1 Chapter 1 Exercise Solutions

Exercise 1.1. List all the members of V_3. Do the same for V_4. How many members will V_n have for $n \in \mathbb{N}$?

Exercise 1.2. Prove for $\alpha < 3$ that $V_{\alpha+1} = V_\alpha \cup \mathcal{P}(V_\alpha)$. (This will turn out to be true for any α.)

Exercise 1.3. We define the *rank* of a set x, $\rho(x)$, to be the least α such that $x \subseteq V_\alpha$. Compute $\rho(\{\{\varnothing\}\})$. Do the same for $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$.

Sol.:1.1 $V_0 = \varnothing$; $V_1 = \{\varnothing\}$; $V_2 = \{\varnothing, \{\varnothing\}\}$; $V_3 = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$;
V_4 has 16 elements. In general if X has k elements $\mathcal{P}(X)$ has 2^k elements. Hence, as can be proven by induction, V_n has $2^{2^{2^{\cdot^{\cdot^{2}}}}}$ elements with a stack of $n-1$ 2's, for $n \geq 1$.

Sol.:1.2 As can be seen $V_3 = \mathcal{P}(V_2) = V_2 \cup \mathcal{P}(V_2)$. (This is because each V_α is transitive: each member of V_α is also a subset of V_α.)

Sol.:1.3 $\mathrm{rank}(\{\{\varnothing\}\}) = 2; \mathrm{rank}(\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}) = 3$ by inspection.

Exercise 1.4. Give examples of sets x, y so that $x \neq y$ but $\bigcup x = \bigcup y$.

Sol: $x = \{\{\varnothing\}\}, y = \{\varnothing, \{\varnothing\}\}$. Then $\bigcup x = \bigcup y = \{\varnothing\}$.

Exercise 1.5. Show that if $a \in X$ then $\mathcal{P}(a) \in \mathcal{P}(\mathcal{P}(\bigcup X))$.

Sol: In general $\mathcal{P}(U) \in \mathcal{P}(V)$ if $\mathcal{P}(U) \subseteq V$. But if $a \in X$ then, if $t \subseteq a$ then $t \in \mathcal{P}(a)$ but also $t \in \mathcal{P}(\bigcup X)$ (the latter because $a \in X$ implies that if $t \subseteq a$ then $t \subseteq \bigcup X$). So $\mathcal{P}(a) \subseteq \mathcal{P}(\bigcup X)$.

Exercise 1.6. Show that for any set X: a) $\bigcup \mathcal{P}(X) = X$ b) $X \subseteq \mathcal{P}(\bigcup X)$; when do we have $=$ here?

Sol: a) $a \in \bigcup \mathcal{P}(X) \Longleftrightarrow \exists Y \in \mathcal{P}(X)(a \in Y) \Longleftrightarrow a \in X$.
b) $a \in X \Rightarrow \forall b \in a(b \in \bigcup X) \Rightarrow a \subseteq \bigcup X \Rightarrow a \in \mathcal{P}(\bigcup X)$. If $X = 1 = \{\varnothing\}$ or $X = 2 = \{\varnothing, \{\varnothing\}\}$ then we have equality. (If $X = 2$ then $\bigcup X = 1 = \{\varnothing\}$. $\mathcal{P}(1) = \{\varnothing, \{\varnothing\}\} = 2$.) More generally if $X = \mathcal{P}(Y)$, then $\bigcup X = Y$, hence $\mathcal{P}(\bigcup X) = X$.

Exercise 1.7. Show that the distributive laws (iv) above are valid.

Sol.: For example: $\bigcap_{i \in I} (A \cup B_i) = A \cup (\bigcap_{i \in I} B_i)$.
$x \in \bigcap_{i \in I} (A \cup B_i) \Longleftrightarrow \forall i \in I(x \in A \vee x \in B_i) \Longleftrightarrow x \in A \vee \forall i \in I(x \in B_i) \Longleftrightarrow$
$\Longleftrightarrow x \in A \cup (\bigcap_{i \in I} B_i)$.

