P}$ forces that x is recognized by a program P from some ordinal γ. Further suppose that $x \notin \mathrm{V}$ and that $\dot{x}$ is a $\mathbb{P}$-name for x. Since p does not decide $\dot{x}$, there is some $\alpha < \mu$ and conditions $q, r \leq p$ such that $q \Vdash \dot{x}(\alpha) = 0$ and $r \Vdash \dot{x}(\alpha) = 1$. Let π be an automorphism of $\mathbb{P}$ such that $q \parallel \pi(r)$ and suppose that $s \leq \pi(q), r$.

Now suppose that G is $\mathbb{P}$-generic over V with $s \in G$. Since s forces that $\dot{x}$ is recognized by P in the parameter γ, we have $q \Vdash P^{\dot{x}}(\gamma)\downarrow$ and $\pi(r) \Vdash P^{\dot{x}}(\gamma)\downarrow$. Since r forces that $\dot{x}$ is recognized by P in the parameter γ, we have $\pi(r) \Vdash P^{\pi(\dot{x})}(\gamma)\downarrow$. We have $q \Vdash \dot{x}(\alpha) = 0$ and $\pi(r) \Vdash \pi(\dot{x})(\alpha) = 1$. Let $x = \dot{x}^G$ and $y = \pi(\dot{x})^G$. We work in $\mathrm{V}[G]$. Since $q \in G$, P recognizes x from γ. Since $\pi(r) \in G$, P recognizes y from γ. However $x \neq y$, contradicting the uniqueness of x. □

We now see that it is consistent that relativized recognizability is not transitive.

Lemma 3.16. If $0^\sharp$ exists, then the relation that x is recognizable in y is intransitive.

Proof. As a counterexample to transitivity take x to be the $\mathrm{L}[0^\sharp]$-least Cohen real over L, $y = 0^\sharp$ and $z = 0$. By Lemma 3.8, x is recognizable in y, and y in z, but by the last lemma x is not recognizable in z. □

3.4. **Separating the recognizable closure from** HOD**.** Any recognizable set of ordinals and hence also any set of ordinals in $\mathcal{C}$ is Δ_1-definable from an ordinal. The next result shows that the recognizable closure is strictly contained in HOD in some forcing extension of L.

Lemma 3.17. Let $\mathbb{P}$ denote Cohen forcing and suppose that $\dot{x}$ is a $\mathbb{P}$-name for the $\mathbb{P}$-generic real. Suppose that $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for the finite support product $\prod_{n \in x} \mathrm{Add}(\omega_n, 1)$, where x is the interpretation of $\dot{x}$. Suppose that x is $\mathbb{P}$-generic over L and that G is $\dot{\mathbb{Q}}^x$-generic over $\mathrm{L}[x]$. Then in $\mathrm{L}[x, G]$, x is Σ_1-definable from an ordinal, but not Π_1-definable from an ordinal. Hence $\mathcal{C} \subsetneq \mathrm{HOD}$ holds in $\mathrm{L}[x, G]$.

Proof. Suppose that φ is a Π_1-formula, $\delta \in \mathrm{Ord}$ and $(p, \dot{q})$ is a condition which forces $\forall n (n \in \dot{x} \Leftrightarrow \varphi(n, \delta))$ Suppose that $s \subseteq \omega$ is finite and $\mathrm{dom}(p) \subseteq s$. We can assume that $p \Vdash \mathrm{supp}(\dot{q}) \subseteq s$.

Suppose that (x, G) is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over L below $(p, \dot{q})$ and $G = \prod_{i \in x} G_i$. Suppose that $n \in \omega \setminus (x \cup s)$ and $y = x \cup \{n\}$. Suppose that G_n is $\mathrm{Add}(\omega_n, 1)$-generic over $\mathrm{L}[x, G]$. Suppose that $\dot{q}^x$, $\dot{q}^y$ are given by the sequences $\langle q_i^x \mid i \in s \cap x \rangle$ and $\langle q_i^y \mid i \in s \cap x \rangle$, where $q_i^x, q_i^y \in \mathrm{Add}(\omega_i, 1)$. Suppose that $\pi_i \colon \mathrm{Add}(\omega_i, 1) \to \mathrm{Add}(\omega_i, 1)$ is an automorphism such that $\pi_i(p_i)$ is compatible with q_i for all $i \in s \cap x$. Let H denote the $\dot{\mathbb{Q}}^y$-generic filter over $\mathrm{L}[y]$ which is equivalent to $\prod_{i \in s \cap x} \pi[G_i] \times \prod_{i \in x \setminus s} G_i \times G_n$ modulo the order of the indices in the product. Then (y, H) is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over L below $(p, \dot{q})$ and $\mathrm{L}[x, G, G_n] = \mathrm{L}[y, H]$. Then $\varphi(n, \delta)$ holds in $\mathrm{L}[y, H]$ by Σ_1-upwards absoluteness, contradicting the assumption on $(p, \dot{q})$. □

