Based on a previous question, but adjusted to make it "Open book".

Let Trans(x) abbreviate 'x is transitive '.

(a) [11 marks]

(i) Prove that $\mathrm{Trans}(x) \leftrightarrow x \cup \bigcup x = x$.

(iii) Give an example of a set which is *not* transitive.

(b) [4 marks]

(i) Prove that if Trans(x) then Trans($\mathcal{P}(x)$).

(ii) Prove that if x is a set of transitive sets, then $\bigcup x$ is transitive.

(c) [7 marks]

For the V_α hierarchy prove carefully that for any $\alpha < \beta$ that $V_\alpha \subseteq V_\beta$. (Yu may assume all V_α are transitive.)

(d) Show that if two sets x, y have ranks, $\rho(x), \rho(y)$ both less than or equal to α then

(i) $\rho(\langle x, x, y\rangle) \leq \alpha + 4$;

(ii) $\rho({}^x y) \leq \alpha + 3$.

Solution: (a) (i) Suppose Trans(x). Let $y \in \bigcup x$. Then let $z \in x$ be such that $y \in z$. Trans(x) implies that $z \subseteq x$. Hence $y \in x$. Thus $x \cup \bigcup x \subseteq x$. That $\supseteq$ holds is trivial.

Conversely now suppose $x \cup \bigcup x = x$. But $\bigcup x \subseteq x$ implies that Trans(x) outright.

(b) (i) Let $y \in \mathcal{P}(x)$, then $y \subseteq x$; thus every $z \in y$ is in x as Trans(x). Similarly every $z \in y$ is also a subset of x. Hence every such $z \in \mathcal{P}(x)$. Thus $y \subseteq \mathcal{P}(x)$ and Trans($\mathcal{P}(x)$).

(ii) Let $z \in \bigcup x$. the for some $t \in x$, we have $z \in t$. As Trans(t), then $z \subseteq t$. But then $z \subseteq \bigcup x$.

(c) By induction on β: suppose true for all $\beta' < \beta$.

If $\beta = \beta' + 1$ then $V_\beta = \mathcal{P}(V_{\beta'})$. But if $u \in V_{\beta'}$ as the latter is transitive, $u \subseteq V_{\beta'}$. Hence $u \in \mathcal{P}(V_{\beta'}) = V_\beta$. Hence $V_{\beta'} \subseteq V_\beta$. If $\alpha < \beta'$ then by the Ind. Hyp. $V_\alpha \subseteq V_{\beta'} \subseteq V_\beta$.

Suppose now Lim(α). Then $V_\alpha = \bigcup_{\beta < \alpha} V_\beta$ so trivially $\forall \beta < \alpha (V_\beta \subseteq V_\alpha)$.

(d) We use that $\rho(x) = \sup\{\rho(y) + 1 \mid y \in x\}$.

Note that $\rho(x) \leq \alpha \to \rho(\{x\}) \leq \alpha + 1$, $\rho(\{\{x, x\}\}) = \rho(\{x\}, \{x, x\}) = \rho(\langle x, x\rangle) \leq \alpha + 2$ etc.

(i)

$\langle x, x, y\rangle = \langle\langle x, x\rangle, y\rangle = \{\{\langle x, x\rangle\}\{\langle x, x\rangle, y\}\}$

So $\rho(\{\langle x, x\rangle, y\}) \leq \sup\{\rho(\langle x, x\rangle) + 1, \rho(y) + 1\} \leq \sup(\alpha + 3, \alpha + 1) = \alpha + 3$.

But then $\rho(\langle x, x, y\rangle) \leq \sup\{\rho(\{\langle x, x\rangle\} + 1\}, (\alpha + 3) + 1\} = \alpha + 4$.

(ii)

For $\rho({}^x y)$: ${}^x y = \{f \mid \mathrm{Func}(f) \wedge f\colon x \to y\}$ so $\rho({}^x y) = \sup\{\rho(f) + 1 \mid f \in {}^x y\}$. But such a function consists of ordered pairs $\langle u, f(u)\rangle$ where $u \in x$, and $f(u) \in y$. Thus both $\rho(u) < \alpha$, $\rho(f(u)) < \alpha$. As $\langle u, f(u)\rangle = \{\{u\}, \{u, f(u)\}\}$, then $\rho(\langle u, f(u)\rangle) \leq \alpha + 1$. But then $\rho(f) = \sup\{\rho(\langle u, f(u)\rangle) + 1 \mid u \in x\} \leq \alpha + 2$. This makes $\rho({}^x y) \leq \alpha + 3$.