Exercise 1.8. Let $I = \mathbb{Q} \cap (0, 1/2)$ be the set of rationals p with $0 < p < 1/2$. Let $A_p = \mathbb{R} \cap (1/2 - p, 1/2 + p)$. Show that $\bigcup_{p \in I} A_p = (0, 1)$; $\bigcap_{p \in I} A_i = \{1/2\}$.

Sol: $x \in \bigcup_{p \in I} A_p \Longleftrightarrow \exists p \in I(1/2 - p < x < 1/2 + p) \Rightarrow 0 < x < 1$. Conversely suppose $x \in (0, 1)$. As the rationals are "dense" in the reals, find $p \in \mathbb{Q}$ with $0 < |1/2 - x| < p < 1/2$. Then $x \in A_p$.

Exercise 1.9. Let $X_0 \supseteq X_1 \supseteq \cdots$ and $Y_0 \supseteq Y_1 \supseteq$ be two infinite sequences of shrinking sets. Show that $\bigcap_{i\in\mathbb{N}} (X_i \cup Y_i) = \bigcap_{i\in\mathbb{N}} X_i \cup \bigcap_{i\in\mathbb{N}} Y_i$. If we take away the requirement that the sequences be shrinking, does this hold in general for any infinite sequences X_i and Y_i?

Sol: $x \in \bigcap_{i\in\mathbb{N}} (X_i \cup Y_i) \Longleftrightarrow \forall i \in \mathbb{N}(x \in X_i \vee x \in Y_i)$. So suppose $x \in \bigcap_{i\in\mathbb{N}} (X_i \cup Y_i)$. If $x \notin \bigcap_{i\in\mathbb{N}} X_i$ then, because the sequence X_i of sets is decreasing, there is some i_0 so that for all $i \geq i_0\, x \notin X_i$. Then we must have $\forall i \in \mathbb{N}(x \in Y_i)$, *i.e.* $x \in \bigcap_{i\in\mathbb{N}} Y_i$. We reason similarly if $x \notin \bigcap_{i\in\mathbb{N}} Y_i$.

Conversely if $x \in \bigcap_{i\in\mathbb{N}} X_i \cup \bigcap_{i\in\mathbb{N}} Y_i$, assume without loss of generality that $x \in \bigcap_{i\in\mathbb{N}} X_i$. The trivially $x \in \bigcap_{i\in\mathbb{N}} (X_i \cup Y_i)$.

Clearly the first part of this argument worked because the sequences were decreasing. We can show that this is an essential requirement by providing a non-decreasing sequence counterexample: take $X_{2i} = \mathbb{R} \cap (0, \frac{1}{2i})$; $X_{2i+1} = \mathbb{R} \cap (\frac{1}{2i+1}, 1)$; $Y_i = \mathbb{R} \cap (0,1) - X_i$. Then $(0,1) = \bigcap_{i\in\mathbb{N}} (X_i \cup Y_i) \neq \bigcap_{i\in\mathbb{N}} X_i \cup \bigcap_{i\in\mathbb{N}} Y_i = \varnothing$.

Exercise 1.10. Think about how you would frame an alternative, but equivalent definition of partial order in terms of the non-strict ordering $\preceq$. Which of the defining conditions above do we need?

Sol: If using $\preceq$ rather than $\prec$, one would only have to substitute $\preceq$ for $\prec$ in (ii) and (iii); and change (i) to: $x \in X \to x \preceq x$, that is, it should be *reflexive*; then add the *anti-symmetric* condition, but remain *transitive*.

Exercise 1.11. If $(A, \prec)$ is a total ordering and A is finite, show that it is a wellordering.