Another model of $\mathcal{C} \subsetneq \mathrm{HOD}$ is the following. Suppose that $\langle \kappa_i \mid i < \omega \rangle$ is a strictly increasing sequence of measurable cardinals and $\vec{\mu} = \langle \mu_i \mid i < \omega \rangle$ is a sequence of normal ultrafilters with $\mathrm{crit}(\mu_i) = \kappa_i$ for all i. Suppose that $\mathrm{V} = \mathrm{L}[\vec{\mu}]$. It can now be seen as in Lemma 4.11 below that $\vec{\mu}$ is not coded by any set in $\mathcal{C}$.

Moreover, the previous lemma motivates the question whether it is consistent that $\mathcal{C} = \mathrm{L}$ and $\mathcal{C} \subsetneq \mathrm{HOD}$.

We now show that the recognizable sets are highly variable in forcing extensions. More precisely, every set of ordinals that is generic over L can be coded into the recognizable closure $\mathcal{C}$ in a further generic extension.

Lemma 3.18. Suppose that G is $\mathbb{P}$-generic over L and $X \in L[G]$ is a set of ordinals. Then there is a cofinality-preserving forcing $\mathbb{Q}$ in $\mathrm{L}[G]$ such that for every $\mathbb{Q}$-generic filter H over $\mathrm{L}[G]$, X is computable from a recognizable set in $\mathrm{L}[G, H]$.

Proof. Let $\lambda = |\mathbb{P}|^+$. Then $\mathbb{P}$ adds no Cohen subset to any regular cardinal $\nu \geq \lambda$, since otherwise some $p \in \mathbb{P}$ decides unboundedly many $\alpha < \nu$. Suppose that μ is a cardinal with $X \subseteq \mu$.

The forcing $\mathbb{Q} \in \mathrm{L}[G]$ is an iteration with full support. We define the iteration separately in the intervals between the ordinals in a strictly increasing sequence $\langle \mu_n \mid n \in \omega \rangle$ of length ω which is defined as follows.

Let $\mu_0 = \mu$. In the first interval $[0, \mu_0)$, we add a Cohen subset to $[\lambda^{+(2\cdot\alpha)}, \lambda^{+(2\cdot\alpha+1)})$ if $\alpha \in X_0 := X$ and a Cohen subset to $[\lambda^{+(2\cdot\alpha+1)}, \lambda^{+(2\cdot\alpha+2)})$ if $\alpha \notin X_0$ for $\alpha < \mu_0$. This iteration of length μ_0 adds a subset X_1 of $\mu_1 := \lambda^{+\mu}$. We similarly define μ_{n+1} from μ_n and X_{n+1} from X_n in the interval $[\mu_n, \mu_{n+1})$ for all n. Let $Y = \bigcup_{n \in \omega} X_n$ and let H denote the generic filter over $\mathrm{L}[G]$ defined by this sequence. Then $\mathrm{L}[G, Y] = \mathrm{L}[G, H]$.

This iteration adds Cohen subsets over L only to the successor cardinals $\nu \geq \lambda$ specified in the iteration.

Claim. Y is recognizable in $\mathrm{L}[G, H]$.

Proof. Suppose that $\bar{Y}$ is a subset of $\sup_{n \in \omega} \mu_n$. We can determine in $\mathrm{L}[\bar{Y}]$ whether $\bar{Y}$ is consistent with the coding described above. Suppose that $\bar{Y}$ is consistent with the coding. Then $\bar{Y} \cap [\mu_n, \mu_{n+1}) = Y \cap [\mu_n, \mu_{n+1})$, since this set is determined by the set of successor cardinals $\nu \geq \mu_{n+1}$ such that there is a Cohen subset of ν over L. Hence $\bar{Y} = Y$. □

