Revision Topics Set Theory

To help with revision, the following is a simple list of the things we have covered and that should be revised for the exam. It is only supposed to bring the topics into some kind of focus, as a revision aid, and is not a definitive list of musts and must-nots.

Chapter 1

- *Basic sets*: $\varnothing$,*singleton* set $\{x\}$, *unordered pair* $\{x,y\}$, *ordered pair* $\langle x,y\rangle$; *ordered n-tuples;* $\bigcup x$; $\bigcap x$; $\mathcal{P}(x)$.
- Sets and classes; *proper classes* such as Russell class $R=\{x\,|\,x\notin x\}$ which are not sets.
- $V=\{x\,|\,x=x\}$ the *universe of all sets*; V is a proper class.
- *Ordering relations: strict partial* orders and *strict total* orders $\prec$. *Wellorders. Isomorphism* between orderings.
- *Relations* R and *functions* F as sets of ordered pairs; $F``x$ and $F\restriction x$ notation. ${}^{X}Y=\{f\,|\,f\colon X\longrightarrow Y\}$
- *Transitive* sets; Trans(x); Know the Def.1.34 of TC(x) - *transitive closure* of a set x - and its basic properties: Lemma 1.35 and Ex. 1.26(i) & (ii).

Ch. 2

- The *von Neumann natural numbers* represented as $0=\varnothing; n=\{0,...n-1\},...$.
- *Inductive* sets; ω, the set of natural numbers as the smallest inductve set
- *Principle of Mathematical Induction*
- **Omit:** the Section 2.2 *Peano's Axioms* for *Dedekind Systems* (**Warning:** also known as *Peano systems* - on much older past Examinations papers both terms have been used. We shall omit these from Exams. this year.)
- *Recursion Theorem* on ω (Thm 2.14). **Omit:** the isomorphism theorem between any two Dedekind systems $\langle\omega,\sigma,0\rangle\cong\langle N,s,e\rangle$ (Thm.2.18).

Ch. 3.

- *Principle of Transfinite Induction* for a wellorder. (Thm. 3.3).
- Elementary facts about wellorders (Lemmas 3.5-3.9).
- Def. of *ordinal*; elementary facts about ordinals (3.11-3.17) and the *Classification Theorem for Ordinals* 3.16. On forms a proper class (Burali-Forti Lemma 3.25).
- *Representation theorem for wellordersings* (3.20) and Def. of ot, *order type*.
- Basic properties of ordinals, *principle of transfinite induction* for On. Lemma 3.24.
- *Recursion Theorem for Ordinals Thm.3.35*.
- Be able to give definition of *ordinal arithmetic* operations A_α, M_α, E_α and prove elementary facts about them (Lemma 3.38-44 and the interleaved exercises). You may
- **Omit**: **Lemma 3.42 and so the Exercise 3.18**
- **Omit:** also the Cantor Normal Form Theorem, Thm 3.45, Ex3.28.

Ch. 4

- *Equinumerosity*; $f\colon A\approx B$, *finite* and *infinite.* $\preceq,\prec$
- *Cantor's Theorems* (4.9-.10) $\omega\not\approx\mathbb{R}$ and $\forall X\,(X\not\approx\mathcal{P}(X))$. *Cantor-Schröder-Bernstein* Thm.
- *Denumerably (= countably) infinite and countable* set.
- *Wellordering Principle* (WP)
- Countable union of countable sets is countable (Lemma 4.18)).
- *Cardinality* of a set, *cardinal number* (Def.4.22) and basic properties of cardinals.

• Be familiar with *cardinal arithmetic* operations $\oplus, \otimes$, and *cardinal exponentiation* κ^λ (Defs 4.24 & 30). Be able to prove basic properties of these, so know Hessenberg's Theorem 4.26, Cor. 4.27, Lemma 4.32, and the accompanying Exercise 4.15.

• The definitions of the *cardinal enumeration function* $\alpha \rightarrowtail \omega_\alpha$, *successor* and *limit* cardinal. **Omit:** Hartogs' Theorem, 4.33.

• Know the definition and meaning of the Continuum Hypothesis (CH) and GCH. **Know that $\aleph_\alpha$ is an alternative notation for ω_α.** Know the beth numbers $\beth_\alpha$.

• **Omit:** p53, "A note on Dedekind finite sets."

Ch. 5

• You should not memorise the principles on p.58 ("*Uniformisation principle .... Tychonoff-Kelley property*") if you are asked about them you will be given definitions.

• **Omit:** Subsection 5.2.1.

Ch. 6

• The *wellfounded hierarchy* of sets $V_\alpha\,(\alpha \in \mathrm{On})$, $V = WF = \bigcup_{\alpha\in\mathrm{On}} V_\alpha$. $\rho(x)$ *rank* of x.

• Basic properties of the V_α-hierarchy (Lemma 6.3-6.6). Be able to calculate the ranks of some simple sets. Principle of $\in$-induction.

Essentially everything in this section is relevant except: **Omit:** Proof of the $\in$-Recursion Thm. 6.8.

Please note that the past 3 years Exam papers are on:

Blackboard (–> Student Resources –> Examinations –> Past Examinations)

and the Questions there, although emphasising different aspects of the course when they were taught by different lecturers are all relevant (unless mentioned as **omitted** as above). It perhaps goes without saying, that there will not be questions on every topic.

• Also on the website is a sample Exam Q & A from a past year, amended to be in an "open book" format.

Errata: 20-21

• In Def.1.23 add a sentence at the end, so Line 333: " $R^{-1} =_{\mathrm{df}} \{\langle y, x\rangle \mid \langle x, y\rangle \in R|\}$. "

• In the statement of the Uniformisation Principle (UP p. 58): "there is a function f: $X \longrightarrow Y$ " should read " there is a function $f{:}\,\mathrm{dom}(f) \longrightarrow Y$ with (i) dom(f)=dom(R) ..."

• Line 1881 (p.64) should read: " $...v = u \cup \{\langle z, G(u \restriction z)\rangle\}$."

• Corollary 3.21. In the proof, line 2, delete the "isomorphic to" occurring before "an initial segment".

• Proof of Thm 3.37: Third line should read:
"$F(u) = F_0(u(\beta))$ if $\mathrm{Func}(u) \wedge \mathrm{dom}(u)$ is a successor ordinal $\beta + 1$,"