Sol. As we are given that $\langle A, \prec\rangle$ is a total ordering, the only thing left to show is that (b) holds: let $\varnothing \neq Y \subseteq A$. Then Y only has finitely many elements. If $a_0 \in Y$ i any element, then $Y_0 = \{a \in Y \mid a \prec a_0\}$ is also finite and has, *e.g.* k elements. Pick $a_1 \in Y_0$ and let $Y_1 = \{a \in Y_1 | \, a \prec a_1\}$. Then Y_1 has $k_1 < k_0$ many elements. Proceeding in this way, after finitely many stages we reach $k_n = 0$, for some n. Then a_{n-1} is the $\prec$-least element of A.

Exercise 1.12. Suppose for no sets x, u do we have $x \in u \in x$. Then if we define $\langle x, y\rangle_1 = \{x, \{x, y\}\}$ then $\langle x, y\rangle_1$ also satisfies the Uniqueness statement of Lemma 1.7.

Sol: Suppose $\langle x, y\rangle_1 = \langle u, v\rangle_1$. *Case 1* $x = y$. Then $\langle x, x\rangle_1 = \{x, \{x, x\}\} = \{x, \{x\}\} = \langle u, v\rangle_1 = \{u, \{u, v\}\}$. Suppose $u \neq v$. Then $\{x\} \neq \{u, v\}$ since $\{x\}$ has one member and the latter set two. So $\{x\} = u$ and $x = \{u, v\}$. But then $x \in \{x\} \in x$! Contradiction. Hence $u = v$. If $x \neq u$ then $x = \{u\}$ and $\{x\} = u$. But again this implies $u \in x \in u$! So $x = u$ and we are done in this case. *Case 2* $x \neq y$. Then similarly we must have $u \neq v$. If $x = \{u, v\}$ then $u = \{x, y\}$. But then $x \in u \in x$ - a contradiction. So $x = u$ and then $\{x, y\} = \{u, v\}$.

Exercise 1.13. Does $\{\{x\}, \{x, y\}, \{x, y, z\}\}$ give a good definition of ordered triple? Does $\{\langle x, y\rangle, \langle y, z\rangle\}$?

Sol: For the first question: no. Consider $\langle x, y, x\rangle$ and $\langle x, y, y\rangle$. The second definition also fails. If $x = z$, but $x \neq y$ we have $\langle x, y, x\rangle = \{\langle x, y\rangle, \langle y, x\rangle\} = \{\langle y, x\rangle, \langle x, y\rangle\} = \langle y, x, y\rangle$.

Exercise 1.14. Let $\mathfrak{P}$ be the class of all ordered pairs. Show that $\mathfrak{P}$ is a proper class - that is - it is not a set. [Hint: suppose for a contradiction it was a set; apply the axiom of union.]

Sol: Suppose for a contradiction that $z \in V$ (so z is a set) but $z = \mathfrak{P}$. Consider $U = \bigcup \bigcup z$. By the Ax. of Union (applied twice) U is a set. But now note $U = V$: clearly $U \subseteq V$, but if $x \in V$ then $\langle x, x\rangle = \{\{x\}, \{x, x\}\} \in \mathfrak{P}$. And then $x \in U$. So $V \subseteq U$ and with the above $V = U$, But we know that V is a *proper* class: it cannot be any set U.

Exercise 1.15. Show that if $x \in A$, $y \in A$ then $\langle x, y\rangle \in \mathcal{P}(\mathcal{P}(A))$. Deduce that if $x, y \in V_n$ then $\langle x, y\rangle \in V_{n+2}$.

Sol: Both $\{x\}, \{x, y\} \in \mathcal{P}(A)$, hence $\langle x, y\rangle = \{\{x\}, \{x, y\}\} \in \mathcal{P}(\mathcal{P}(A))$. If $x, y \in V_n$ then with V_n as A, we have $\langle x, y\rangle \in V_{n+2} = \mathcal{P}(\mathcal{P}(V_n))$.

Exercise 1.16. Show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$. Show that if $A \times B = A \times C$ and $A \neq \varnothing$, then $B = C$.