This completes the proof of the lemma. □

4. The connection with Woodin cardinals

In this section, we determine the recognizable closure for subsets of a countable ordinal α, assuming the existence of sufficient large cardinals. We will work with *premice* of the form $M = (\mathrm{J}_\alpha[\vec{E}], \in, \vec{E})$ as defined in [Zem02, MS91] and their relativizations. We will further assume that all premice have Jensen indexing; this will be used in the proof of Lemma 4.6. Note that all results that we use about premice with Steel indexing also work for Jensen indexing; a correspondence between these indexing schemes can be found in [Fuc11b, Fuc11a]. In our notation, the top extender on the sequence is not necessarily an element of the structure, but all other extenders are. The *initial segment* $M|\beta$ of M is defined as the structure $(\mathrm{J}_\beta[\vec{E}\upharpoonright\beta], \in, \vec{E}\upharpoonright\beta)$ for any ordinal β; the extender at β is not included. Moreover, a premouse M is 1-*small* if for every extender E on the sequence of

M, $M|\mathrm{crit}(E)$ does not have Woodin cardinals. The premouse $M_1^\sharp$ is defined in [Sch10, Section 5.1] and [Ste10, page 1660].

Definition 4.1. $M_1^\sharp(x)$ denotes any sound ω_1-iterable x-premouse M with first projectum ω and a top measure with critical point κ above a Woodin cardinal in M such that $M|\kappa$ is 1-small. Moreover, $M_1^\sharp$ denotes $M_1^\sharp(\emptyset)$.

The premouse $M_1^\sharp$ exists and is ω_1+1-iterable if there is a measurable cardinal above a Woodin cardinal (see [MS91, Ste93] or [Ste10, Corollary 3.12]). Moreover, it is well-known that $M_1^\sharp$ is unique if it is ω_1+1-iterable or if $M_1^\sharp(z)$ exists for every real z (see e.g. [Sch14b, Lemma 2.41]). We can then define the inner models M_1 and M^∞ as follows.

Definition 4.2. (a) Let

$$M_1 = \bigcup_{\alpha \in \mathrm{Ord}} M^{(\alpha)}|\lambda_\alpha,$$

where λ_α denotes the image of $\lambda_0 = \lambda$ in the α'th iterate $M^{(\alpha)}$ of $M_1^\sharp$ by the top measure and its images.

(b) Let

$$M^\infty = \bigcup_{\alpha \in \mathrm{Ord}} M^\alpha|\kappa_\alpha,$$

where κ_α denotes the image of $\kappa_0 = \kappa$ in the α-th iterate M^α of $M_1^\sharp$ by the unique normal measure on its least measurable cardinal κ and its images.

If M is a premouse, we always understand $\mathrm{L}(M)$ as a class premouse with the same extender sequence as M.

4.1. **The countable case.** In this section, we assume that $M_1^\sharp$ exists and is ω_1+1-iterable or an ω_1-iterable $M_1^\sharp(z)$ exists for every real z as explained above. The latter condition is equivalent to $\mathbf{\Pi}^1_2$-determinacy [Sch10, Theorem 5.3]. Our aim is to see that $\mathcal{C}_\alpha$ is equal to the power set of α in M^∞ for countable ordinals α. The next result proves one direction of this claim.

Lemma 4.3. If x is a recognizable subset of a countable ordinal α, then $x \in M^\infty$.

Proof. It is sufficient to show that $x \in M^\alpha$, since $M^\alpha|\mu_\alpha = M^\infty|\mu_\alpha$ and $\alpha \leq \mu_\alpha$. We thus assume otherwise. There is a forcing $\mathbb{P} \in M^\alpha$ and countable iteration tree on M^α with no drops on the main branch, last model N and iteration map $j\colon M^\alpha \to N$ such that x is $j(\mathbb{P})$-generic over N. If we use Neeman's genericity iteration [Nee95, Corollary 1.8], then $\mathbb{P}$ can be chosen as $\mathrm{Col}(\omega, \delta)$, where δ is the unique Woodin cardinal of M^α. Or we can use Woodin's genericity iteration [Ste10, Theorem 7.14], [NZ01, Section 2] and then choose $\mathbb{P}$ as Woodin's extender algebra at the same δ (see also [Sch14b, Lemma 2.44] for the case that only ω_1-iterability is assumed). Assuming that P recognizes x from β, this also holds in $N[x]$ and we can find a condition $p \in \mathbb{P}$ that forces this. In particular, it is forced by p that the $\mathbb{P}$-generic real x is not in the ground model. We further choose some $\mathbb{P}^N$-generic real y below p over $N[x]$ in V. In particular, x and y are mutually generic and hence distinct. Now y is recognized by P from β in $N[y]$ by the choice of p. Hence the computation of P halts on input y with parameter β in V by the absoluteness of computations. However, this contradicts the uniqueness of x. $\square$

We will use the following notions. The *index* ν_E of an extender E in a fine extender sequence denotes its index in the sequence. Moreover, an iteration tree is *normal* if the indices of the extenders are non-decreasing.

