Revision Topics Set Theory

To help with revision, the following is a simple list of the things we have covered and that should be revised for the exam. It is only supposed to bring the topics into some kind of focus, as a revision aid, and is not a definitive list of musts and must-nots.

Chapter 1

- *Basic sets*: $\varnothing$,*singleton* set $\{x\}$, *unordered pair* $\{x, y\}$, *ordered pair* $\langle x, y\rangle$; *ordered* n*-tuples;* $\bigcup x$; $\bigcap x$; $\mathcal{P}(x)$.
- Sets and classes; *proper classes* such as Russell class $R = \{x \mid x \notin x\}$ which are not sets.
- $V = \{x \mid x = x\}$ the *universe of all sets*; V is a proper class.
- *Ordering relations: strict partial* orders and *strict total* orders $\prec$. *Wellorders. Isomorphism* between orderings.
- *Relations* R and *functions* F as sets of ordered pairs; $F``x$ and $F \upharpoonright x$ notation. ${}^XY = \{f \mid f: X \longrightarrow Y\}$
- *Transitive* sets; Trans(x); Know the Def.1.34 of TC(x) - *transitive closure* of a set x - and its basic properties: Lemma 1.35 and Ex. 1.27(i) & (ii).

Ch. 2

- The *von Neumann natural numbers* represented as $0 = \varnothing; n = \{0, ...n - 1\}, \ldots$.
- *Inductive* sets; ω, the set of natural numbers as the smallest inductve set
- *Principle of Mathematical Induction*
- **Omit:** the Section 2.2 *Peano's Axioms* for *Dedekind Systems* (**Warning:** also known as *Peano systems* - on much older past Examinations papers both terms have been used. We shall omit these from Exams. this year.)
- *Recursion Theorem* on ω (Thm 2.14). **Omit:** the isomorphism theorem between any two Dedekind systems $\langle \omega, \sigma, 0\rangle \cong \langle N, s, e\rangle$, Thm.2.18. Subsection 2.4.1: this is voluntary so to speak: we use this form of recursion when we come to ordinal recursion, so it introduces it. But the subsection itself we did not cover in lectures and is not examinable.

Ch. 3.

- *Principle of Transfinite Induction* for a wellorder. (Thm. 3.3).
- Elementary facts about wellorders (Lemmas 3.5-3.9).
- Def. of *ordinal*; elementary facts about ordinals (3.11-3.17) and the *Classification Theorem for Ordinals* 3.16. On forms a proper class (Burali-Forti Lemma 3.25).
- *Representation theorem for wellordersings* (3.20) and Def. of ot, *order type*.
- Basic properties of ordinals, *principle of transfinite induction* for On. Lemma 3.24.
- *Recursion Theorem for Ordinals Thm.3.35* and the *Second Form* Thm 3.38.
- Be able to give definition of *ordinal arithmetic* operations A_α, M_α, E_α and prove elementary facts about them (Lemma 3.40-44 and the interleaved exercises). You may
- **Omit**: **Lemma 3.43 and so the Exercise 3.19.**

Ch. 4

- *Equinumerosity*; $f: A \approx B$, *finite* and *infinite.* $\preceq, \prec$
- *Cantor's Theorems* (4.9-.10) $\omega \not\approx \mathbb{R}$ and $\forall X\ (\ X \not\approx \mathcal{P}(X)\ )$. *Cantor-Schröder-Bernstein* Thm.
- *Denumerably (= countably) infinite and countable* set.
- *Wellordering Principle* (WP)
- Countable union of countable sets is countable (Lemma 4.18)).
- *Cardinality* of a set, *cardinal number* (Def.4.22) and basic properties of cardinals.

• Be familiar with *cardinal arithmetic* operations $\oplus, \otimes$, and *cardinal exponentiation* κ^λ (Defs 4.24 & 30). Be able to prove basic properties of these, so know Hessenberg's Theorem 4.26, Cor. 4.27, Lemma 4.32, and the accompanying Exercise 4.15.

• The definitions of the *cardinal enumeration function* $\alpha \rightarrowtail \omega_\alpha$, *successor* and *limit* cardinal. **Omit:** Hartogs' Theorem, 4.33.

• Know the definition and meaning of the Continuum Hypothesis (CH) and GCH. **Know that $\aleph_\alpha$ is an alternative notation for ω_α.** Know the beth numbers $\beth_\alpha$ (Def 4.39.)

• **Omit:** On p53, "A note on Dedekind finite sets."

Ch. 5

• The Axiom of Replacement.

• The Axiom of Choice and its equivalence to WP; Zorn's Lemma.

• You should not memorise the principles on p.58 ("*Uniformisation principle .... Tychonoff-Kelley property*") if you are asked about them you will be given definitions.

• **Omit:** Subsection 5.2.1.

Ch. 6

• The *wellfounded hierarchy* of sets $V_\alpha\,(\alpha \in \mathrm{On})$, $V = WF = \bigcup_{\alpha\in\mathrm{On}} V_\alpha$. $\rho(x)$ *rank* of x.

• Basic properties of the V_α-hierarchy (Lemma 6.2-6.6). Be able to calculate the ranks of some simple sets, as in the Examples at the top of p62. Principle of $\in$-induction.

Essentially everything in this section is relevant except: **Omit:** The proof of the $\in$-Recursion Thm. 6.8.

• Also on the website is a sample Exam Q & A from a past year, amended to be in a somewhat "open book" format.

Note: The exam will be 2 and 1/2 hours, all 4 questions to be attempted.

Past papers: see Blackboard (–>Organisations -> Resources for Students –> Marks and Assessment–>Examinations). The Questions there, although emphasising different aspects of the course when they were taught by different lecturers are all relevant (unless mentioned as **omitted** as above). It perhaps goes without saying, that there will not be questions on every topic.