Sol: $x \in A \times (B \cup C) \Longleftrightarrow \exists u \in A \exists v \in B \cup C (x = \langle u, v \rangle)$.
$\Longleftrightarrow \exists u \in A (\exists v \in B (x = \langle u, v \rangle) \vee \exists v \in C (x = \langle u, v \rangle))$
$\Longleftrightarrow x \in (A \times B) \cup (A \times C)$.
Suppose $A \times B = A \times C$ and $A \neq \varnothing$. Pick $x \in A$. Hence if $A \times B = A \times C$ then
$\forall v \in B \; \langle x, v \rangle \in A \times B \longleftrightarrow \langle x, v \rangle \in A \times C$. And
$\forall v \in C \; \langle x, v \rangle \in A \times B \longleftrightarrow \langle x, v \rangle \in A \times C$. But these two imply that $B = C$.

Exercise 1.17. Show that $A \times \bigcup B = \bigcup \{A \times X \mid X \in B\}$.

Sol: $x \in A \times \bigcup B \Longleftrightarrow \exists u \in A \exists v \in \bigcup B (x = \langle u, v \rangle)$
$\Longleftrightarrow \exists u \in A \exists X \in B \exists v \in X (x = \langle u, v \rangle)$
$\Longleftrightarrow \exists u \in A \exists X \in B (x \in A \times X)$
$\Longleftrightarrow x \in \bigcup \{A \times X \mid X \in B\}$.

Exercise 1.18. We define the 'unpairing functions' $(u)_0$ and $(u)_1$ so that if $u = \langle x, y \rangle$ then $(u)_0 = x$ and $(u)_1 = y$ (or in other words $u = \langle (u)_0, (u)_1 \rangle$). Show that these can be expressed as: $(u)_0 = \bigcup \bigcap u$; $(u)_1 = \bigcup (\bigcup u - \bigcap u)$ if $\bigcup u \neq \bigcap u$; and $(u)_1 = \bigcup \bigcup u$ otherwise.

Sol: The first element $(u)_0$ of the ordered pair $u = \langle x, y \rangle$ is the only one common to both elements of u. That is $z = (u)_0$ iff z is in both elements of $\{\{x\}, \{x, y\}\}$. But $\bigcap u = \{x\}$. And then $\bigcup \{x\} = x$. The second element $z = (u)_1$ iff z is in the second element but not both elements of $\{\{x\}, \{x, y\}\}$, *i.e.* $z \in \{z\} = (\bigcup u - \bigcap u)$ except when there is no such z, in which case z is the unique element ofthe unique set in u, *i.e.* $z \in \{z\} = \bigcup u$.

Exercise 1.19. (i) Find a counterexample to the assertion $F \cap A^2$ equals $F \restriction A$. (ii) Show $F \restriction A = F \cap (A \times \mathrm{ran}(F))$.

Sol: (i) Let $F = \{\langle 0, 2 \rangle\}$; then with $A = \{0\}$, $A^2 = \{\langle 0, 0 \rangle\}$; hence $F \cap A^2 = \varnothing$, but $F \restriction A = F$. (ii) ($\subseteq$) follows immediately from the definitions of $F \restriction A$ *etc.* If $u \in \langle x, y \rangle \in F \cap (A \times \mathrm{ran}(F))$, then $\langle x, y \rangle \in F$ implies $F(x) = y$, and $\langle x, y \rangle \in A \times \mathrm{ran}(F)$ implies $\langle x, y \rangle \in F \restriction A$.

Exercise 1.20. As a further exercise in using this notation, suppose T is a class of functions, with the property that that for any two $f, g \in T$, $f \restriction (\mathrm{dom}(f) \cap \mathrm{dom}(g)) = g \restriction (\mathrm{dom}(f) \cap \mathrm{dom}(g))$ (more simply put: they both agree on the part of their domains they have in common). Then check a) $F = \bigcup T$ is a function, and b) $\mathrm{dom}(F) = \bigcup \{\mathrm{dom}(g) \mid g \in T\}$.