Definition 4.4. Suppose that T is a normal iteration tree ([Ste10, Section 3.1], see also [Sch14a, Section 10.4] for coarse iteration trees) of limit length with the sequence $\langle M_i \mid i < \lambda \rangle$ of models and the sequence $\langle E_i \mid i < \lambda \rangle$ of extenders.

(a) $M_T = \bigcup_{i<\lambda} M_i | \nu_{E_i}$ is the *common-part model of* T.
(b) $\delta_T = \sup_{i<\lambda} \nu_{E_i}$ is the height of M_T.
(c) T is *maximal* if δ_T is a Woodin cardinal in $\mathrm{L}(M_T)$ and *short* otherwise.

The following definition is a variant of [Ste95, Definition 1.6] (see also [MS94, Def. 6.11]) that is slightly weaker, but is sufficient for our purpose.

Definition 4.5. A premouse M is Π^1_2*-iterable* if there is an iteration strategy σ such that the following conditions hold for every countable iteration tree T on M following σ.

(a) If T has successor length, then every ultrapower with an extender from the last model of T according to the rules of the iteration game is well-founded.
(b) If T has limit length, then for every $\alpha < \omega_1$, there is a cofinal branch b in T such that M^T_b is wellfounded and has height at most α, or α is contained in the well-founded part of the direct limit M^T_b along b.

Before we state the next result, we introduce some notation. By definition, a co-iteration of two premice *terminates* if at some stage, one of the models occurring in the coiteration is ill-founded or if there is a stage where the models on both sides are both well-founded and comparable. We shall also say here that a co-iteration *succeeds* if the latter case holds: the coiteration terminates with comparable and well-founded target models. The next proof uses ideas from [Ste95, Lemma 2.2].

Lemma 4.6. If $x \in M^\infty$ is a subset of a countable ordinal, then it is computable from a recognizable subset of a countable ordinal.

Proof. Let κ denote the least measurable cardinal in M_1, μ its successor in M_1 and $M = M_1|\mu$: thus the unique normal measure on κ with index μ is omitted in M. We can assume that $x \in M$, since the general case can be proved by replacing M with an iterated ultrapower by this unique measure on κ such that κ is sent above the supremum of x. We will also assume that N is a countable sound (see [MS91, Definition 2.8.3]) and solid (see [MS91, Definition 2.7.4]) Π^1_2-iterable 1-small premouse with no top extender such that N is a model of ZF^- of height μ with no measurable cardinals and with κ as its largest cardinal.

We first claim that it suffices to prove that $M = N$ for any such N. To see this, let y be the $<_{\mathrm{L}(M)}$-least subset of μ that codes M, where $<_{\mathrm{L}(M)}$ denotes the canonical well-order of $\mathrm{L}(M)$. By Shoenfield absoluteness and the fact that Π^1_2-iterability is a Π^1_2 condition and hence absolute, we can thus recognize y with the parameters μ and ω_1 as the unique $<_{\mathrm{L}(N)}$-least code of such a structure N as above. Moreover x is computable from y.

To prove that M and N are equal, we will co-iterate them. Since both of them are 1-small, the proof of [Ste95, Lemma 2.2 (3)] or the proof of [MS91, Theorem 6.2] show that every countable iteration tree on M or N has at most one cofinal well-founded branch. Therefore, the co-iteration of M and N to any ordinal $\alpha \leq \omega_1$ is unique, if it exists. We denote the iteration trees arising on M, N by T, U with resultant sequences of models $\langle M_i \mid i < \theta \rangle$ and $\langle N_i \mid i < \theta \rangle$ for some $\theta \leq \omega_1$.

We first assume that both $\theta = \omega_1$ and there is a well-founded direct limit model at stage θ. If $M_1^\sharp$ exists and is $\omega_1 + 1$-iterable, then the usual proof of the comparison lemma [Ste10, Theorem 3.11] shows that the co-iteration in fact succeeds at a countable stage, but this is impossible since otherwise the co-iteration would have terminated at that stage. If $M_1^\sharp$ is only assumed to be ω_1-iterable and $M_1^\sharp(z)$ exists for all reals z, then the co-iteration is absolute to $M_1^\sharp(M, N)$ (see [Sch14b, Lemma 2.38]) and we then obtain the same contradiction for the co-iteration in this model.

So in the next claim, we assume that the co-iteration terminates successfully at a countable stage, i.e. $\theta < \omega_1$ is a successor.

Claim. If the co-iteration succeeds at a countable stage, then both of T, U are trivial and $M = N$.

Proof. By a standard argument (see [MS91, Lemma 7.2]) at most one of the trees T, U has a truncation on its main branch, which exists by the case assumption. We shall assume that U has no truncation on its main branch; the argument in the case that T has no truncation on its main branch is virtually symmetric.

Suppose that N moves for the first time in the ultrapower from $N = N_\alpha$ to $N_{\alpha+1}$ by an extender E. Since N has only partial measures, N is truncated in this step and $\nu_E = i_E(\mathrm{crit}(E))^{+N_{\alpha+1}}$ is not an N-cardinal, where i_E is the ultrapower of the truncate of N_α that is given by U (by Jensen indexing). In particular $N_{\alpha+1}$ is not on the main branch by our assumption on U. Now suppose that F is the extender on the sequence of N_β that is applied to form the ultrapower $N_{\beta+1}$ of N, where $N_{\beta+1}$ is first on the main branch of U after $N = N_0$. Since this happens after step α, we have $\nu_E < \nu_F$. Since ν_E is not an N-cardinal and by our assumption N is not truncated when the ultrapower $N_{\beta+1}$ of N by F is formed, it is also not a cardinal in $N_{\beta+1}$. However, this contradicts the fact that it remains a cardinal in all further iterates by the agreement of the models in iteration trees (see e.g. [Zem02, Lemma 4.2.1], [MS91, Lemma 5.1]).

We now show that M does not move either. There are only finitely many truncations on the main branch of T by the rules of the iteration game (see [Ste10, Section 3.1]) and the comparison process (see [Ste10, Theorem 3.11]). Hence we obtain a strictly increasing finite sequence of ordinals $\beta_0, \ldots, \beta_{n+1}$ with $\beta_0 = 0$ and fine-structure preserving embeddings $M^*_{\beta_i} \to M_{\beta_{i+1}}$ induced by T such that $M^*_{\beta_i}$ is a truncate of M_{β_i} for all $i \leq n$ and $M_{\beta_{n+1}}$ is the last model of T. Since N does not move and there are only partial extenders on the sequence of N, it follows from the agreement of the models in T that M and its iterates can only truncate to use a measure of order 0 and in each such case, the iteration can only use the images of the same measure before the next truncation. Therefore, the sequence of iterates of M along the main branch of T is definable over M. In particular, the linear iteration from $M^*_{\beta_n}$ to $M_{\beta_{n+1}}$ is definable over M and we now argue that this leaves behind a club of inaccessible cardinals in N below μ. Since M is a model of ZFC^-,[2] the height of M_δ is strictly less than μ for all $\delta < \beta_{n+1}$. Hence the set of critical points in the iteration from $M^*_{\beta_n}$ to $M_{\beta_{n+1}}$ forms a club in μ and each such critical point is inaccessible in the final iterate $M_{\beta_{n+1}}$ of M. However, N does not move in this co-iteration and there are no N-cardinals in the interval (κ, μ). This is a contradiction. □

The above cases assumed that N is sufficiently iterable for these arguments to work. We finally consider the case that comparison fails because there is no well-founded direct limit model at stage θ. This is covered by the next claim.

Claim. If θ is a limit and U has no cofinal well-founded branches, then both of T, U are trivial and $M = N$.

Proof. We first argue that T is trivial and thus M does not move. To this end, we work in a $\mathrm{Col}(\omega, \omega_1)$-generic extension of V and use that N is Π^1_2-iterable there by Shoenfield absoluteness. Since U has no cofinal well-founded branches, for any countable ordinal α there is a cofinal branch b_α in U such that α is contained in the well-founded part of the direct limit model $N^U_{b_\alpha}$ along b_α. Then δ_U is Woodin in $\mathrm{L}_\alpha(M_U)$ for all countable α by [Ste10, Theorem 6.10]. Since ω_1 is a cardinal in $\mathrm{L}(M_U)$, it follows that δ_U is Woodin in $\mathrm{L}(M_U)$. Moreover we have $\delta_T = \delta_U$ and $\mathrm{L}(M_T) = \mathrm{L}(M_U)$.

[2]It is sufficient for this argument to assume that M is admissible (i.e. satisfies Σ_1-replacement), since linear iterations are Σ_1-definable.

We now work in V again. Otherwise, there is an extender with critical point strictly above δ_T in the ultrapower of M_θ with E. By coherence of the extender sequence, the extender with least index satisfying this condition would then witness that the ultrapower of M_θ with E is not 1-small. Hence $M_\theta = \mathrm{L}_{\theta^*}(M_T)$ for some ordinal θ^*.