Sol: a) Note that F is a union of a set of functions, so F is certainly a set of ordered pairs, and thus has the potential to be a function. It would only fail to be a function if it was not single-valued: that is for some x, y, y' both $\langle x, y \rangle$ and $\langle x, y' \rangle$ were in F, but $y \neq y'$. But this would require two different functions $f, g \in T$ with, *e.g.*, $f(x) = y$ and $g(x) = y'$. But this is explcitly ruled out by supposition. b) As $F = \bigcup T$, x is the first element of an ordered pair in F iff for some $g \in T$ x is the first element of an ordered pair in g. In mathematical notation that is the given equation.

Exercise 1.21. Suppose X, Y both have rank n ("$\rho(X) = n$"). Compute a) $\rho(X \times Y)$; *b*) $\rho({}^Y X)$ [Hint for *b*): show first if $X, Y \in Z$ show that ${}^Y X \in \mathcal{PPPP}(Z)$.

Sol: Recall that $\rho(Z) =_{\mathrm{df}}$ the least α such that $Z \subseteq V_\alpha$ (equivalently, the least α so that $Z \in V_{\alpha+1}$). (a) If $X, Y \subseteq V_n$ then for any $u \in X, v \in Y u, v \in V_n$. Then $\{u\}, \{u, v\} \in V_{n+1}$, and $\langle u, v \rangle = \{\{u\}, \{u, v\}\} \in V_{n+2}$. Hence $X \times Y$, being the set of all such ordered pairs is a subset of V_{n+2}. Hence $\rho(X \times Y) = n + 2$.

(b): If $f \in {}^Y X$ then f is a subset of $Y \times X$. Hence $f \in V_{n+3}$. Hence ${}^Y X \subseteq V_{n+3}$. Hence $\rho({}^Y X) = n + 3$.

Exercise 1.22. (i) Let $\mathrm{Trans}(Z) \wedge x \subseteq Z$. Then $Z \cup \{x\}$ is transitive.

(ii) If x, y are transitive, then so are: $x \cup y, x \cap y, S(x), \bigcup x$.

(iii) Let X be a set of transitive sets. then $\bigcup X$ and $\bigcap X$ are transitive.

(iv) Show that $\mathrm{Trans}(x) \longleftrightarrow \mathrm{Trans}(\mathcal{P}(x))$. Deduce that each V_n is transitive.

Sol: (i) is immediate.

(ii) Assume $\mathrm{Trans}(x), \mathrm{Trans}(y)$. $x \cup y$: let $z \in x \cup y$; if $z \in x$ say, then $z \subseteq x \subseteq x \cup y$. $x \cap y$: if $z \in x \cap y$ then $z \subseteq x \wedge z \subseteq y$, so $z \subseteq x \cap y$.

$S(x)$: This is just a special case of (i) with $Z = x$. Hence $\mathrm{Trans}(S(x))$.

Suppose $u \in \bigcup x$. Then for some $z \in x$ $u \in z$. $\mathrm{Trans}(x) \longrightarrow u \in x$ hence $\bigcup x \subseteq x$ (this is only reproving the comment after Def. 1.29) and a second time, $u \subseteq x$. Then $t \in u \in \bigcup x \longrightarrow t \in u \in x \longrightarrow t \in \bigcup x \longrightarrow \mathrm{Trans}(\bigcup x)$.

(iii) Easy: if $z \in \bigcup X$ and $t \in z$, then for some $y \in X$, we have $z \in y$. But $\mathrm{Trans}(y)$ so $z \subseteq y$. So $t \in y \in X$. Assume $X \neq \varnothing$, then $z \in \bigcap X \leftrightarrow \forall y \in X (z \subseteq y) \leftrightarrow z \subseteq \bigcap X$.