We now show that T has no truncation on its main branch. Towards a contradiction, assume that there is such a truncation and at the last such, the model M_α is truncated to M_α^* to apply the extender E. Then T induces a fine-structure preserving elementary embedding : $M_\alpha^* \to M_\theta$. By virtue of the truncation, the ultimate projectum of M_α drops to $\mathrm{crit}(E)$ or below. Thus there is a new subset A of this ordinal that is definable over M_α^* but not an element of M_α^*. But by the agreement of the models, A is also definable over $M_\theta = \mathrm{L}_{\theta^*}(M_T)$. We thus obtain a bounded subset of δ_T that is not measured by any of the extenders in $\mathrm{L}_{\delta_T}(M_T)$. Thus δ_T cannot be Woodin in $\mathrm{L}(M_T)$, contradicting the statement above.

Since T has no truncation on its main branch, the proof of the previous claim shows that both M and N are not moved as required. □

As we argued above, this covers all cases and we have thus completed the proof of Lemma 4.6. □

Together with Lemma 4.3, the previous lemma finishes the proof of Theorem 1.2, since these results imply the following.

Theorem 4.7. If α is countable and x is a subset of α, then $x \in \mathcal{C}_\alpha$ if and only if $x \in M^\infty$. In particular $\mathcal{C}_\omega = P(\omega) \cap M_1 = Q_3$ [KMS83].

4.2. **The uncountable case.** In this section, we will show that it is consistent with $\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$ that every recognizable subset of ω_1 is in M^∞.

Theorem 4.8. Assume that $\mathsf{ZF} + \mathsf{DC}$ holds, $M_1^\sharp(x)$ exists for every real x and for every $A \subseteq \omega_1$ there is a real x with $A \in \mathrm{L}[x]$. Moreover, assume that $\mathbb{P}$ is a homogeneous forcing that preserves ω_1 and G is a $\mathbb{P}$-generic filter over V. Then in $\mathrm{V}[G]$, every recognizable subset of ω_1 is an element of M^∞.

Proof. Under the hypothesis the structures $M_1^\sharp(x)$ are uniquely determined, since the comparison can be done in a model of the same form (see [Sch14b, Lemma 1.2.20]).

Let $\langle \kappa_\alpha \mid \alpha \in \mathrm{Ord} \rangle$ denote the sequence of images of the least measurable cardinal κ of M_1 obtained by iterating the unique measure on κ and its images and $\pi_{\alpha,\beta} \colon M^\alpha \to M^\beta$ the induced elementary embeddings. Assuming that $A \in \mathrm{V}[G]$ is a recognizable subset of ω_1, we have $A \in \mathrm{V}$ by Lemma 3.15. We now work in V and assume that x is a real with $A \in \mathrm{L}[x]$. For each countable ordinal α, we fix a forcing $\mathbb{P}_\alpha \in M^\alpha$ and a countable iteration tree on M^α with all critical points strictly above κ_α that induces an elementary embedding $\pi_\alpha \colon M^\alpha \to N^\alpha$ to some model N^α such that x is $\rho_\alpha(\mathbb{P}_\alpha)$-generic over N^α. The forcing $\mathbb{P}_\alpha$ can be chosen as $\mathrm{Col}(\omega, \delta_\alpha)$ for Neeman's genericity iteration [Nee95, Corollary 1.8], where δ_α is the Woodin cardinal of M^α, or as Woodin's extender algebra at δ_α for Woodin's genericity iteration [Ste10, Theorem 7.14], [NZ01, Section 2] (see also [Sch14b, Lemma 2.44] for the case that only ω_1-iterability is assumed).

We first claim that $A \in N^\alpha$ for all countable α. To prove this, we assume towards a contradiction that this fails for some α. Then there is a $\rho_\alpha(\mathbb{P}_\alpha)$-name $\dot{A} \in N^\alpha$ for A and a condition $p \in \rho_\alpha(\mathbb{P}_\alpha)$ which forces that $\dot{A} \notin N^\alpha$ and some program P recognizes $\dot{A}$ from an ordinal parameter β. We now pick a $\rho_\alpha(\mathbb{P}_\alpha)$-generic real y over $N^\alpha[x]$ in V such that the generic filter contains p. In particular x and y are mutually generic over N^α and hence $N^\alpha[x] \cap N^\alpha[y] = N^\alpha$. Thus the interpretation $\dot{A}^y$ of $\dot{A}$ in $N^\alpha[y]$ is distinct from A. Then the program P halts on input $\dot{A}^y$ with the parameter β and output 1, but this contradicts the uniqueness of A.

Therefore $A \cap \kappa_\alpha \in P(\kappa_\alpha) \cap N^\alpha = P(\kappa_\alpha) \cap M^\alpha = P(\kappa_\alpha) \cap M^\infty$ for all countable ordinals α. Since M^α is a direct limit for all countable limit ordinals α, there is some $f(\alpha) < \alpha$ such that $A \cap \kappa_\alpha$ has a preimage in $M^{f(\alpha)}$. Let S be a stationary subset of ω_1 such that f is constant on S with value α_0. Since $P(\kappa_{\alpha_0}) \cap M^\alpha$ is countable in V, there is a subset B of κ_{α_0} in M^{α_0} such that $\pi_{\alpha_0,\alpha}(B) = A \cap \kappa_\alpha$ for stationarily many $\alpha \geq \alpha_0$. To see that this equality actually holds for all countable ordinals $\alpha \geq \alpha_0$, assume that such an ordinal α is given and find some β above it that satisfies the equality. We then have $\pi_{\alpha_0,\alpha}(B) = \rho_{\alpha_0,\beta}(B) \cap \kappa_\alpha = A \cap \kappa_\alpha$. Since $\kappa_{\omega_1} = \omega_1$, we thus obtain $A = \pi_{\alpha_0,\omega_1+1}(B) \cap \omega_1 \in P(\omega_1) \cap M^{\omega_1+1} = P(\omega_1) \cap M^\infty$ as required. □

The assumptions in the previous theorem hold for instance for the $\mathbb{P}_{\max}$-extension of $L(\mathbb{R})$ (see [Woo10, Definition 4.33]) if the ground model $L(\mathbb{R})$ satisfies the axiom of determinacy [Kan09, Theorem 28.5]. In particular the conclusion is consistent with the existence of inner models with n Woodin cardinals for all $n \in \omega$ by [Sch10, Corollary 5.4].

Lemma 4.9. The existence of a Woodin cardinal does not imply the statement: *For any inner model M with a Woodin cardinal that is iterable for countable short trees, any recognizable subset of ω_1 is in M.*

Proof. Assuming that $M_1^\sharp$ exists and is $\omega_1 + 1$-iterable, we claim that there is a cardinal-preserving generic extension N of M_1 such that in N, there is a recognizable set of ordinals that is not an element of M_1. Working in M_1, we pick a Suslin tree T with the unique branch property. The existence of such a tree is proved from $\diamondsuit$ in [FH09, Theorem 1.1]. Moreover, suppose that b is a T-generic branch over M_1 and let $N = M_1[b]$. Then there is a unique branch in T of length ω_1 in N and since T is $<\omega_1$-distributive, the forcing does not add reals. Suppose that κ is the least measurable cardinal in M_1 and x is the subset of $(\kappa^+)^{M_1}$ that codes $M_1|(\kappa^+)^{M_1}$ with its canonical well-order via Gödel pairing. Now the proof of Lemma 4.6 shows that x is recognizable. Therefore the join $x \oplus b$ is recognizable, but it is not an element of M_1. □

4.3. **Further observations.** The main open problem is to characterize the recognizable closure for subsets of uncountable ordinals. It is worthwhile to note that sets that code transitive models with infinitely many measurable cardinals cannot be recognizable. To prove this, we use the next lemma, which follows immediately from a result of Kunen (see [Kan09, Theorem 19.17]).

Lemma 4.10. Suppose that $\langle \kappa_n \mid n \in \omega \rangle$ is a strictly increasing sequence of measurable cardinals, U_n is a $<\kappa_n$-complete ultrafilter on κ_n for each $n \in \omega$ and $j_n \colon V \to N_n$ is the ultrapower embedding with respect to U_n. Then for any ordinal α, we have that $j_n(\alpha) = \alpha$ holds for all but finitely many n.

Lemma 4.11. Suppose that M is a transitive model of ZFC containing the ordinals and $\langle \kappa_n \mid n \in \omega \rangle$ is a strictly increasing sequence of measurable cardinals in M with supremum κ. Any set $x \in M$ of ordinals with $V_\kappa^M \in L[x]$ is not recognizable.