(iv) $(\longrightarrow)$ $t \in \mathcal{P}(x) \longrightarrow t \subseteq x \longrightarrow \forall u \in t (u \in x) \longrightarrow \forall u \in t (u \subseteq x) \longrightarrow t \subseteq \mathcal{P}(x) \longrightarrow \mathrm{Trans}(\mathcal{P}(x))$.

$(\longleftarrow)$ $u \in x \longrightarrow \{u\} \in \mathcal{P}(x) \longrightarrow \{u\} \subseteq \mathcal{P}(x) \longrightarrow u \subseteq x \longrightarrow \mathrm{Trans}(x)$.

Since $\mathrm{Trans}(V_0)$ and $V_{n+1} = \mathcal{P}(V_n)$ we have by a mathematical induction that for any n $\mathrm{Trans}(V_n)$ using the above.

Exercise 1.23. Prove the $(\longleftarrow)$ direction of the Lemma 1.32: $\mathrm{Trans}(x) \longleftrightarrow \bigcup S(x) = x$. Prove that $x \subseteq y \wedge \mathrm{Trans}(y) \longrightarrow \bigcup x \subseteq y$.

Sol: $\bigcup (x \cup \{x\}) = \bigcup x \cup \bigcup \{x\} = \bigcup x \cup x$. If this equals x by hypothesis, then $\bigcup x \subseteq x$. Hence $\forall u \in x (u \subseteq x) \longrightarrow \mathrm{Trans}(x)$. Assume $x \subseteq y \wedge \mathrm{Trans}(y)$. If $z \in u \in x$ then also $z \in u \in y$. As $\mathrm{Trans}(y)$ then $u \subseteq y$. So $z \in y$. So $\bigcup x \subseteq y$.

Exercise 1.24. (i) What sets would you have to add to $\{\{\{\varnothing\}\}\}$ to make it transitive?

(ii) In general given a set x think about how a transitive y could be found with $y \supseteq x$. (It will turn out that for any set x there is a smallest $y \supseteq x$ with $\mathrm{Trans}(y)$.) [Hint: consider repeated applications of $\bigcup$: $\bigcup x$, $\bigcup \bigcup x$, $\bigcup \bigcup \bigcup x$,..., $\bigcup^n x$, ...]

Sol: (i) $X_0 = \{\{\{\varnothing\}\}\}$ is intransitive because it has "$\in$-holes" : $\{\{\varnothing\}\} \in \{\{\{\varnothing\}\}\}$, but $\{\{\varnothing\}\} \not\subseteq \{\{\{\varnothing\}\}\}$, because $\{\varnothing\} \notin \{\{\{\varnothing\}\}\}$. So enlarge X_0 to $X_1 = X_0 \cup \{\{\varnothing\}\}$. However X_1 is still intransitive (because $\varnothing \notin X_1$). So let $X_2 = X_1 \cup \{\varnothing\} = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}\}$. Then $\mathrm{Trans}(X_2)$. Note that $X_1 = \bigcup X_0 \cup X_0$ and $X_2 = \bigcup \bigcup X_0 \cup \bigcup X_0 \cup X_0$.

(ii) We could follow the idea of part (i) throwing into X_n all those members of members of ... members of X_0 for any $n+1$-length chain $x_n \in x_{n-1} \in \ldots x_1 \in x_0 \in X_0$. Then we should have $X_n = \bigcup^n X_0 \cup \bigcup^{n-1} X_0 \cup \ldots \cup \bigcup X_0 \cup X_0$. If $\tilde{X}$ is, informally speaking, the union of this process: $\tilde{X} = \ldots \bigcup^n X_0 \cup \bigcup^{n-1} X_0 \cup \ldots \cup \bigcup X_0 \cup X_0$ for every n then $\mathrm{Trans}(\tilde{X})$.

Exercise 1.25. Show that $y \in \bigcup^n x \leftrightarrow \exists x_n, x_{n-1}, \ldots, x_1, x_0 (y \in x_n \in x_{n-1} \in \cdots \in x_1 \in x_0 = x)$.