Proof. Suppose that x is recognized by P with the parameter α. Since M contains all ordinals, this also holds in M. By Lemma 4.10, we can find a $<\kappa_n$-complete ultrafilter U on κ_n in M such that $j(\alpha) = \alpha$ holds for the ultrapower embedding $j \colon M \to N$ with respect to U. Since $U \notin N$ but $U \in V_\kappa^M \in L[x]$, we have $j(x) \neq x$. Moreover P halts with input $j(x)$, parameter α and output 1 in N and hence in V by the absoluteness of computations. However, this contradicts the uniqueness of x. □

It is moreover natural to define the following generic version of the recognizable closure. Assuming that there is a proper class of Woodin cardinals, the next theorem shows that this class contains exactly the sets of ordinals in M^∞.

Definition 4.12. (a) A subset x of an ordinal α is *generically recognizable* if it is recognizable in all $\mathrm{Col}(\omega, \beta)$-generic extensions for all sufficiently large ordinals β.
(b) The *generic recognizable closure* $\mathcal{C}_{\text{gen}}$ is the class of all sets x of ordinals such that x is computable from a generically recognizable set of ordinals.

We further define the *recognizable hull* $\mathcal{R}$ as the union of all $\mathrm{L}[x]$, where x is recognizable and moreover, the *generic recognizable hull* $\mathcal{R}_{\text{gen}}$ is defined similarly for the generically recognizable sets.

Theorem 4.13. Suppose that $M_1^\sharp(X)$ exists for all sets X of ordinals. Then the elements of $\mathcal{C}_{\text{gen}}$ are exactly the sets of ordinals in M^∞ and therefore $\mathcal{R}_{\text{gen}} = M^\infty$.

Proof. Since $M_1^\sharp$ is absolute to all generic extension by the assumptions (see [Sch14b, Lemma 3.4], the same holds for both M_1 and M^∞. Thus for any generically recognizable subset x of any ordinal α and any $\mathrm{Col}(\omega, \alpha)$-generic filter G over V, we have that $x \in (M^\infty)^{\mathrm{V}[G]} = M^\infty$ by Theorem 4.7. Conversely, suppose that $x \in M^\infty$ is a subset of an ordinal α. Let $\langle \kappa_\beta \mid \beta \in \mathrm{Ord} \rangle$ denote the sequence of images of the least measurable cardinal κ under the ultrapower of M_1 with the unique normal measure on κ and and its images. Let M denote the initial segment of M^∞ of height $\beta = (\kappa_\alpha^+)^{M^\infty}$. If $\gamma \geq \beta$ is arbitrary and G is any $\mathrm{Col}(\omega, \gamma)$-generic filter over V, then the proof of Lemma 4.6 shows that M has a recognizable code in $\mathrm{V}[G]$. Hence x is an element of the generic recognizable closure as required. □

5. Questions

We have seen in Theorem 3.12 that the recognizable closure can be characterized as the constructible closure of the class of sets that are implicitly definable over L. Moreover, Theorem 4.7 shows that the recognizable closure is closely connected to the inner model M^∞ if we assume the existence of a sufficient large cardinals. This motivates our main open question.

Question 5.1. Assuming that there is a proper class of Woodin cardinals, is every recognizable set of ordinals an element of M^∞?

Moreover, it seems natural to ask about general properties of the recognizable hull $\mathcal{R}$, for instance the following.

Question 5.2. Which axioms of set theory hold in $\mathcal{R}$ and in particular, does $\mathcal{R}$ always satisfy Σ_1-collection?

About the previous question, we know that Σ_2-replacement fails in the recognizable hull of the least iterable fine structural model $\mathrm{L}[\vec{\mu}]$ with infinitely many measurable cardinals. Moreover, since the set of recognizable reals is countable if $M_1^\sharp$ exists and has size ω_1 in L, we ask the following question.

Question 5.3. Is it consistent that there are ω_2 many recognizable reals?

While the proof of Lemma 3.12 shows that every subset of an ordinal that is implicitly definable over L is recognizable, the converse is open. In particular, Hamkins and Leahy asked (after Corollary 8 in [HL13]) whether $0^\sharp$ is implicitly definable over L. More generally, we can ask the following.

Question 5.4. Is is consistent that there is a recognizable set of ordinals that is not implicitly definable over L?

We can further consider the reducibility defined by $x \leq_{\text{rec}} y$ if x is in the recognizable closure of y. As we remarked, this generalizes Δ^1_2-reducibility and hence we ask the following question that is analogous to a result about Δ^1_2-degrees [Fri74].

Question 5.5. Is there a minimal $\leq_{\text{rec}}$-degree?

In a different direction, we can strengthen the model of computation to allow countable sets of ordinals as parameters, motivating the following question.

Question 5.6. Assuming that countable sets of ordinals are allowed as parameters, is the recognizable closure contained in the Chang model $\mathrm{L}(\mathrm{Ord}^\omega)$?

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