Sol: Very similar to the last: by induction on n: for $n = 0$ it is trivial. Suppose true for k, then $y \in \bigcup^{k+1} x \leftrightarrow y \in \bigcup (\bigcup^k x) \leftrightarrow \exists x_{k+1} (y \in x_{k+1} \in \bigcup^k x) \leftrightarrow$ (by the I.H.)

$\leftrightarrow \exists x_{\mathrm{nk+1}}, x_k, \ldots, x_1, x_0 (y \in x_{k+1} \in x_k \in \cdots \in x_1 \in x_0 = x)$.

Exercise 1.26. Show that (i) $y\in x\longrightarrow \mathrm{TC}(y)\subseteq \mathrm{TC}(x)$; (ii) $\mathrm{TC}(x)=x\cup\bigcup\{\mathrm{TC}(y)\,|\,y\in x\}$

Sol: (i) Suppose $y\in x$. Then by induction $\bigcup^n y\subseteq\bigcup^{n+1}x$. Hence $\mathrm{TC}(y)\subseteq\mathrm{TC}(x)$. (ii) By this first part we have that $\mathrm{TC}(x)\supseteq x\cup\bigcup\{\mathrm{TC}(y)\,|\,y\in x\}$ (as $\mathrm{TC}(x)\supseteq\mathrm{TC}(y)$ for any $y\in x$). To show $\subseteq$: (1st Method) by induction on k we show $\bigcup^k x\subseteq x\cup\bigcup\{\mathrm{TC}(y)\,|\,y\in x\}$. For $k=0$ this is trivial. Suppose true for k. let $z\in\bigcup^{k+1}x$. Then for some $t\in\bigcup^k x$, $z\in t$. By Ind. Hyp. either $t\in x$ (and so in which case $z\in\mathrm{TC}(t)$, and we are done) or there is $y\in x$ with $t\in\mathrm{TC}(y)$. But then $z\in t\subseteq\mathrm{TC}(y)$). This suffices. (2nd Method) Just argue that $T=_{\mathrm{df}}x\cup\bigcup\{\mathrm{TC}(y)\,|\,y\in x\}$ is transitive, and then $\mathrm{TC}(x)\subseteq T$ is automatic, as $\mathrm{TC}(x)$ we've already seen is the smallest transitive set containing x. Let $z\in T$. If $z\in x$, then $z\subseteq\mathrm{TC}(z)\subseteq T$, so $z\subseteq T$. If $z\in\bigcup\{\mathrm{TC}(y)\,|\,y\in x\}$ then for some $y\in x$, $z\in\mathrm{TC}(y)$. As the latter is transitive $z\subseteq\mathrm{TC}(y)\subseteq T$ again.

Exercise 1.27. If f is a (1-1) function show that $f^{-1}\subseteq\mathcal{P}\mathcal{P}(\bigcup\{\mathrm{dom}(f),\mathrm{ran}(f)\})$.

Sol: Note that $\langle x,\,y\rangle\in f\longleftrightarrow\langle y,\,x\rangle\in f^{-1}$. That is $\mathrm{dom}(f^{-1})=\mathrm{ran}(f)$. But $\langle y,\,x\rangle=\{\{y\},\{x,y\}\}\in f^{-1}$ implies $\{y\}\in\mathcal{P}(\mathrm{dom}(f^{-1}))\wedge\{x,y\}\in\mathcal{P}(\mathrm{dom}(f)\cup\mathrm{ran}(f))$. Using that $\mathrm{dom}(f)\cup\mathrm{ran}(f)=\bigcup\{\mathrm{dom}\{f\},\mathrm{ran}\{f\}\}$ we get that $\{\{y\},\{x,y\}\}\in\mathcal{P}\mathcal{P}(\bigcup\{\mathrm{dom}\{f\},\mathrm{ran}\{f\}\})$. [This did not use that f was even a function, but merely a relation.]